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Research Article

Strong Convergence of Parallel Iterative Algorithm with Mean Errors for Two Finite Families of Ćirić Quasi-Contractive Operators

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The purpose of this paper is to establish a strong convergence of a new parallel iterative algorithm with mean errors to a common fixed point for two finite families of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalize and improve the corresponding results of Berinde, Gu, Rafiq, Rhoades, and Zamfirescu.

1. Introduction and Preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *a-contraction*, if $d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, where $a \in (0, 1)$.

The mapping $T : X \to X$ is said to be Kannan mapping [1], if there exists $b \in (0, 1/2)$ such that $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$.

A mapping $T : X \to X$ is said to be Chatterjea mapping [2], if there exists $c \in (0, 1/2)$ such that $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

Combining these three definitions, Zamfirescu [3] proved the following important result.

Theorem Z (see [3]). *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist the real numbers a, b, and c satisfying $a \in (0, 1)$, $b, c \in (0, 1/2)$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$,

(z_3) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N} \tag{1.1}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

Remark 1.1. An operator T satisfying the contractive conditions (z_1)–(z_3) in the above theorem is called Z-operator.

Remark 1.2. The conditions $(z_1)-(z_3)$ can be written in the following equivalent form:

$$d(Tx,Ty) \le h \max\left\{ d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2} \right\}, \tag{1.2}$$

for all $x, y \in X, 0 < h < 1$. Thus, a class of mappings satisfying the contractive conditions $(z_1)-(z_3)$ is a subclass of mappings satisfying the following condition:

$$d(Tx,Ty) \le h \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \right\}, \tag{CG}$$

$0 < h < 1$. The class of mappings satisfying (CG) is introduced and investigated by Ćirić [4] in 1971.

Remark 1.3. A mapping satisfying (CG) is commonly called Ćirić generalized contraction.

In 2000, Berinde [5] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \le \rho\|x - y\| + L\|Tx - x\|, \tag{1.3}$$

for any $x, y \in E$, $0 \le \delta < 1$ and $L \ge 0$.

Note that (1.3) is equivalent to

$$\|Tx - Ty\| \le \rho\|x - y\| + L \min\{\|Tx - x\|, \|Ty - y\|\}, \tag{1.4}$$

for any $x, y \in E$, $0 \le \rho < 1$ and $L \ge 0$.

Berinde [5] proved that this class is wider than the class of Zamfiresu operators and used the Mann [6] iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem.

Theorem B (see [5]). *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.3) *and $F(T) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \ge 0, \tag{1.5}$$

where $\{\alpha_n\}$ *be a real sequence in* $[0,1]$. *If* $\sum_{n=1}^{\infty}\alpha_n = \infty$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T.

In 2006, Rafiq [7] considered a class of mappings satisfying the following condition:

$$\|Tx - Ty\| \le h \max\left\{\|x-y\|, \frac{\|x-Tx\| + \|y-Ty\|}{2}, \|x-Ty\|, \|y-Tx\|\right\}, \tag{CR}$$

$0 < h < 1$. This class of mappings is a subclass of mappings satisfying the following condition:

$$\|Tx - Ty\| \le h \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}, \tag{CQ}$$

$0 < h < 1$. The class of mappings satisfying (CQ) was introduced and investigated by Ćirić [8] in 1974 and a mapping satisfying is commonly called Ćirić quasi-contraction.

Rafiq [7] proved the following result.

Theorem R (see [7]). *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* $T : C \to C$ *be an operator satisfying the condition* (CR). *For given* $x_0 \in C$, *let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \ge 0, \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be three real sequences in* $[0,1]$ *satisfying* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \ge 1$, $\{u_n\}$ *is a bounded sequences in* C. *If* $\sum_{n=1}^{\infty}\beta_n = \infty$ *and* $\gamma_n = o(\alpha_n)$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T.

In 2007, Gu [9] proved the following theorem.

Theorem G (see [9]). *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* $\{T_i\}_{i=1}^{N} : C \to C$ *be* N *operators satisfying the condition* (CR) *with* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ *(the set of common fixed points of* $\{T_i\}_{i=1}^{N}$*). Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be three real sequences in* $[0,1]$ *satisfying* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \ge 1$, $\{u_n\}$ *a bounded sequences in* C *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}\beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty}\gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

Suppose further that $x_0 \in C$ *is any given point and* $\{x_n\}$ *is generated by the algorithm*

$$x_{n+1} = \alpha_n x_n + \beta_n T_n x_n + \gamma_n u_n, \quad n \ge 0, \tag{1.7}$$

where $T_n = T_{n(\bmod N)}$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^{N}$.

Remark 1.4. It should be pointed out that Theorem G extends Theorem R from a Ćirić quasi-contractive operator to a finite family of Ćirić quasi-contractive operators.

Inspired and motivated by the facts said above, we introduced a new two-step parallel iterative algorithm with mean errors for two finite family of operators $\{S_i\}_{i=1}^m$ and $\{T_j\}_{j=1}^k$ as follows:

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n \sum_{i=1}^{m} \lambda_i S_i y_n + \gamma_n u_n, \quad n \geq 1,$$
$$y_n = (1 - \beta_n - \delta_n)x_n + \beta_n \sum_{j=1}^{k} \mu_i T_j x_n + \delta_n v_n, \quad n \geq 1, \tag{1.8}$$

where $\{\lambda_i\}_{i=1}^m$, $\{\mu_j\}_{j=1}^k$ are two finite sequences of positive number such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{j=1}^k \mu_j = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are four real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C and x_0 is a given point.

Especially, if $\{\alpha_n\}$, $\{\gamma_n\}$ are two sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^m \subset [0,1]$ satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, $\{u_n\}$ is a bounded sequence in C and x_0 is a given point in C, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n \sum_{i=1}^{m} \lambda_i S_i x_n + \gamma_n u_n, \quad n \geq 1 \tag{1.9}$$

is called the one-step parallel iterative algorithm with mean errors for a finite family of operators $\{S_i\}_{i=1}^m$.

The purpose of this paper is to study the convergence of two-steps parallel iterative algorithm with mean errors defined by (1.8) to a common fixed point for two finite family of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalized and extend the corresponding results of Berinde [5], Gu [9], Rafiq [7], Rhoades [10], and Zamfirescu [3]. Even in the case of $\beta_n = \delta_n = 0$ or $\gamma_n = \delta_n = 0$ for all $n \geq 1$ or $m = k = 1$ are also new.

In order to prove the main results of this paper, we need the following Lemma.

Lemma 1.5 (see [11]). *Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq n_0, \tag{1.10}$$

where n_0 is some nonnegative integer, $t_n \in [0,1]$, $\sum_{n=0}^\infty t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^\infty c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let C be a nonempty closed convex subset of a normed space E. Let $\{S_i\}_{i=1}^m : C \to C$ be m operators satisfying the condition (CR) and $\{T_j\}_{j=1}^k : C \to C$ be k operators satisfying the* *condition* (*CR*) *with* $F = (\bigcap_{i=1}^{m} F(S_i)) \cap (\bigcap_{j=1}^{k} F(T_j)) \neq \emptyset$, *where* $F(S_i)$ *and* $F(T_j)$ *are the set of fixed points of* S_i *and* T_j *in* C, *respectively. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be four real sequences in [0, 1] satisfying* $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$, $\{\lambda_i\}_{i=1}^{m}$, $\{\mu_j\}_{j=1}^{k}$ *two finite sequences of positive number such that* $\sum_{i=1}^{m} \lambda_i = 1$ *and* $\sum_{j=1}^{k} \mu_j = 1$, $\{u_n\}$ *and* $\{v_n\}$ *two bounded sequences in* C *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

Suppose further that $x_0 \in C$ *is any given point and* $\{x_n\}$ *is an iteration sequence with mane errors defined by* (1.8), *then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{S_i\}_{i=1}^{m}$ *and* $\{T_j\}_{j=1}^{k}$.

Proof. Since $\{S_i\}_{i=1}^{m} : C \to C$ is m Ćirić operator satisfying the condition (CR), hence there exists $0 < h_i < 1$ ($i \in I = \{1, 2, \ldots, m\}$) such that

$$\|S_i x - S_i y\| \leq h_i \max\left\{ \|x - y\|, \frac{\|x - S_i x\| + \|y - S_i y\|}{2}, \|x - S_i y\|, \|y - S_i x\| \right\}. \tag{2.1}$$

For each fixed $i \in I = \{1, 2, \ldots, m\}$. Denote $h = \max\{h_1, h_2, \ldots, h_m\}$, then $0 < h < 1$ and

$$\|S_i x - S_i y\| \leq h \max\left\{ \|x - y\|, \frac{\|x - S_i x\| + \|y - S_i y\|}{2}, \|x - S_i y\|, \|y - S_i x\| \right\} \tag{2.2}$$

hold for each fixed $i \in I = \{1, 2, \ldots, m\}$. If from (2.2) we have

$$\|S_i x - S_i y\| \leq \frac{h}{2}\left[\|x - S_i x\| + \|y - S_i y\|\right], \tag{2.3}$$

then

$$\begin{aligned} \|S_i x - S_i y\| &\leq \frac{h}{2}\left[\|x - S_i x\| + \|y - S_i y\|\right] \\ &\leq \frac{h}{2}\left[\|x - S_i x\| + \|y - x\| + \|x - S_i x\| + \|S_i x - S_i y\|\right]. \end{aligned} \tag{2.4}$$

Hence

$$\left(1 - \frac{h}{2}\right)\|S_i x - S_i y\| \leq \frac{h}{2}\|x - y\| + h\|x - S_i x\|, \tag{2.5}$$

which yields (using the fact that $0 < h < 1$)

$$\|S_i x - S_i y\| \leq \frac{h/2}{1 - h/2}\|x - y\| + \frac{h}{1 - h/2}\|x - S_i x\|. \tag{2.6}$$

Also, from (2.2), if

$$\left\|S_i x - S_i y\right\| \leq h \max\left\{\left\|x - S_i y\right\|, \left\|y - S_i x\right\|\right\} \tag{2.7}$$

holds, then

(a) $\|S_i x - S_i y\| \leq h\|x - S_i y\|$, which implies $\|S_i x - S_i y\| \leq h\|x - S_i x\| + h\|S_i x - S_i y\|$ and hence, as $h < 1$,

$$\left\|S_i x - S_i y\right\| \leq \frac{h}{1-h}\|x - S_i x\|, \tag{2.8}$$

or

(b) $\|S_i x - S_i y\| \leq h\|y - S_i x\|$, which implies

$$\left\|S_i x - S_i y\right\| \leq h\left\|y - x\right\| + h\|x - S_i x\|. \tag{2.9}$$

Thus, if (2.7) holds, then from (2.8) and (2.9) we have

$$\left\|S_i x - S_i y\right\| \leq h\left\|y - x\right\| + \frac{h}{1-h}\|x - S_i x\|. \tag{2.10}$$

Denote

$$\rho_1 = \max\left\{h, \frac{h/2}{1-h/2}\right\} = h, \qquad L_1 = \max\left\{h, \frac{h}{1-h/2}, \frac{h}{1-h}\right\} = \frac{h}{1-h}. \tag{2.11}$$

Then we have $0 < \rho_1 < 1$ and $L_1 \geq 0$. Combining (2.2),(2.6), and (2.10) we get

$$\left\|S_i x - S_i y\right\| \leq \rho_1\left\|x - y\right\| + L_1\|x - S_i x\| \tag{2.12}$$

holds for all $x, y \in C$ and $i \in I$.

On the other hand, since $\{T_j\}_{j=1}^{k} : C \to C$ is k Ćirić operator satisfying the condition (CR), similarly, we can prove

$$\left\|T_j x - T_j y\right\| \leq \rho_2\left\|x - y\right\| + L_2\|x - S_i x\|, \tag{2.13}$$

for all $x, y \in C$ and $j \in J = \{1, 2, \ldots, k\}$, where $0 < \rho_2 < 1$ and $L_2 \geq 0$.

Let $p \in F = (\bigcap_{i=1}^{m} F(S_i)) \cap (\bigcap_{j=1}^{k} F(T_j))$; using (1.8) we have

$$\begin{aligned}\|x_{n+1} - p\| &= \left\|(1-\alpha_n-\gamma_n)(x_n - p) + \alpha_n \sum_{i=1}^{m} \lambda_i (S_i y_n - p) + \gamma_n (u_n - p)\right\| \\ &\le (1-\alpha_n-\gamma_n)\|x_n - p\| + \alpha_n \sum_{i=1}^{m} \lambda_i \|S_i y_n - p\| + \gamma_n \|u_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \sum_{i=1}^{m} \lambda_i \|S_i y_n - p\| + \gamma_n M,\end{aligned} \tag{2.14}$$

where $M = \sup_{n\ge1}\{\|u_n - p\|, \|v_n - p\|\}$. Now for $y = y_n$ and $x = p$, (2.12) gives

$$\|S_i y_n - p\| = \|S_i y_n - S_i p\| \le \rho_1 \|y_n - p\|. \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain that

$$\|x_{n+1} - p\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n \rho_1 \|y_n - p\| + \gamma_n M. \tag{2.16}$$

Again it follows from (1.8) that

$$\begin{aligned}\|y_n - p\| &= \left\|(1-\beta_n-\delta_n)(x_n - p) + \beta_n \sum_{j=1}^{k} \mu_j (T_j x_n - p) + \delta_n (v_n - p)\right\| \\ &\le (1-\beta_n-\delta_n)\|x_n - p\| + \beta_n \sum_{j=1}^{k} \mu_j \|T_j x_n - p\| + \delta_n \|v_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n \sum_{j=1}^{k} \mu_j \|T_j x_n - p\| + \delta_n M.\end{aligned} \tag{2.17}$$

Now for $y = x_n$ and $x = p$, (2.13) gives

$$\|T_j x_n - p\| = \|T_j x_n - T_j p\| \le \rho_2 \|x_n - p\|. \tag{2.18}$$

Combining (2.17) and (2.18) we get

$$\|y_n - p\| \le [1 - \beta_n(1-\rho_2)]\|x_n - p\| + \delta_n M \le \|x_n - p\| + \delta_n M. \tag{2.19}$$

Substituting (2.19) into (2.16), we obtain that

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \rho_1 (\|x_n - p\| + \delta_n M) + \gamma_n M \\ &= [1 - \alpha_n(1-\rho_1)]\|x_n - p\| + \alpha_n \delta_n \rho_1 M + \gamma_n M \\ &= (1-t_n)\|x_n - p\| + b_n + c_n,\end{aligned} \tag{2.20}$$

where

$$t_n = \alpha_n(1-\rho_1), \quad b_n = \alpha_n\delta_n\rho_1 M, \quad c_n = \gamma_n M \tag{2.21}$$

or

$$t_n = \alpha_n(1-\rho_1), \quad b_n = \alpha_n\delta_n\rho_1 M + \gamma_n M, \quad c_n = 0. \tag{2.22}$$

From the conditions (i)–(iii) it is easy to see that $t_n \in [0,1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=1}^{\infty} c_n < \infty$. Thus using (2.20) and Lemma 1.5 we have $\lim_{n\to\infty}\|x_n - p\| = 0$, and so $\lim_{n\to\infty} x_n = p$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a normed space E. Let $\{S_i\}_{i=1}^m : C \to C$ be m operators satisfying the condition (2.12) and let $\{T_j\}_{j=1}^k : C \to C$ be k operators satisfying the condition (2.13) with $F = (\bigcap_{i=1}^m F(S_i)) \cap (\bigcap_{j=1}^k F(T_j)) \neq \emptyset$, where $F(S_i)$ and $F(T_j)$ are the set of fixed points of S_i and T_j in C, respectively. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be four real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^m$, $\{\mu_j\}_{j=1}^k$ two finite sequences of positive number such that $\sum_{i=1}^m \lambda_i = 1$, and $\sum_{j=1}^k \mu_j = 1$, $\{u_n\}$ and $\{v_n\}$ two bounded sequences in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty}\delta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence defined by (1.8), then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^m$ and $\{T_j\}_{j=1}^k$.

Theorem 2.3. *Let C be a nonempty closed convex subset of a normed space E. Let $\{S_i\}_{i=1}^m : C \to C$ be m operators satisfying the condition (CR) with $F = \bigcap_{i=1}^m F(S_i) \neq \emptyset$ (the set of common fixed points of $\{S_i\}_{i=1}^m$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^m$ a finite sequence of positive number such that $\sum_{i=1}^m \lambda_i = 1$, and $\{u_n\}$ a bounded sequence in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence with mane errors defined by (1.9), then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^m$.

Theorem 2.4. *Let C be a nonempty closed convex subset of a normed space E. Let $\{S_i\}_{i=1}^m : C \to C$ be m operators satisfying the condition (2.12) with $F = \bigcap_{i=1}^m F(S_i) \neq \emptyset$ (the set of common fixed points of $\{S_i\}_{i=1}^m$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^m$ a finite sequence of positive number such that $\sum_{i=1}^m \lambda_i = 1$, and $\{u_n\}$ a bounded sequence in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence defined by (1.9), *then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^m$.*

Corollary 2.5 (see [7]). *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operators satisfying the condition* (CR)*. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $\{u_n\}$ a bounded sequences in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an explicit iteration sequence as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 1, \tag{2.23}$$

then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. By Ćirić [8], we know that T has a unique fixed point in C. Taking $m = 1$ in Theorem 2.3, then the conclusion of Corollary 2.5 can be obtained from Theorem 2.3 immediately. This completes the proof of Corollary 2.5. □

Remark 2.6. Theorems 2.2–2.4 and Corollary 2.5 improve and extend the corresponding results of Berinde [5], Gu [9], Rafiq [7], Rhoades [10], and Zamfirescu [3].

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