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Research Article

Common Fixed Point Theorems of Altman Integral Type Mappings in G-Metric Spaces

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We introduce the concept of φ-weakly commuting self-mapping pairs in G-metric space. Using this concept, we establish a new common fixed point theorem of Altman integral type for six self-mappings in the framework of complete G-metric space. An example is provided to support our result. The results obtained in this paper differ from the recent relative results in the literature.

1. Introduction and Preliminaries

Metric fixed point theory is an important mathematical discipline because of its applications in areas as variational and linear inequalities, optimization theory. Many results have been obtained by many authors considering different contractive conditions for self-mappings in metric space. In 1975, Altman [1] proved a fixed point theorem for a mapping which satisfies the condition $d(fx, fy) \le Q(d(x, y))$, where $Q : [0, +\infty) \to [0, +\infty)$ is an increasing function satisfying the following conditions:

(i) $0 < Q(t) < t, t \in (0, \infty)$;

(ii) $p(t) = t/(t - Q(t))$ is a decreasing function;

(iii) for some positive number t_1, there holds $\int_0^{t_1} p(t)dt < +\infty$.

Remark 1.1. By condition (i) and that Q is increasing, we know that $Q(0) = 0$ and $Q(t) = t \Leftrightarrow t = 0$.

Gu and Deng [2], Liu [3], Zhang [4], and Li and Gu [5] discussed common fixed point theorems for Altman type mappings in metric space. In 2006, a new structure of generalized metric space was introduced by Mustafa and Sims [6] as an appropriate notion of generalized metric space called G-metric space. Abbas and Rhoades [7] initiated the study of common fixed point in generalized metric space. Recently, many fixed point and common fixed point theorems for certain contractive conditions have been established in G-metric spaces, and for more details, one can refer to [8–42]. Coupled fixed point problems have also been considered in partially ordered G-metric spaces (see [43–56]). However, no one has discussed the common fixed point theorems for two or three pairs combining Altman type mappings recently.

Inspired by that, the purpose of this paper is to study common fixed point problem of Altman integral type for six self-mappings in G-metric space. We introduce a new concept of φ-weakly commuting self-mapping pairs in G-metric space, and a new common fixed point theorem for six self-mappings has been established through this concept. The results obtained in this paper differ from the recent relative results in the literature.

Throughout the paper, we mean by $\mathbb{N}$ the set of all natural numbers.

Definition 1.2 (see [6]). Let X be a nonempty set, and let $G : X \times X \times X \to R^{+}$ be a function satisfying the following axioms:

($G1$) $G(x, y, z) = 0$ if $x = y = z$,

($G2$) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

($G3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality),

then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.3 (see [6]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 1.4 (see [6]). *Let (X, G) be a G-metric space, then the followings are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$,*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Definition 1.5 (see [6]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.6 (see [6]). A G-metric space (X, G) is said to be G-complete, if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.7 (see [6]). *Let (X, G) be a G-metric space. Then the followings are equivalent.*

(1) *The sequence $\{x_n\}$ is G-Cauchy;*

(2) *For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.*

Proposition 1.8 (see [6]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.9 (see [6]). Let (X, G) and (X', G') be G-metric space, and $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ imply $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.10 (see [6]). *Let (X, G) and (X', G') be G-metric space. Then $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Proposition 1.11 (see [6]). *Let (X, G) be a G-metric space. Then, for any x, y, z, a in X it follows that:*

(i) *if $G(x, y, z) = 0$, then $x = y = z$,*

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq (2/3)(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 1.12 (see [8]). Self-mappings f and g of a G-metric space (X, G) are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ and $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2010, Manro et al. [9] introduced the concept of weakly commuting mappings, R-weakly commuting mappings into G-metric space as follows.

Definition 1.13 (see [9]). A pair of self-mappings (f, g) of a G-metric space is said to be weakly commuting if

$$G(fgx, gfx, gfx) \leq G(fx, gx, gx), \quad \forall x \in X. \tag{1.1}$$

Definition 1.14 (see [9]). A pair of self-mappings (f, g) of a G-metric space is said to be R-weakly commuting, if there exists some positive real number R such that

$$G(fgx, gfx, gfx) \leq RG(fx, gx, gx), \quad \forall x \in X. \tag{1.2}$$

Remark 1.15. If $R \leq 1$, then R-weakly commuting mappings are weakly commuting.

Now we introduce the new concept of φ-weakly commuting mappings as follow.

Definition 1.16. A pair of self-mappings (f, g) of a G-metric space is said to be φ-weakly commuting, if there exists a continuous function $\varphi : [0, \infty) \to [0, \infty), \varphi(0) = 0$, such that

$$G(fgx, gfx, gfx) \leq \varphi(G(fx, gx, gx)), \quad \forall x \in X. \tag{1.3}$$

Remark 1.17. Commuting mappings are weakly commuting mappings, but the reverse is not true. For example: let $X = [0, 1/2]$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$, define $f(x) = x/2$, $g(x) = x^2/2$, through a straightforward calculation, we have: $fgx = x^2/4$, $gfx = x^2/8$, $G(fgx, gfx, gfx) = G(x^2/4, x^2/8, x^2/8) = x^2/4$, but $G(fx, gx, gx) = |x - x^2| = x - x^2$, hence, $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$, but $fgx \neq gfx$.

Remark 1.18. Weakly commuting mappings are R-weakly commuting mappings, but the reverse is not true. For example: let $X = [-1, 1]$, define $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$, $f(x) = |x|$, $g(x) = |x| - 1$, then $gfx = |x| - 1$, $fgx = 1 - |x|$, $|fx - gx| = 1$, $|fgx - gfx| = 2(1 - |x|)$, $G(fgx, gfx, gfx) = 2|fgx - gfx| = 4(1 - |x|) \leq 4 = 4|fx - gx| = G(fx, gx, gx)$, when $R = 2$, we get that f and g are R-weakly commuting mappings, but not weakly commuting mappings.

Remark 1.19. R-weakly commuting mappings are φ-weakly commuting mappings but the reverse is not true. For example: let $X = [0, +\infty)$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$, $f(x) = x^2/4, g(x) = x^2$, thus, we have $fgx = x^4/4$, $gfx = x^4/16$, $G(fgx, gfx, gfx) = (3/8)x^4$, $G(fx, gx, gx) = (3/2)x^2$. Let $\varphi(x) = (1/2)x^2$, then

$$G(fgx, gfx, gfx) = \frac{3}{8}x^4 \leq \frac{9}{8}x^4 = \frac{1}{2}\left(\frac{3}{2}x^2\right)^2 = \varphi\left(\frac{3}{2}x^2\right) = \varphi(G(fx, gx, gx)). \tag{1.4}$$

For any given $R > 0$, since $\lim_{x \to +\infty}(1/4)x^2 = +\infty$, there exists $x \in X$ such that $(1/4)x^2 > R$, so we get $G(fgx, gfx, gfx) = (1/4)x^2 G(fx, gx, gx) > RG(fx, gx, gx)$. Therefore, f and g are φ-weakly commuting mappings, but not R-weakly commuting mappings.

Lemma 1.20. *Let $\delta(t)$ be Lebesgue integrable, and $\delta(t) > 0$, for all $t > 0$, let $F(x) = \int_0^x \delta(t)dt$, then $F(x)$ is an increasing function in $[0, +\infty)$.*

Definition 1.21. Let f and g be self-mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called s point of coincidence of f and g.

2. Main Results

In this paper, we denote $\phi : [0, +\infty) \to [0, +\infty)$ the function satisfying $0 < \phi(t) < t$, for all $t > 0$.

Theorem 2.1. *Let (X, G) be a complete G-metric space and let S, T, R, f, g, and h be six mappings of X into itself. If there exists an increasing function $Q : [0, +\infty) \to [0, +\infty)$ satisfying the conditions (i)~(iii) and the following conditions:*

(iv) $S(X) \subseteq g(X), T(X) \subseteq h(X), R(X) \subseteq f(X)$,

(v) $\int_0^{G(Sx,Ty,Rz)} \delta(t)dt \leq \phi(\int_0^{Q(G(fx,gy,hz))} \delta(t)dt)$, *for all $x, y, z \in X$,* *where $\delta(t)$ is a Lebesgue integrable function which is summable nonnegative such that*

$$\int_0^{\epsilon} \delta(t)dt > 0, \quad \forall \epsilon > 0. \tag{2.1}$$

Then,

(a) *one of the pairs (S, f), (T, g), and (R, h) has a coincidence point in X,*

(b) *if (S, f), (T, g), and (R, h) are three pairs of continuous φ-weakly commuting mappings, then the mappings S, T, R, f, g, and h have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X, from the condition (iv), there exist $x_1, x_2, x_3 \in X$ such that

$$y_1 = Sx_0 = gx_1, \qquad y_2 = Tx_1 = hx_2, \qquad y_3 = Rx_2 = fx_3. \tag{2.2}$$

By induction, there exist two sequences $\{x_n\}$, $\{y_n\}$ in X, such that

$$y_{3n+1} = Sx_{3n} = gx_{3n+1}, \quad y_{3n+2} = Tx_{3n+1} = hx_{3n+2}, \quad y_{3n+3} = Rx_{3n+2} = fx_{3n+3}, \quad n \in \mathbb{N}. \tag{2.3}$$

If $y_n = y_{n+1}$ for some n, with $n = 3m$, then $p = x_{3m+1}$ is a coincidence point of the pair (S, f); if $y_{n+1} = y_{n+2}$ for some n, with $n = 3m$, then $p = x_{3m+2}$ is a coincidence point of the pair (T, g); if $y_{n+2} = y_{n+3}$ for some n, with $n = 3m$, then $p = x_{3m+3}$ is a coincidence point of the pair (R, h).

On the other hand, if there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1} = y_{n_0+2}$, then $y_n = y_{n_0}$ for any $n \geq n_0$. This implies that $\{y_n\}$ is a G-Cauchy sequence.

In fact, if there exists $p \in \mathbb{N}$ such that $y_{3p} = y_{3p+1} = y_{3p+2}$, then applying the contractive condition (v) with $x = y_{3p}$, $y = y_{3p+1}$, and $z = y_{3p+2}$, and the property of ϕ, we get

$$\begin{aligned}
\int_0^{G(y_{3p+1},y_{3p+2},y_{3p+3})} \delta(t)dt &= \int_0^{G(Sx_{3p},Tx_{3p+1},Rx_{3p+2})} \delta(t)dt \\
&\leq \phi\left(\int_0^{Q(G(fx_{3p},gx_{3p+1},hx_{3p+2}))} \delta(t)dt\right) \\
&= \phi\left(\int_0^{Q(G(y_{3p},y_{3p+1},y_{3p+2}))} \delta(t)dt\right) \\
&\leq \int_0^{Q(G(y_{3p},y_{3p+1},y_{3p+2}))} \delta(t)dt.
\end{aligned} \tag{2.4}$$

From Lemma 1.20 and the property of Q, we have

$$G(y_{3p+1}, y_{3p+2}, y_{3p+3}) \leq Q(G(y_{3p}, y_{3p+1}, y_{3p+2})) = Q(0) = 0. \tag{2.5}$$

Which implies that $y_{3p+3} = y_{3p+1} = y_{3p}$. So we find $y_n = y_{3p}$ for any $n \geq 3p$. This implies that $\{y_n\}$ is a G-Cauchy sequence. The same conclusion holds if $y_{3p+1} = y_{3p+2} = y_{3p+3}$, or $y_{3p+2} = y_{3p+3} = y_{3p+4}$ for some $p \in \mathbb{N}$.

Without loss of generality, we can assume that $y_n \neq y_m$ for all $n, m \in \mathbb{N}$ and $n \neq m$.

Now we prove that $\{y_n\}$ is a G-Cauchy sequence in X.

Let $t_n = G(y_n, y_{n+1}, y_{n+2})$, then we have

$$t_{n+1} \leq Q(t_n) < t_n \tag{2.6}$$

for all $n \in \mathbb{N}$. Actually, from the condition (v), (2.3) and the property of ϕ, we have

$$\begin{aligned}
\int_0^{t_{3n}} \delta(t)dt &= \int_0^{G(y_{3n}, y_{3n+1}, y_{3n+2})} \delta(t)dt = \int_0^{G(Rx_{3n-1}, Sx_{3n}, Tx_{3n+1})} \delta(t)dt \\
&= \int_0^{G(Sx_{3n}, Tx_{3n+1}, Rx_{3n-1})} \delta(t)dt \leq \phi\left(\int_0^{Q(G(fx_{3n}, gx_{3n+1}, hx_{3n-1}))} \delta(t)dt\right) \\
&= \phi\left(\int_0^{Q(G(y_{3n}, y_{3n+1}, y_{3n-1}))} \delta(t)dt\right) = \phi\left(\int_0^{Q(t_{3n-1})} \delta(t)dt\right) \\
&\leq \int_0^{Q(t_{3n-1})} \delta(t)dt.
\end{aligned} \tag{2.7}$$

By Lemma 1.20 and the property of Q, we have

$$t_{3n} \leq Q(t_{3n-1}) < t_{3n-1}. \tag{2.8}$$

Again, using condition (v), (2.3) and the property of ϕ, we get

$$\begin{aligned}
\int_0^{t_{3n+1}} \delta(t)dt &= \int_0^{G(y_{3n+1}, y_{3n+2}, y_{3n+3})} \delta(t)dt = \int_0^{G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})} \delta(t)dt \\
&\leq \phi\left(\int_0^{Q(G(fx_{3n}, gx_{3n+1}, hx_{3n+2}))} \delta(t)dt\right) = \phi\left(\int_0^{Q(G(y_{3n}, y_{3n+1}, y_{3n+2}))} \delta(t)dt\right) \\
&= \phi\left(\int_0^{Q(t_{3n})} \delta(t)dt\right) \leq \int_0^{Q(t_{3n})} \delta(t)dt.
\end{aligned} \tag{2.9}$$

From Lemma 1.20 and the property of Q we have

$$t_{3n+1} \leq Q(t_{3n}) < t_{3n}. \tag{2.10}$$

Similarly, we can get

$$\begin{aligned}
\int_0^{t_{3n+2}} \delta(t)dt &= \int_0^{G(y_{3n+2},y_{3n+3},y_{3n+4})} \delta(t)dt = \int_0^{G(Sx_{3n+3},Tx_{3n+1},Rx_{3n+2})} \delta(t)dt \\
&\leq \phi\left(\int_0^{Q(G(fx_{3n+3},gx_{3n+1},hx_{3n+2}))} \delta(t)dt\right) = \phi\left(\int_0^{Q(G(y_{3n+3},y_{3n+1},y_{3n+2}))} \delta(t)dt\right) \\
&= \phi\left(\int_0^{Q(t_{3n+1})} \delta(t)dt\right) \leq \int_0^{Q(t_{3n+1})} \delta(t)dt.
\end{aligned} \tag{2.11}$$

From Lemma 1.20 and property of Q, we have

$$t_{3n+2} \leq Q(t_{3n+1}) < t_{3n+1}. \tag{2.12}$$

Combining (2.8), (2.10), and (2.12), we know that the (2.6) holds. This implies that $\{t_n\}$ is a nonnegative sequence which is strictly decreasing, hence, $\{t_n\}$ is convergent and $t_{n+1} \leq Q(t_n) < t_n$, for all $n \in \mathbb{N}$.

For any $n, m \in \mathbb{N}$, $m > n$, by combining $(G5)$, $(G3)$, and (2.6), we have

$$\begin{aligned}
G(y_n, y_m, y_m) &\leq \sum_{i=n}^{m-1} G(y_i, y_{i+1}, y_{i+1}) \leq \sum_{i=n}^{m-1} G(y_i, y_{i+1}, y_{i+2}) = \sum_{i=n}^{m-1} t_i \\
&= \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - t_{i+1}} \leq \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - Q(t_i)} \leq \sum_{i=n}^{m-1} \int_{t_{i+1}}^{t_i} \frac{t}{t - Q(t)} dt = \int_{t_m}^{t_n} p(t)dt.
\end{aligned} \tag{2.13}$$

From the convergence of the sequence $\{t_n\}$ and the condition (iii) we assure that

$$\lim_{n,m \to \infty} \int_{t_m}^{t_n} p(t)dt = 0. \tag{2.14}$$

Thus, $\{y_n\}$ is a G-Cauchy sequence in X, since (X, G) is a complete G-metric space, there exists $u \in X$ such that $\lim_{n \to \infty} y_n = u$, hence

$$\begin{aligned}
\lim_{n \to \infty} y_{3n+1} &= \lim_{n \to \infty} Sx_{3n} = \lim_{n \to \infty} gx_{3n+1} = u, \\
\lim_{n \to \infty} y_{3n+2} &= \lim_{n \to \infty} Tx_{3n+1} = \lim_{n \to \infty} hx_{3n+2} = u, \\
\lim_{n \to \infty} y_{3n+3} &= \lim_{n \to \infty} Rx_{3n+2} = \lim_{n \to \infty} fx_{3n+3} = u.
\end{aligned} \tag{2.15}$$

Since (S, f) are φ-weakly commuting mappings, thus we have

$$G(Sfx_{3n}, fSx_{3n}, fSx_{3n}) \leq \varphi(G(Sx_{3n}, fx_{3n}, fx_{3n})). \tag{2.16}$$

On taking $n \to \infty$ at both sides, noting that S and f are continuous mappings, we have

$$G(Su, fu, fu) \leq \varphi(G(u,u,u)) = \varphi(0) = 0. \tag{2.17}$$

Which gives that $Su = fu$. Similarly, we can get $Tu = gu, Ru = hu$.

By using condition (v) and the property of ϕ, we get

$$\int_0^{G(Su,Tu,Ru)} \delta(t)dt \leq \phi\left(\int_0^{Q(G(fu,gu,hu))} \delta(t)dt\right) \leq \int_0^{Q(G(fu,gu,hu))} \delta(t)dt. \tag{2.18}$$

Thus, by Lemma 1.20, $(G5)$ and the property of Q, noting that $Su = fu$, $Tu = gu$, $Ru = hu$, we have

$$\begin{aligned} G(Su,Tu,Ru) &\leq Q(G(fu,gu,hu)) \leq G(fu,gu,hu) \\ &\leq G(fu,Su,Su) + G(Su,gu,hu) \\ &\leq G(fu,Su,Su) + G(hu,Ru,Ru) + G(Ru,gu,Su) \\ &\leq G(fu,Su,Su) + G(hu,Ru,Ru) + G(Ru,Tu,Su) + G(Tu,Tu,gu) \\ &= G(Su,Tu,Ru). \end{aligned} \tag{2.19}$$

Which implies that

$$Q(G(fu,gu,hu)) = G(Su,Tu,Ru) = G(fu,gu,hu). \tag{2.20}$$

By Remark 1.1, we have $G(fu,gu,hu) = 0$, therefore, $fu = gu = hu$. So, immediately, we can have $Su = Tu = Ru = fu = gu = hu$. Setting

$$z = Su = Tu = Ru = fu = gu = hu. \tag{2.21}$$

Since (S, f) are φ-weakly commuting mappings, we have

$$G(Sz, fz, fz) = G(Sfu, fSu, fSu) \leq \varphi(G(Su, fu, fu)) = \varphi(0) = 0. \tag{2.22}$$

Which gives that $Sz = fz$. By the same argument, we can get $Tz = gz$, $Rz = hz$, So we have $Sfu = fSu$, $Tgu = gTu$, $Rhu = hRu$. Again, by condition (v), we have

$$\int_0^{G(Sz,z,z)} \delta(t)dt = \int_0^{G(S^2u,Tu,Ru)} \delta(t)dt \leq \phi\left(\int_0^{Q(G(fSu,gu,hu))} \delta(t)dt\right) \leq \int_0^{Q(G(fSu,gu,hu))} \delta(t)dt. \tag{2.23}$$

By the Lemma 1.20 and the property of Q, we have

$$G(Sz,z,z) \leq Q(G(fSu,gu,hu)) \leq G(fSu,gu,hu) = G(Sfu,gu,hu) = G(Sz,z,z). \tag{2.24}$$

Which implies that $Q(G(Sfu, gu, hu)) = G(Sfu, gu, hu)$. Thus, by the property of Q, we have $Sfu = gu = hu$, hence, $z = Sz = fz$. Similarly, we can prove that $z = Tz = gz$, $z = Rz = hz$, so we get $z = Sz = Tz = Rz = fz = gz = hz$, which means that z is a common fixed point of S, T, R, f, g, and h.

Now, we will show the common fixed point of S, T, R, f, g, and h is unique. Actually, assume $w \neq z$ is another common fixed point of S, T, R, f, g, and h, then by condition (v), we have

$$\int_0^{G(z,w,w)} \delta(t)dt = \int_0^{G(Sz,Tw,Rw)} \delta(t)dt \leq \phi\left(\int_0^{Q(G(fz,gw,hw))} \delta(t)dt\right) \leq \int_0^{Q(G(fz,gw,hw))} \delta(t)dt. \tag{2.25}$$

By Lemma 1.20 and the property of Q, we have

$$G(z, w, w) \leq Q\big(G\big(fz, gw, hw\big)\big) = Q(G(z, w, w)) < G(z, w, w). \tag{2.26}$$

It is a contradiction, unless $z = w$, that is, S, T, R, f, g, and h have a unique common fixed point in X. This completes the proof of Theorem 2.1. □

Remark 2.2. If we take: (1) $S = T = R$; (2) $f = g = h$; (3) $f = g = h = I$ (I is identity mapping); (4) $T = R$ and $g = h$; (5) $T = R$, $g = h = I$; (6) $y = z$ in Theorem 2.1, then several new results can be obtained.

Corollary 2.3. *Let (X, G) be a complete G-metric space and let S, T, R, f, g, and h be six mappings of X into itself. If there exists an increasing function $Q : [0, +\infty) \to [0, +\infty)$ satisfying the conditions (i)~(iii) and the following conditions:*

(iv) $S(X) \subseteq g(X)$, $T(X) \subseteq h(X)$, $R(X) \subseteq f(X)$,

(v) $G(Sx, Ty, Rz) \leq \phi(Q(G(fx, gy, hz)))$, $\forall x, y, z \in X$.

Then,

(a) *one of the pairs (S, f), (T, g), and (R, h) has a coincidence point in X.*

(b) *if (S, f), (T, g), and (R, h) are three pairs of continuous φ-weakly commuting mappings, then the mappings S, T, R, f, g, and h have a unique common fixed point in X.*

Proof. Taking $\delta(t) = 1$ in Theorem 2.1, the conclusion of Corollary 2.3 can be obtained from Theorem 2.1 immediately. This completes the proof of Corollary 2.3. □

Now we give an example to support Corollary 2.3.

Example 2.4. Let $X = [0, \infty)$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Let $S, T, R, f, g, h : X \to X$ be defined by $Sx = x/8$, $Tx = x/16$, $Rx = x/32$, $fx = x$, $gx = x/2$,

$hx = x/4$. Clearly, we can get $S(X) \subseteq g(X)$, $T(X) \subseteq h(X)$, $R(X) \subseteq f(X)$. Through calculation, we have

$$\begin{aligned} G(Sx,Ty,Rz) &= G\left(\frac{x}{8},\frac{y}{16},\frac{z}{32}\right) \\ &= \left|\frac{x}{8}-\frac{y}{16}\right| + \left|\frac{y}{16}-\frac{z}{32}\right| + \left|\frac{x}{8}-\frac{z}{32}\right| \\ &= \frac{1}{8}\left(\left|x-\frac{y}{2}\right| + \left|\frac{y}{2}-\frac{z}{4}\right| + \left|\frac{z}{4}-x\right|\right), \\ G(fx,gy,hz) = G\left(x,\frac{y}{2},\frac{z}{4}\right) &= \left|x-\frac{y}{2}\right| + \left|\frac{y}{2}-\frac{z}{4}\right| + \left|\frac{z}{4}-x\right|. \end{aligned} \tag{2.27}$$

Thus, we have

$$G(Sx,Ty,Rz) = \frac{1}{8}G(fx,gy,hz). \tag{2.28}$$

Now we choose $\phi(t) = 3t/4$, and $Q(t) = t/2$, then we have $\phi(t) < t$ and $Q(t)$ satisfies (i)~(iii). Thus, we have

$$\begin{aligned} G(Sx,Ty,Rz) &= \frac{1}{8}G(fx,gy,hz) \le \frac{3}{4}\cdot\frac{1}{2}G(fx,gy,hz) \\ &= \frac{3}{4}Q(G(fx,gy,hz)) = \phi(Q(G(fx,gy,hz))). \end{aligned} \tag{2.29}$$

On the other hand, let $\varphi(u) = u/2$ for all $u \in [0,\infty)$, we have

$$\begin{aligned} G(Sfx,fSx,fSx) &= G\left(\frac{x}{8},\frac{x}{8},\frac{x}{8}\right) \le \frac{1}{2}\cdot\frac{7x}{4} = \frac{1}{2}G(Sx,fx,fx) = \varphi(G(Sx,fx,fx)), \\ G(Tgx,gTx,gTx) &= G\left(\frac{x}{32},\frac{x}{32},\frac{x}{32}\right) \le \frac{1}{2}\cdot\frac{7x}{8} = \frac{1}{2}G(Tx,gx,gx) = \varphi(G(Tx,gx,gx)), \\ G(Rhx,hRx,hRx) &= G\left(\frac{x}{128},\frac{x}{128},\frac{x}{128}\right) \le \frac{1}{2}\cdot\frac{7x}{16} = \frac{1}{2}G(Rx,hx,hx) = \varphi(G(Rx,hx,hx)), \end{aligned} \tag{2.30}$$

for all $x \in X$. Which means that (S,f), (T,g), and (R,h) are three pairs of continuous φ-weakly commuting mappings in X. So that all the conditions of Corollary 2.3 are satisfied. Moreover, 0 is the unique common fixed point for all of the mappings S, T, R, f, g, and h.

Corollary 2.5. *Let (X,G) be a complete G-metric space, $S,T,R : X \to X$ are three self-mappings in X, and function $Q : [0,+\infty) \to [0,+\infty)$ satisfies conditions (i)~(iii) and the following condition:*

$$G(Sx,Ty,Rz) \le \phi(Q(G(x,y,z))), \quad \forall x,y,z \in X. \tag{2.31}$$

Then S, T, and R have a unique common fixed point in X.

Proof. Taking $h = g = f = I$ in Corollary 2.3, where I is an identity mapping. Then the conclusion of Corollary 2.5 can be obtained from Corollary 2.3 immediately. This completes the proof of Corollary 2.5. □

Corollary 2.6. *Let (X, G) be a complete G-metric space, $S : X \to X$ is a self-mapping in X, and function $Q : [0, +\infty) \to [0, +\infty)$ satisfies conditions (i)~(iii) and the following condition:*

$$G(Sx, Sy, Sz) \leq \phi Q(G(x, y, z)), \quad \forall x, y, z \in X. \tag{2.32}$$

Then S has a unique fixed point in X.

Proof. Taking $S = T = R$ in Corollary 2.5, the conclusion of Corollary 2.6 can be obtained from Corollary 2.5 immediately. This completes the proof of Corollary 2.6. □

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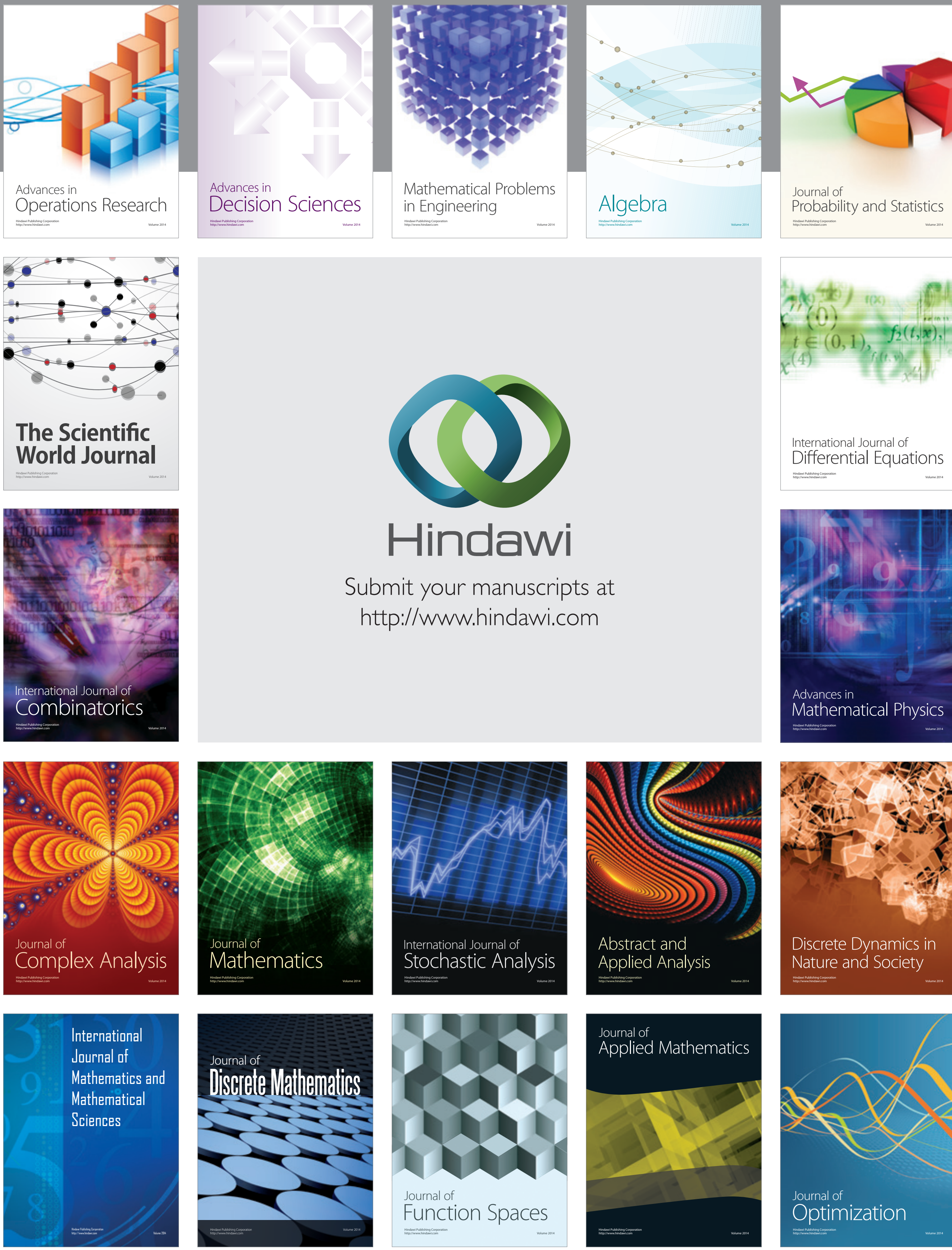
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