Research Article

Convergence of the Euler Method of Stochastic Differential Equations with Piecewise Continuous Arguments

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The main purpose of this paper is to investigate the strong convergence of the Euler method to stochastic differential equations with piecewise continuous arguments (SEPCAs). Firstly, it is proved that the Euler approximation solution converges to the analytic solution under local Lipschitz condition and the bounded pth moment condition. Secondly, the Euler approximation solution converge to the analytic solution is given under local Lipschitz condition and the linear growth condition. Then an example is provided to show which is satisfied with the monotone condition without the linear growth condition. Finally, the convergence of numerical solutions to SEPCAs under local Lipschitz condition and the monotone condition is established.

1. Introduction

Recently, differential equations with piecewise continuous arguments (EPCAs) have attracted much attention, and many useful conclusions have been obtained. These systems have applications in certain biomedical models, control systems with feedback delay in the work of Cooke and Wiener [1]. The general theory and basic results for EPCAs have by now been thoroughly investigated in the book of Wiener [2]. Song et al. [3] deal with the stability analysis of numerical methods for the solution of advanced differential equations with piecewise continuous arguments. A typical EPCA contains arguments that are constant on certain intervals. The solutions are determined by a finite set of initial data, rather than by an initial function, as in the case of general functional differential equation. A solution

is defined as a continuous, sectionally smooth function that satisfies the equation within these intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence, EPCAs represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

However, up to now there are few people who have considered the influence of noise to EPCAs. Actually, the environment and accidental events may greatly influence the systems. Thus analyzing SEPCAs is an interesting topic both in theory and applications. There is in general no explicit solution to an SEPCA, hence numerical solutions are required in practice. Numerical solutions to stochastic differential equations (SDEs) have been discussed under the local Lipschitz condition and the linear growth condition by many authors (see [4, 5]). Mao [6] discusses numerical solutions to stochastic differential delay equations (SDDEs) under the local Lipschitz condition and the linear growth condition. Mao and Sabanis [7] discuss numerical solutions to SDDEs with variable delay under the local Lipschitz condition and the linear growth condition. Mao discusses numerical solutions to SDEs and SDDEs under the local Lipschitz condition and the monotone condition (see [8]). Dai and Liu [9] give the mean-square stability of the numerical solutions of linear SEPCAs. However, SEPCAs do not have the convergence results. The main aim of this paper is to establish convergence of numerical solution for SEPCAs under the differential conditions.

The paper is organized as follows. In Section 2, we introduce necessary notations and the Euler method. In Section 3, the strong convergence of the Euler-Maruyama method to SEPCAs under local Lipschitz condition and the bounded pth moment condition will be given. In Section 4, the strong convergence of the Euler-Maruyama method to SEPCAs under local Lipschitz condition and the linear growth condition will be presented. In Section 5, an example is provided to show which is satisfied with the monotone condition without the linear growth condition. In Section 6, we obtain the convergence of numerical solutions to SEPCAs under local Lipschitz condition and the monotone condition is established.

2. Preliminary Notation and Euler Method

In this paper, unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in R^n$. If A is a vector or matrix, its transpose is defined by A^T. If A is a matrix, its trace norm is defined by $|A| = \sqrt{\text{trace}(A^T A)}$. For simplicity, we also have to denote by $a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, satisfying the usual conditions. $\mathcal{L}^1([0, \infty), R^n)$ and $\mathcal{L}^2([0, \infty), R^n)$ denote the family of all real valued $\mathcal{F}_t$-adapted process $f(t)_{t \geq 0}$, such that for every $T > 0, \int_0^T |f(t)| dt < \infty$ a.s. and $\int_0^T |f(t)|^2 dt < \infty$ a.s., respectively. For any $a, b \in R$ with $a < b$, denote $C([a, b]; R^n)$ the family of continuous functions ϕ from $[a, b]$ to R^n with the norm $\|\phi\| = \sup_{a \leq \theta \leq b} |\phi(\theta)|$. Denote $C^b_{\mathcal{F}_t}([a, b]; R^n)$ the family of all bounded $\mathcal{F}_t$-measurable $C([a, b]; R^n)$-valued random variables. Let $B(t) = (B_1(t), \ldots, B_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space.

Throughout this paper, we consider stochastic differential equations with piecewise continuous arguments:

$$dx(t) = f(x(t), x([t])) dt + g(x(t), x([t])) dB(t) \quad \forall t \geq 0, \tag{2.1}$$

with initial data $x(0) = x_0$, where $f: R^n \times R^n \to R^n$, $g: R^n \times R^n \to R^{n \times d}$, x_0 is a vector, and $[\cdot]$ denotes the greatest-integer function. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x(0) + \int_0^t f(x(s), x([s]))ds + \int_0^t g(x(s), x([s]))dB(s) \quad \forall t \geq 0. \tag{2.2}$$

Moreover, we also require the coefficients f and g to be sufficiently smooth.

To be precise, let us state the following conditions.

(H1) The local Lipschitz condition: for every integer $i \geq 1$, there exists a positive constant L_i such that

$$\left| f(x, y) - f(\overline{x}, \overline{y}) \right|^2 \vee \left| g(x, y) - g(\overline{x}, \overline{y}) \right|^2 \leq L_i \left(|x - \overline{x}|^2 + |y - \overline{y}|^2 \right), \tag{2.3}$$

for those $x, \overline{x}, y, \overline{y} \in R^n$ with $|x| \vee |\overline{x}| \vee |y| \vee |\overline{y}| \leq i$.

(H2) Linear growth condition: there exists a positive constant K such that

$$\left| f(x, y) \right|^2 \vee \left| g(x, y) \right|^2 \leq K \left(1 + |x|^2 + |y|^2 \right), \tag{2.4}$$

for all $(x, y) \in R^n \times R^n$.

(H3) Monotone condition: there exists a positive constant K_1 such that

$$x^T f(x, y) + \frac{p-1}{2} \left| g(x, y) \right|^2 \leq K_1 \left(1 + |x|^2 + |y|^2 \right), \tag{2.5}$$

for all $(x, y) \in R^n \times R^n$.

(H4) The bounded pth moment condition: there exists a pair of constants $p > 2$ and $K_2 > 0$ such that

$$E \left[\sup_{0 \leq t \leq T} |x(t)|^p \right] \vee E \left[\sup_{0 \leq t \leq T} |y(t)|^p \right] \leq K_2. \tag{2.6}$$

Let us first give the definition of the solution.

Definition 2.1 (see [10]). An R^n-valued stochastic process $\{x(t)\}$ is called a solution of (2.1) on $[0, \infty)$, if it has the following properties:

(1) $\{x(t)\}$ is continuous on $[0, \infty)$ and $\mathcal{F}_t$ adapted;

(2) $\{f(x(t), x([t]))\} \in \mathcal{L}^1([0, \infty), R^n)$ and $\{g(x(t), x([t]))\} \in \mathcal{L}^2([0, \infty), R^{n \times d})$;

(3) Equation (2.2) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end points almost surely. A solution $\{x(t)\}$ is said to be unique if any other solution $\{\overline{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is,

$$P\{x(t) = \overline{x}(t) \ \forall t \in [0, \infty)\} = 1. \tag{2.7}$$

Let $h = 1/m$ be a given stepsize with integer $m \geq 1$ and the grid points t_n defined by $t_n = nh$ $(n = 0, 1, 2, \ldots)$. For simplicity, we assume $T = Nh$. We consider the Euler-Maruyama method to (2.1),

$$y_{n+1} = y_n + f\left(y_n, y^h([nh])\right)h + g\left(y_n, y^h([nh])\right)\Delta B_n, \tag{2.8}$$

for $n = 0, 1, 2, \ldots$, where $\Delta B_n = B(t_n) - B(t_{n-1})$, $y^h([nh])$ is approximation to the exact solution $x([nh])$. Let $n = km + l$ $(k = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots, m - 1)$. The adaptation of the Euler method to (2.1) leads to a numerical process of the following type:

$$y_{km+l+1} = y_{km+l} + f\left(y_{km+l}, y_{km}\right)h + g\left(y_{km+l}, y_{km}\right)\Delta B_{km+l}, \tag{2.9}$$

where $\Delta B_{km+l} = B(t_{km+l}) - B(t_{km+l-1})$, y_{km+l} and y_{km} are approximations to the exact solution $x(t_{km+l})$ and $x([t_{km+l}])$, respectively. The continuous Euler-Maruyama approximate solution is defined by

$$y(t) = y(0) + \int_0^t f(z(s), z([s]))ds + \int_0^t g(z(s), z([s]))dB(s), \tag{2.10}$$

where $z(t) = y_{km+l}$ and $z([t]) = y_{km}$ for $t \in [t_{km+l}, t_{km+l+1})$. It is not difficult to see that $y(t_{km+l}) = z(t_{km+l}) = y_{km+l}$ for $k = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots, m - 1$. For sufficiently large integer i, define the stopping times $\eta_i = \inf\{t \geq 0 : |x(t)| \geq i\}$, $\theta_i = \inf\{t \geq 0 : |y(t)| \geq i\}$, $\tau_i = \eta_i \wedge \theta_i$.

3. Convergence of the Euler-Maruyama Method under the Bounded pth Moment

We will show the strong convergence of the EM method on (2.1) under local Lipschitz condition and the bounded pth moment condition. The following lemma shows that both $y(t)$ and $z(t)$ are close to each other.

Lemma 3.1. *Under the condition (H1), let $T > 0$ be arbitrary. Then*

$$E \sup_{0 \leq t \leq T} |y(t) - z(t)|^2 \leq C_1(x_0, i)h, \tag{3.1}$$

where $C_1(x_0, i) = 4L_i(T + 4)K_2$.

Proof. For $t \in [0, T)$, there are two integers k and l such that $t \in [t_{km+l}, t_{km+l+1})$. By the Hölder inequality, we compute

$$
\begin{aligned}
\left| y(t) - z(t) \right|^2 &= \left| \int_{t_{km+l}}^{t} f(z(s), z([s])) ds + \int_{t_{km+l}}^{t} g(z(s), z([s])) dB(s) \right|^2 \\
&\leq 2 \left| \int_{t_{km+l}}^{t} f(z(s), z([s])) ds \right|^2 + 2 \left| \int_{t_{km+l}}^{t} g(z(s), z([s])) dB(s) \right|^2 \\
&\leq 2T \int_{t_{km+l}}^{t} \left| f(z(s), z([s])) \right|^2 ds + 2 \left| \int_{t_{km+l}}^{t} g(z(s), z([s])) dB(s) \right|^2.
\end{aligned}
\tag{3.2}
$$

This implies that, for any $0 \leq t_1 \leq T$,

$$
\begin{aligned}
E \sup_{0 \leq t \leq t_1} \left| y(t) - z(t) \right|^2 &\leq 2TE \sup_{0 \leq t \leq t_1} \int_{t_{km+l}}^{t} \left| f(z(s), z([s])) \right|^2 ds \\
&\quad + 2E \sup_{0 \leq t \leq t_1} \left| \int_{t_{km+l}}^{t} g(z(s), z([s])) dB(s) \right|^2.
\end{aligned}
\tag{3.3}
$$

By the Doob martingale inequality, we have

$$
E \sup_{0 \leq t \leq t_1} \left| y(t) - z(t) \right|^2 \leq 2TE \int_{t_{km+l}}^{t_1} \left| f(z(s), z([s])) \right|^2 ds + 8E \int_{t_{km+l}}^{t_1} \left| g(z(s), z([s])) \right|^2 ds.
\tag{3.4}
$$

Using the local Lipschitz conditions

$$
\begin{aligned}
E \sup_{0 \leq t \leq t_1} \left| y(t) - z(t) \right|^2 &\leq 2L_i(T + 4)E \int_{t_{km+l}}^{t_1} \left(|z(s)|^2 + |z([s])|^2 \right) ds \\
&\leq 4L_i(T + 4) \int_{t_{km+l}}^{t_1} \left(E \sup_{0 \leq u \leq s} |y(u)|^2 \right) ds \\
&\leq C_1(x_0, i) h,
\end{aligned}
\tag{3.5}
$$

where $C_1(x_0, i) = 4L_i(T + 4)K_2$. The proof is completed. $\square$

Theorem 3.2. *Under the conditions (H1) and (H4), the EM approximate solution converges to the exact solution of* (2.1) *in the sense that*

$$
\lim_{h \to 0} E \left[\sup_{0 \leq t \leq T} \left| y(t) - x(t) \right|^2 \right] = 0.
\tag{3.6}
$$

Proof. Fix a $p > 2$; let $e(t) = x(t) - y(t)$; it is easy to see that

$$E\left[\sup_{0 \le t \le T} |e(t)|^2\right] = E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i > T \text{ and } \eta_i > T\}}\right] + E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i \le T \text{ or } \eta_i \le T\}}\right]$$

$$= E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\tau_i > T\}}\right] + E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i \le T \text{ or } \eta_i \le T\}}\right] \tag{3.7}$$

$$\le E\left[\sup_{0 \le t \le T} |e(t \wedge \tau_i)|^2\right] + E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i \le T \text{ or } \eta_i \le T\}}\right].$$

By the Young inequality $xy \le (x^p/p) + (y^q/q)$, for any $a, b, p, q, \delta > 0, (1/p) + (1/q) = 1$, we have

$$ab \le a\delta^{1/p} \frac{b}{\delta^{1/p}} \le \frac{(a\delta^{1/p})^p}{p} + \frac{b^q}{q\delta^{q/p}} = \frac{a^p \delta}{p} + \frac{b^q}{q\delta^{q/p}}. \tag{3.8}$$

Thus for any $\delta > 0$, we have

$$E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i \le T \text{ or } \eta_i \le T\}}\right] \le \frac{2\delta}{p} E\left[\sup_{0 \le t \le T} |e(t)|^p\right] + \frac{1 - (2/p)}{\delta^{2/(p-2)}} P\{\theta_i \le T \text{ or } \eta_i \le T\}. \tag{3.9}$$

By condition (H4), we have

$$P(\theta_i \le T) = E\left[1_{\{\theta_i \le T\}} \frac{|y(\theta_i)|^p}{i^p}\right] \le \frac{1}{i^p} E\left[\sup_{0 \le t \le T} |y(t)|^p\right] \le \frac{K_2}{i^p}. \tag{3.10}$$

Similarly, the result is

$$P(\eta_i \le T) \le \frac{K_2}{i^p}. \tag{3.11}$$

So that

$$P(\theta_i \le T \text{ or } \eta_i \le T) \le P(\theta_i \le T) + P(\eta_i \le T) \le \frac{2K_2}{i^p}. \tag{3.12}$$

Using these bounds, then

$$E\left[\sup_{0 \le t \le T} |e(t)|^p\right] \le 2^{p-1}\left(E\left[\sup_{0 \le t \le T} |x(t)|^p\right] + E\left[\sup_{0 \le t \le T} |y(t)|^p\right]\right)$$

$$\le 2^p K_2. \tag{3.13}$$

$$E\left[\sup_{0 \le t \le T} |e(t)|^2 1_{\{\theta_i \le T \text{ or } \eta_i \le T\}}\right] \le \frac{2^{p+1}\delta K_2}{p} + \frac{2(p-2)K_2}{p\delta^{2/(p-2)} i^p}.$$

By the definitions of $x(t)$ and $y(t)$, we have

$$|x(t \wedge \tau_i) - y(t \wedge \tau_i)|^2 \leq 2 \left| \int_0^{t \wedge \tau_i} [f(x(s), x([s]) - f(z(s), z([s]))] \, ds \right|^2$$

$$+ 2 \left| \int_0^{t \wedge \tau_i} [g(x(s), x([s]) - g(z(s), z([s]))] \, dB(s) \right|^2. \tag{3.14}$$

Thus, for any $t_1 \in [0, T]$

$$E \sup_{0 \leq t \leq t_1} |x(t \wedge \tau_i) - y(t \wedge \tau_i)|^2 \leq 2E \sup_{0 \leq t \leq t_1} \left| \int_0^{t \wedge \tau_i} [f(x(s), x([s]) - f(z(s), z([s]))] \, ds \right|^2$$

$$+ 2E \sup_{0 \leq t \leq t_1} \left| \int_0^{t \wedge \tau_i} [g(x(s), x([s]) - g(z(s), z([s]))] \, dB(s) \right|^2. \tag{3.15}$$

By the Hölder inequality, condition (H1), and Lemma 3.1, one gets

$$E \sup_{0 \leq t \leq t_1} \left| \int_0^{t \wedge \tau_i} [f(x(s), x([s]) - f(z(s), z([s]))] \, ds \right|^2$$

$$\leq T E \int_0^{t_1 \wedge \tau_i} |f(x(s), x([s]) - f(z(s), z([s]))|^2 \, ds$$

$$= T E \int_0^{t_1} |f(x(s \wedge \tau_i), x([s \wedge \tau_i]) - f(z(s \wedge \tau_i), z([s \wedge \tau_i]))|^2 \, ds$$

$$\leq T L_i E \int_0^{t_1} \left(|x(s \wedge \tau_i) - z(s \wedge \tau_i)|^2 + |x([s \wedge \tau_i]) - z([s \wedge \tau_i])|^2 \right) ds$$

$$\leq 2 T L_i \int_0^{t_1} \left(E|x(s \wedge \tau_i) - y(s \wedge \tau_i)|^2 + E|x([s \wedge \tau_i]) - y([s \wedge \tau_i])|^2 \right) ds$$

$$+ 2 T L_i \int_0^{t_1} \left(E|y(s \wedge \tau_i) - z(s \wedge \tau_i)|^2 + E|y([s \wedge \tau_i]) - z([s \wedge \tau_i])|^2 \right) ds$$

$$\leq 2 T L_i \int_0^{t_1} \left(E|x(s \wedge \tau_i) - y(s \wedge \tau_i)|^2 + E|x([s \wedge \tau_i]) - y([s \wedge \tau_i])|^2 \right) ds + 4 T^2 L_i C_1(x_0, i) h$$

$$\leq 4 T L_i \int_0^{t_1} E \sup_{0 \leq v \leq s \wedge \tau_i} |x(v) - y(v)|^2 \, ds + 4 T^2 L_i C_1(x_0, i) h. \tag{3.16}$$

Similarly, by the Burkhôlder-Davis-Gundy inequality, then

$$
\begin{aligned}
E\sup_{0\le t\le t_1}\left|\int_0^{t\wedge\tau_i}\left[g(x(s),x([s]))-g(z(s),z([s]))\right]dB(s)\right|^2 & \\
\le 4E\int_0^{t\wedge\tau_i}\left|g(x(s),x([s]))-g(z(s),z([s]))\right|^2ds & \\
\le 16L_i\int_0^{t_1}E\sup_{0\le v\le s\wedge\tau_i}\left|x(v)-y(v)\right|^2ds+16TL_iC_1(x_0,i)h. &
\end{aligned}
\tag{3.17}
$$

Substituting (3.16) and (3.17) into (3.15) gives

$$
E\sup_{0\le t\le t_1}\left|x(t\wedge\tau_i)-y(t\wedge\tau_i)\right|^2\le 8(T+4)L_i\int_0^{t_1}E\sup_{0\le v\le s\wedge\tau_i}\left|x(v)-y(v)\right|^2ds+8T(T+4)L_iC_1(x_0,i)h.
$$

$$
\tag{3.18}
$$

By the Gronwall inequality, we must get

$$
E\sup_{0\le t\le T}\left|x(t\wedge\tau_i)-y(t\wedge\tau_i)\right|^2\le C_2(x_0,i)h,
\tag{3.19}
$$

where $C_2(x_0,i)=8T(T+4)L_iC_1(x_0,i)e^{8T(T+4)L_i}$. So we have

$$
\begin{aligned}
E\left[\sup_{0\le t\le T}|e(t)|^2\right]&\le E\left[\sup_{0\le t\le T}|e(t\wedge\tau_i)|^2\right]+E\left[\sup_{0\le t\le T}|e(t)|^2 1_{\{\theta_i<T\ \text{or}\ \eta_i<T\}}\right]\\
&\le C_2(x_0,i)h+\frac{2^{p+1}\delta K_2}{p}+\frac{2(p-2)K_2}{p\delta^{2/(p-2)}i^p}.
\end{aligned}
\tag{3.20}
$$

Given any $\epsilon>0$, we can choose δ sufficiently small for

$$
\frac{2^{p+1}\delta K_2}{p}<\frac{\epsilon}{3},
\tag{3.21}
$$

then choose i sufficiently large for

$$
\frac{2(p-2)K_2}{p\delta^{2/(p-2)}i^p}<\frac{\epsilon}{3},
\tag{3.22}
$$

and finally choose h sufficiently small, so that

$$C_2(x_0, i)h < \frac{\epsilon}{3}. \tag{3.23}$$

Thus $E[\sup_{0 \le t \le T} |e(t)|^2] < \epsilon$. The proof is completed. $\qquad\square$

4. Convergence of the Euler-Maruyama Method under Linear Growth Condition

We will show the strong convergence of the EM method on (2.1) under local Lipschitz condition and the linear growth condition. In the following we will show that the linear growth condition (H2) implies the bounded pth moment condition (H4).

Lemma 4.1. *Under the linear growth conditions (H2), there exists a positive constant C_3 such that the solution of (2.1) satisfies*

$$E \sup_{0 \le t \le T} |x(t)|^p \le C_3(1 + |x_0|^p), \tag{4.1}$$

where $C_3 = C_3(p, T, K)$ is a constant independent of h.

Proof. It follows from (2.2) that

$$|x(t)|^p = \left| x(0) + \int_0^t f(x(s), x([s]))ds + \int_0^t g(x(s), x([s]))dB(s) \right|^p$$

$$\le 3^{p-1} \left[|x(0)|^p + \left| \int_0^t f(x(s), x([s]))ds \right|^p + \left| \int_0^t g(x(s), x([s]))dB(s) \right|^p \right]. \tag{4.2}$$

By the Hölder inequality, we obtain

$$|x(t)|^p \le 3^{p-1} \left[|x(0)|^p + T^{p-1} \int_0^t |f(x(s), x([s]))|^p ds + \left| \int_0^t g(x(s), x([s]))dB(s) \right|^p \right]. \tag{4.3}$$

This implies that, for any $0 \le t_1 \le T$,

$$E \sup_{0 \le t \le t_1} |x(t)|^p \le 3^{p-1} \left[|x(0)|^p + T^{p-1} E \sup_{0 \le t \le t_1} \int_0^t |f(x(s), x([s]))|^p ds \right.$$

$$\left. + E \sup_{0 \le t \le t_1} \left| \int_0^t g(x(s), x([s]))dB(s) \right|^p \right]. \tag{4.4}$$

By the Burkholder-Davis-Gundy inequality and the Hölder inequality, it is not difficult to show that

$$
E \sup_{0 \le t \le t_1} |x(t)|^p \le 3^{p-1} \left[|x(0)|^p + T^{p-1} E \int_0^{t_1} |f(x(s), x([s]))|^p ds \right.
$$
$$
\left. + C_p T^{p/2-1} E \int_0^{t_1} |g(x(s), x([s]))|^p ds \right],
$$

(4.5)

where C_p is a constant. Note from the linear growth conditions that

$$
E \sup_{0 \le t \le t_1} |x(t)|^p \le 3^{p-1} \left[|x(0)|^p + T^{p-1} E \int_0^{t_1} K^{p/2} \left(1 + |x(s)|^2 + |x([s])|^2 \right)^{p/2} ds \right.
$$
$$
\left. + C_p T^{p/2-1} E \int_0^{t_1} K^{p/2} \left(1 + |x(s)|^2 + |x([s])|^2 \right)^{p/2} ds \right]
$$
$$
\le 3^{p-1} |x(0)|^p + 3^{(3p/2)-2} K^{p/2} \left(T^{p-1} + C_p T^{p/2-1} \right)
$$
$$
\times E \int_0^{t_1} \left(1 + |x(s)|^p + |x([s])|^p \right) ds
$$
$$
\le 3^{p-1} |x(0)|^p + 3^{(3p/2)-2} K^{p/2} \left(T^p + C_p T^{p/2} \right)
$$
$$
+ 2 \times 3^{(3p/2)-2} K^{p/2} \left(T^{p-1} + C_p T^{p/2-1} \right) E \sup_{0 \le u \le s} |x(u)|^p ds.
$$

(4.6)

By the Gronwall inequality, we must get

$$
E \sup_{0 \le t \le T} |x(t)|^p \le C_3 (1 + |x_0|^p),
$$

(4.7)

where $C_3 = C_3(p, T, K)$ is a constant independent of h.   □

The following lemma shows that the continuous Euler-Maruyama approximate solution has bounded pth moments.

Lemma 4.2. *Under the linear growth conditions (H2), there exists a positive constant C_4 such that the continuous approximate solution of the Euler-Maruyama (2.10) satisfies*

$$
E \sup_{0 \le t \le T} |y(t)|^p \le C_4 (1 + |x_0|^p),
$$

(4.8)

where $C_4 = C_4(p, T, K)$ is a constant independent of h.

Proof. By the inequality $|a+b+c|^2 \leq 3|a|^2+3|b|^2+3|c|^2$ and (2.10), in the same way as Lemma 4.1, for any $0 \leq t_1 \leq T$, we can obtain

$$
\begin{aligned}
E \sup_{0 \leq t \leq t_1} |y(t)|^p \leq 3^{p-1} \Bigg[&|y(0)|^p + T^{p-1} E \int_0^{t_1} |f(z(s), z([s]))|^p ds \\
&+ C_p T^{(p/2)-1} E \int_0^{t_1} |g(z(s), z([s]))|^p ds \Bigg],
\end{aligned}
\tag{4.9}
$$

where C_p is a constant. Note from the linear growth conditions that

$$
\begin{aligned}
E \sup_{0 \leq t \leq t_1} |y(t)|^p \leq 3^{p-1} \Bigg[&|y(0)|^p + T^{p-1} E \int_0^{t_1} K^{p/2} \left(1 + |z(s)|^2 + |z([s])|^2\right)^{p/2} ds \\
&+ C_p T^{p/2-1} E \int_0^{t_1} K^{p/2} \left(1 + |z(s)|^2 + |z([s])|^2\right)^{p/2} ds \Bigg] \\
\leq\ & 3^{p-1} |y(0)|^p + 3^{(3p/2)-2} K^{p/2} \left(T^{p-1} + C_p T^{p/2-1}\right) \\
&\times E \int_0^{t_1} \left(1 + |z(s)|^p + |z([s])|^p\right) ds \\
\leq\ & 3^{p-1} |y(0)|^p + 3^{(3p/2)-2} K^{p/2} \left(T^p + C_p T^{p/2}\right) \\
&+ 2 \times 3^{(3p/2)-2} K^{p/2} \left(T^{p-1} + C_p T^{p/2-1}\right) E \sup_{0 \leq u \leq s} |y(u)|^p ds.
\end{aligned}
\tag{4.10}
$$

By the Gronwall inequality, we must get

$$
E \sup_{0 \leq t \leq T} |y(t)|^p \leq C_4 \left(1 + |x_0|^p\right),
\tag{4.11}
$$

where $C_4 = C_4(p, T, K)$ is a constant independent of h. $\qquad\square$

According to Theorem 3.2, we have the following theorem.

Theorem 4.3. *Under the conditions (H1) and (H2), the EM approximate solution converges to the exact solution of* (2.1) *in the sense that*

$$
\lim_{h \to 0} E \left[\sup_{0 \leq t \leq T} |y(t) - x(t)|^2 \right] = 0.
\tag{4.12}
$$

5. A Motivating Example

In the above section, we give the strong convergence numerical solution of SEPCAs under the local Lipschitz condition (H1) and the linear growth condition (H2). However, there are many SEPCAs that do not satisfy the linear growth condition, consider the following SEPCA:

$$dx(t) = \left[-x^3(t) + x([t])\right]dt + \left[\sin x^2(t) + x([t])\right]dB(t) \quad \forall t \geq 0. \tag{5.1}$$

Clearly, the equation do not satisfy the linear growth condition (H2). But the example is analyzed under condition (H3) which covers many nonlinear SEPCAs. On the other hand, we have

$$x\left(-x^3 + y\right) + \frac{1}{2}\left(\sin x^2 + y\right)^2 \leq -x^4 + xy + \left(\sin x^2\right)^2 + y^2 \leq 2\left(1 + x^2 + y^2\right). \tag{5.2}$$

In other words, the equation satisfies condition (H3). Moreover, we also have

$$2x^T f(x, y) + \left|g(x, y)\right|^2 \leq |x|^2 + 2K\left(1 + |x|^2 + |y|^2\right) \leq (1 + 2K)\left(1 + |x|^2 + |y|^2\right). \tag{5.3}$$

We see clearly that (H3) follows from (H2). Therefore, The following result is more general than Theorem 3.2. Let us now turn to establish the convergence of the Euler-Maruyama method to (2.1) under the conditions (H1) and (H3).

6. Convergence of the Euler-Maruyama Method under Monotone Condition

In this section, we give the convergence of the EM method to (2.1) under the local Lipschitz condition (H1) and the monotone condition (H3). We prove the bounded pth moment property of the EM approximate solution and the exact solution to (2.1) under the monotone condition (H3).

Lemma 6.1. *Under the monotone condition (H3), there exists a positive constant C_5 such that the solution of* (2.1) *satisfies*

$$E\sup_{0 \leq t \leq T}|x(t)|^p \vee E\sup_{0 \leq t \leq T}|y(t)|^p \leq C_5, \tag{6.1}$$

where $C_5 = C_5(p, T, K_1, x_0)$ is a constant independent of h.

Proof. By Itô formula, for all $t \geq 0$, we have

$$
\left(1 + |x(t)|^2\right)^{p/2}
$$

$$
= \left(1 + |x(0)|^2\right)^{p/2} + \int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2}
$$

$$
\times \left[x(t)^T f(x(s), x([s])) + \frac{p-1}{2}|g(x(s), x([s]))|^2\right] ds
$$

$$
+ \int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2} x(s)^T g(x(s), x([s])) dB(s)
$$

$$
\leq \left(1 + |x(0)|^2\right)^{p/2} + \int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2}\left[K_1\left(1 + |x(s)|^2 + |x([s])|^2\right)\right] ds
$$

$$
+ \int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2} x(s)^T g(x(s), x([s])) dB(s).
$$

$$(6.2)$$

For any $t_1 \in [0, T]$, we have

$$
E \sup_{0 \leq t \leq t_1} \left(1 + |x(t)|^2\right)^{p/2}
$$

$$
\leq \left(1 + |x(0)|^2\right)^{p/2} + E \sup_{0 \leq t \leq t_1} \int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2}\left[K_1\left(1 + |x(s)|^2 + |x([s])|^2\right)\right] ds \quad (6.3)
$$

$$
+ E \sup_{0 \leq t \leq t_1} \left|\int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2} x(s)^T g(x(s), x([s])) dB(s)\right|.
$$

By the Burkholder-Davis-Gundy inequality and Exercise 2.5 in [8], it is not difficult to show that

$$
E \sup_{0 \leq t \leq t_1} \left|\int_0^t p\left(1 + |x(t)|^2\right)^{(p-2)/2} x(s)^T g(x(s), x([s])) dB(s)\right|
$$

$$
\leq 3E\left(\int_0^{t_1} p^2\left(1 + |x(t)|^2\right)^{p-2}|x(s)|^2|g(x(s), x([s]))|^2 ds\right)^{1/2}
$$

$$
\leq 3E\left(\sup_{0 \leq t \leq t_1}\left(1 + |x(t)|^2\right)^{p/2} \int_0^{t_1} p^2\left(1 + |x(t)|^2\right)^{(p-2)/2}|g(x(s), x([s]))|^2 ds\right)^{1/2}
$$

$$\leq 3E\left(\sup_{0\leq t\leq t_1}\left(1+|x(t)|^2\right)^{p/2}\int_0^{t_1}p^2K_1\left(1+|x(t)|^2\right)^{(p-2)/2}\left(1+|x(s)|^2+|x([s])|^2\right)ds\right)^{1/2}$$

$$\leq 0.5E\sup_{0\leq t\leq t_1}\left(1+|x(t)|^2\right)^{p/2}+4.5E\int_0^{t_1}p^2K_1\left(1+|x(t)|^2\right)^{(p-2)/2}\left(1+|x(s)|^2+|x([s])|^2\right)ds.$$

$$(6.4)$$

Substituting (6.4) into (6.3) and using the Hölder inequality

$$E\sup_{0\leq t\leq t_1}\left(1+|x(t)|^2\right)^{p/2}$$

$$\leq 2\left(1+|x(0)|^2\right)^{p/2}+2pK_1(1+4.5p)E\int_0^{t_1}\left(1+|x(t)|^2\right)^{(p-2)/2}\left(1+|x(s)|^2+|x([s])|^2\right)ds$$

$$\leq 2\left(1+|x(0)|^2\right)^{p/2}$$

$$+2pK_1(1+4.5p)\left(E\int_0^{t_1}\left(1+x(t)|^2\right)^{p/2}ds\right)^{(p-2)/p}$$

$$\times\left(E\int_0^{t_1}\left(1+|x(s)|^2+|x([s])|^2\right)^{p/2}ds\right)^{2/p}$$

$$\leq 2\left(1+|x(0)|^2\right)^{p/2}$$

$$+2pK_1(1+4.5p)\left(TE\sup_{0\leq t\leq t_1}\left(1+|x(t)|^2\right)^{p/2}\right)^{(p-2)/p}$$

$$\times\left(E\int_0^{t_1}\left(1+|x(s)|^2+|x([s])|^2\right)^{p/2}ds\right)^{2/p}.$$

$$(6.5)$$

So we obtain

$$E\sup_{0\leq t\leq t_1}\left(1+|x(t)|^2\right)^{p/2}$$

$$\leq\left[2\left(1+|x(0)|^2\right)^{p/2}+2pK_1(1+4.5p)T^{(p-2)/p}(E\int_0^{t_1}\left(1+|x(s)|^2+|x([s])|^2\right)^{p/2}ds)^{2/p}\right]^{p/2}$$

$$\leq 2^{(p/2)-1}\left[2\left(1+|x(0)|^2\right)^{p/2}\right]^{p/2}$$

$$+2^{(p/2)-1}\left[2pK_1(1+4.5p)T^{(p-2)/p}\left(E\int_0^{t_1}\left(1+|x(s)|^2+|x([s])|^2\right)^{p/2}ds\right)^{2/p}\right]^{p/2}$$

$$\leq 2^{p-1}\left(1+|x(0)|^2\right)^{p^2/4} + 2^{p-1}\left[pK_1(1+4.5p)T^{(p-2)/p}\right]^{p/2}E\int_0^{t_1}\left(1+|x(s)|^2+|x([s])|^2\right)^{p/2}ds$$

$$\leq 2^{p-1}\left(1+|x(0)|^2\right)^{p^2/4} + 2^{(3p/2)-1}\left[pK_1(1+4.5p)T^{(p-2)/p}\right]^{p/2}\int_0^{t_1}E\sup_{0\leq u\leq s}\left(1+|x(u)|^2\right)^{p/2}ds. \tag{6.6}$$

By the Gronwall inequality, we must get

$$E\sup_{0\leq t\leq T}\left(|x(t)|^p\right)\leq C_5, \tag{6.7}$$

where $C_5 = C_5(p, T, K, x_0)$ is a constant independent of h. Similarly, we can show that

$$E\sup_{0\leq t\leq T}\left(|y(t)|^p\right)\leq C_5. \tag{6.8}$$

The proof is completed. $\qquad\square$

According to Theorem 3.2, we obtain the following.

Theorem 6.2. *Under the conditions (H1) and (H3), the EM approximate solution converges to the exact solution of* (2.1) *in the sense that*

$$\lim_{h\to 0}E\left(\sup_{0\leq t\leq T}|y(t)-x(t)|^2\right)=0. \tag{6.9}$$

Acknowledgments

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