Research Article

A New Hybrid Algorithm for λ-Strict Asymptotically Pseudocontractions in 2-Uniformly Smooth Banach Spaces

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A new hybrid projection algorithm is considered for a λ-strict asymptotically pseudocontractive mapping. Using the metric projection, a strong convergence theorem is obtained in a uniformly convex and 2-uniformly smooth Banach spaces. The result presented in this paper mainly improves and extends the corresponding results of Matsushita and Takahashi (2008), Dehghan (2011) Kang and Wang (2011), and many others.

1. Introduction

Let E be a real Banach space and E^* be the dual spaces of E. Assume that J is the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing between E and E^*.

Let C be a nonempty closed convex subset of a real Banach space E.

Definition 1.1. Let $T : C \to C$ be a mapping:

(1) T is said to be nonexpansive if for all $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|, \tag{1.2}$$

(2) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq k_n \|x - y\|, \tag{1.3}$$

(3) T is said to be λ-strictly pseudocontractive in the terminology of Browder-Petryshyn [1] if there exists a constant $\lambda \in (0, 1)$ such that for all $x, y \in C$,

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - T)x - (I - T)y\|^2, \quad \forall j(x - y) \in J(x - y), \tag{1.4}$$

(4) T is said to be $(\lambda, \{k_n\})$-strict asymptotically pseudocontractive if there exist a constant $\lambda \in (0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for all $x, y \in C$ and for all $j(x - y) \in J(x - y)$,

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2 - \lambda \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall n \geq 1, \tag{1.5}$$

(5) T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall n \geq 1. \tag{1.6}$$

It is wellknown that the class of $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mappings was first introduced in Hilbert spaces by Liu [2]. In the case of Hilbert spaces, it is shown by [2] that (1.5) is equivalent to the inequality

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \lambda \|(I - T^n)x - (I - T^n)y\|^2. \tag{1.7}$$

Concerning the convergence problem of iterative sequences for strictly pseudocontractive mappings has been studied by several authors (see [1, 3–20]). Concerning the class of strictly asymptotically pseudocontractive mappings, Liu [2] proved the following results.

Theorem 1.2. *Let H be a real Hilbert space, let C be a nonempty closed convex and bounded subset of H, and let $T : C \to C$ be a completely continuous uniformly L-Lipschitzian $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping such that $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\} \subset (0, 1)$ be a sequence satisfying the following condition:*

$$0 < \epsilon \leq \alpha_n \leq 1 - \lambda - \epsilon, \quad \forall n \geq 1 \text{ and some } \epsilon > 0. \tag{1.8}$$

Then, the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1 \tag{1.9}$$

converges strongly to a fixed point of T.

In 2007, Osilike et al. [21] proved the following theorem.

Theorem 1.3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, and let $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$ be a real sequence satisfying the following condition:*

$$0 < a \leq \alpha_n \leq b < \frac{q(1-k)}{2c_q}(1+L)^{-(q-2)}, \quad \forall n \geq 1. \tag{1.10}$$

Let $\{x_n\}$ be the sequence defined by (1.9). Then, $\{x_n\}$ converges weakly to a fixed point of T.

On the other hand, by using the metric projection, Nakajo and Takahashi [22] introduced the following iterative algorithms for the nonexpansive mapping T in the framework of Hilbert spaces:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= \alpha_n x_{\mathrm{n}} + (1 - \alpha_n) T x_n, \\
C_n &= \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.11}
$$

where $\{\alpha_n\} \subset [0, \alpha], \alpha \in [0, 1)$ and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.11) converges strongly to a fixed point of T.

In 2006, Xu [23] extended Nakajo and Takahashi's theorem to Banach spaces by using the generalized projection.

In 2008, Matsushita and Takahashi [24] presented the following iterative algorithms for the nonexpansive mapping T in the framework of Banach spaces:

$$
\begin{aligned}
x_0 &= x \in C, \\
C_n &= \overline{\mathrm{co}}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}, \\
D_n &= \{z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap D_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.12}
$$

where $\overline{\mathrm{co}}\, C$ denotes the convex closure of the set C, J is normalized duality mapping, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, they proved that $\{x_n\}$ generated by (1.12) converges strongly to a fixed point of nonexpansive mapping T.

Recently, Dehghan [25] introduced the following hybrid projection algorithm for an asymptotically nonexpansive mapping T in the framework of Banach spaces:

$$
\begin{aligned}
x_0 &= x \in C, \qquad C_0 = D_0 = C, \\
C_n &= \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - T^n z\| \leq t_n \|x_n - T^n x_n\|\}, \\
D_n &= \{z \in D_{n-1} : \langle x_n - z, J(x - x_n) \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap D_n} x, \quad n = 1, 2, \ldots,
\end{aligned}
\tag{1.13}
$$

where $\overline{\mathrm{co}}\,C$ denotes the convex closure of the set C, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, he proved that $\{x_n\}$ generated by (1.13) converges strongly to a fixed point of an asymptotically nonexpansive mappings T.

Motivated by the research work going on in this direction, the purpose of this paper is to introduce the following iteration for finding a fixed point of $(\lambda, \{k_n\})$-strict asymptotically pseudocontraction in a uniformly convex and 2-uniformly smooth Banach spaces:

$$
\begin{aligned}
x_0 = x \in C, \qquad C_0 = C, \\
C_n = \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - T^n z\| \le t_n \|x_n - T^n x_n\|\}, \\
x_{n+1} = P_{C_n} x, \quad n = 1, 2, \dots,
\end{aligned}
\tag{1.14}
$$

where $\overline{\mathrm{co}}\,C$ denotes the convex closure of the set C, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and P_{C_n} is the metric projection from E onto C_n. Under suitable conditions some strong convergence theorem for the sequence $\{x_n\}$ defined by (1.14) to converge a fixed point of an asymptotically λ-strictly pseudocontraction. The result presented in the paper extends and improves the main results of Matsushita and Takahashi [24], Dehghan [25], Kang and Wang [26], and others.

2. Preliminaries

In this section, we recall the well-known concepts and results which will be needed to prove our main results. Throughout this paper, we assume that E is a real Banach space and C is a nonempty subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x - y\| \ge \epsilon, \|x + y\| \le 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$
\delta_E(\epsilon) = \inf\left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\|, \|y\| \le 1, \|x - y\| \ge \epsilon \right\},
\tag{2.1}
$$

E is said to be *smooth* if the limit

$$
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\tag{2.2}
$$

exists for all $x, y \in U$. The modulus of smoothness of E is defined by

$$
\rho_E(t) = \sup\left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \le 1, \|y\| \le t \right\}.
\tag{2.3}
$$

A Banach space E is said to be *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. A Banach space E is said to be *q-uniformly smooth*, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$.

If E is a reflexive, strictly convex and smooth Banach space, then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y \in C} \|y - x\|. \tag{2.4}$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. Let $x \in E$ and $u \in C$. Then it is known that $u = P_C x$ if and only if

$$\langle u - y, J(x - u) \rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

For the details on the metric projection, refer to [27–30].

In the sequel, we make use of the following lemmas for our main results.

Lemma 2.1 (see [31]). *Let E be a real Banach space, C a nonempty subset of E, and $T : C \to C$ a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.2 (see [32]). *Let E be a real 2-uniformly smooth Banach spaces with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2\|Ky\|^2, \tag{2.6}$$

for any $x, y \in E$.

Lemma 2.3 (demiclosed principle [21]). *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let C be a nonempty closed convex subset of E and $T : C \to C$ a $(\lambda, \{k_n\})$-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set. Then $I - T$ is demiclosed at zero, where I is the identical mapping.*

Lemma 2.4 (see [33]). *Let C be a closed convex subset of a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|T\left(\sum_{j=0}^{m} \mu_j z_j\right) - \sum_{j=0}^{m} \mu_j T z_j\right\|\right) \leq \max_{0 \leq j < k \leq m} (\|z_j - z_k\| - \|T z_j - T z_k\|), \tag{2.7}$$

for all $m \geq 1$, $\{\mu_j\}_{j=0}^{m} \in \Delta^m, \{z_j\}_{j=0}^{m} \subset C \cap B_r$ and $T \in \text{Lip}\,(C, 1)$, where $\Delta^m = \{\{\mu_0, \mu_1, \ldots, \mu_m\} : 0 \leq \mu_j (0 \leq j \leq m)$ and $\sum_{j=0}^{m} \mu_j = 1\}$, $B_r = \{x \in E : \|x\| \leq r\}$ and $\text{Lip}\,(C, 1)$ is the set of all nonexpansive mappings from C into E.

3. Main Results

Now we are ready to give our main results in this paper.

Lemma 3.1. *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with the best smooth constant K, and $T : C \rightarrow C$ be a $(\lambda, \{k_n\})$-strict asymptotically pseudocontraction. For $\alpha \in (0,1) \cap (0, \lambda/K^2]$, one defines*

$$S_{n,\alpha}x = \frac{1}{\sqrt{2\alpha(k_n - 1) + 1}}[(1 - \alpha)x + \alpha T^n x], \tag{3.1}$$

for all $x \in C$ and each $n \geq 1$. Then $S_{n,\alpha} : C \rightarrow E$ is a nonexpansive.

Proof. For any $x, y \in C$, put $\beta_{n,\alpha} = \sqrt{2\alpha(k_n - 1) + 1}$, we compute

$$
\begin{aligned}
\|S_{n,\alpha}&x - S_{n,\alpha}y\|^2 \\
&= \left\| \frac{1}{\beta_{n,\alpha}}[(1-\alpha)x + \alpha T^n x] - \frac{1}{\beta_{n,\alpha}}[(1-\alpha)y + \alpha T^n y] \right\|^2 \\
&= \frac{1}{\beta_{n,\alpha}^2} \|(x - y) + \alpha(T^n x - T^n y - (x - y))\|^2 \\
&\leq \frac{1}{\beta_{n,\alpha}^2} \Big(\|x - y\|^2 + 2\alpha\langle T^n x - T^n y - (x - y), j(x - y)\rangle \\
&\qquad\qquad + 2K^2\alpha^2\|T^n x - T^n y - (x - y)\|^2 \Big) \\
&= \frac{1}{\beta_{n,\alpha}^2} \Big(\|x - y\|^2 + 2\alpha\langle T^n x - T^n y, j(x - y)\rangle \\
&\qquad\qquad - 2\alpha\|x - y\|^2 + 2K^2\alpha^2\|T^n x - T^n y - (x - y)\|^2 \Big) \\
&\leq \frac{1}{\beta_{n,\alpha}^2} \Big(\|x - y\|^2 + 2\alpha\big(k_n\|x - y\|^2 - \lambda\|T^n x - T^n y - (x - y)\|^2\big) \\
&\qquad\qquad - 2\alpha\|x - y\|^2 + 2K^2\alpha^2\|T^n x - T^n y - (x - y)\|^2 \Big) \\
&= \frac{1}{\beta_{n,\alpha}^2} \Big\{ (1 + 2\alpha k_n - 2\alpha)\|x - y\|^2 - 2\alpha\lambda\|T^n x - T^n y - (x - y)\|^2 \\
&\qquad\qquad + 2K^2\alpha^2\|T^n x - T^n y - (x - y)\|^2 \Big\} \\
&\leq \|x - y\|^2,
\end{aligned}
\tag{3.2}
$$

which shows that $S_{n,\alpha}$ is a nonexpansive mapping. This completes the proof. $\qquad\square$

Theorem 3.2. *Let C be a nonempty bounded and closed convex subset of a uniformly convex and 2-uniformly smooth Banach spaces E with the best smooth constant $K > 0$, assume that $T : C \rightarrow C$*

is a $(\lambda, \{k_n\})$-strict asymptotically pseudocontraction such that $\mathcal{F} = \mathcal{F}(T) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0, 1)$ with $t_n \to 0$. Let $\{x_n\}$ be a sequence generated by (1.14), where

$$\overline{\text{co}}\{z \in C_{n-1} : \|z - T^n z\| \leq t_n \|x_n - T^n x_n\|\} \tag{3.3}$$

denotes the convex closure of the set $\{z \in C_{n-1} : \|z - T^n z\| \leq t_n \|x_n - T^n x_n\|\}$ and P_{C_n} is the metric projection from E onto C_n. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x$.

Proof. (I) First we prove that $\{x_n\}$ is well defined and bounded.

It is easy to check that C_n is closed and convex and $\mathcal{F} \subset C_n$ for all $n \geq 0$. Therefore, $\{x_n\}$ is well defined.

Put $p = P_{\mathcal{F}} x$. Since $\mathcal{F} \subset C_n$ and $x_{n+1} = P_{C_n} x$, we have that

$$\|x_{n+1} - x\| \leq \|p - x\|, \tag{3.4}$$

for all $n \geq 0$. Hence, $\{x_n\}$ is bounded.

(II) Now we prove that $\|x_n - T^{n-k} x_n\| \to 0$ as $n \to \infty$ for any $k \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of all positive integers).

Fix $k \in \mathbb{N}$ and put $l = n - k, n > k$. Since $x_n = P_{C_{n-1}} x$, we have $x_n \in C_{n-1} \subset \cdots \subset C_l$. Then there exist some positive integer $m \in \mathbb{N}$, $\{\mu_i\} \in \Delta^m$ and $\{z_i\}_{i=0}^m \{z_i\}_{i=1}^m \subset C_{l-1}$ such that

$$\left\| x_n - \sum_{j=0}^m \mu_j z_j \right\| < t_l, \tag{3.5}$$

$$\left\| z_j - T^l z_j \right\| \leq t_l \left\| x_l - T^l x_l \right\|, \tag{3.6}$$

for all $j \in \{0, 1, \ldots, m\}$. Take $\alpha \in (0, 1) \cap (0, \lambda/K^2]$. Put $M = \sup_{x \in C} \|x\|$, $p = P_{\mathcal{F}} x$, $r_0 = \sup_{n \geq 1} \|x_n - p\|$ and $\beta_{l,\alpha} = \sqrt{2\alpha(k_l - 1) + 1}$. we define

$$S_{n,\alpha} x = \frac{1}{\sqrt{2\alpha(k_n - 1) + 1}} [(1 - \alpha)x + \alpha T^n x], \tag{3.7}$$

for all $x \in C$ and each $n \geq 1$, then $S_{l,\alpha} p = (1/\beta_{l,\alpha}) p$. It follows from Lemma 3.1 and (3.5) that

$$\left\| x_l - T^l x_l \right\| = \frac{\beta_{l,\alpha}}{\alpha} \left\| \frac{1}{\beta_{l,\alpha}} \left(\alpha T^l x_l + (1-\alpha) x_l \right) - \frac{1}{\beta_{l,\alpha}} x_l \right\|$$

$$= \frac{\beta_{l,\alpha}}{\alpha} \left\| \left(S_{l,\alpha} x_l - S_{l,\alpha} p \right) + \frac{1}{\beta_{l,\alpha}} \left(p - x_l \right) \right\|$$

$$\leq \frac{\beta_{l,\alpha} + 1}{\alpha} r_0,$$

$$\left\| T^l \left(\sum_{j=0}^{m} \mu_j z_j \right) - T^l x_n \right\|$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \left(\left\| S_{l,\alpha} \left(\sum_{j=0}^{m} \mu_j z_j \right) - S_{l,\alpha} x_n \right\| + \frac{1-\alpha}{\beta_{l,\alpha}} \left\| \sum_{j=0}^{m} \mu_j z_j - x_n \right\| \right)$$

$$\leq \left(\frac{\beta_{l,\alpha} + 1}{\alpha} - 1 \right) \left\| \sum_{j=0}^{m} \mu_j z_j - x_n \right\|$$

$$\leq \left(\frac{\beta_{l,\alpha} + 1}{\alpha} - 1 \right) t_l. \tag{3.8}$$

Moreover, from Lemmas 2.4 and 3.1, we have

$$\left\| T^l \left(\sum_{j=0}^{m} \mu_j z_j \right) - \sum_{j=0}^{m} \mu_j T^l z_j \right\|$$

$$= \frac{\beta_{l,\alpha}}{\alpha} \left\| S_{l,\alpha} \left(\sum_{j=0}^{m} \mu_j z_j \right) - \sum_{j=0}^{m} \mu_j S_{l,\alpha} z_j \right\|$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left[\max_{0 \leq j < k \leq m} \left(\| z_j - z_k \| - \| S_{l,\alpha} z_j - S_{l,\alpha} z_k \| \right) \right]$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left[\max_{0 \leq j < k \leq m} \left(\| z_j - S_{l,\alpha} z_j \| + \| z_k - S_{l,\alpha} z_k \| \right) \right]$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ \max_{0 \leq j < k \leq m} \left[\frac{\alpha}{\beta_{l,\alpha}} \left(\| z_j - T^l z_j \| + \| z_k - T^l z_k \| \right) \right. \right. \tag{3.9}$$

$$\left. \left. + \left(1 - \frac{1}{\beta_{l,\alpha}} \right) \left(\| z_j \| + \| z_k \| \right) \right] \right\}$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ \frac{2\alpha t_l}{\beta_{l,\alpha}} \left\| x_l - T^l x_l \right\| + 2 \left(1 - \frac{1}{\beta_{l,\alpha}} \right) M \right\}$$

$$\leq \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ \frac{2\alpha t_l}{\beta_{l,\alpha}} \left\{ \frac{\beta_{l,\alpha} + 1}{\alpha} r_0 \right\} + 2 \left(1 - \frac{1}{\beta_{l,\alpha}} \right) M \right\}$$

$$= \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ 2 \left(1 + \frac{1}{\beta_{l,\alpha}} \right) r_0 t_l + 2 \left(1 - \frac{1}{\beta_{l,\alpha}} \right) M \right\}.$$

Observe that $\beta_{l,\alpha} \to 1$ as $n \to \infty$, it follows from (3.5)–(3.9) that

$$
\begin{aligned}
\left\| x_n - T^l x_n \right\| &\leq \left\| x_n - \sum_{j=0}^{m} \mu_j z_j \right\| + \left\| \sum_{j=0}^{m} \mu_j \left(z_j - T^l z_j \right) \right\| \\
&\quad + \left\| \sum_{j=0}^{m} \mu_j T^l z_j - T^l \left(\sum_{j=0}^{m} \mu_j z_j \right) \right\| + \left\| T^l \left(\sum_{j=0}^{m} \mu_j z_j \right) - T^l x_n \right\| \\
&\leq t_l + t_l \left\| x_l - T^l x_l \right\| + \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ 2\left(1 + \frac{1}{\beta_{l,\alpha}}\right) r_0 t_l + 2\left(1 - \frac{1}{\beta_{l,\alpha}}\right) M \right\} \\
&\quad + \left(\frac{\beta_{l,\alpha} + 1}{\alpha} - 1 \right) t_l \\
&\leq t_l + \frac{\beta_{l,\alpha} + 1}{\alpha} r_0 t_l + \frac{\beta_{l,\alpha}}{\alpha} \gamma^{-1} \left\{ 2\left(1 + \frac{1}{\beta_{l,\alpha}}\right) r_0 t_l + 2\left(1 - \frac{1}{\beta_{l,\alpha}}\right) M \right\} \\
&\quad + \left(\frac{\beta_{l,\alpha} + 1}{\alpha} - 1 \right) t_l \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
\tag{3.10}
$$

This shows that

$$
\left\| x_n - T^l x_n \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\tag{3.11}
$$

(III) we prove that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Since T is a uniformly L-Lipschitzian, we have

$$
\begin{aligned}
\|x_n - Tx_n\| &\leq \left\| x_n - T^{n-1} x_n \right\| + \left\| T^{n-1} x_n - Tx_n \right\| \\
&\leq \left\| x_n - T^{n-1} x_n \right\| + L \left\| T^{n-2} x_n - x_n \right\| \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
\tag{3.12}
$$

(IV) Finally, we prove that $x_n \to p = P_{\mathcal{F}} x$.

It follows from the boundedness of $\{x_n\}$ that for each subsequence $\{x_{n_i}\} \subset \{x_n\}$ there exists a subsequence (without loss of generality we can still denote it by) $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup v$ as $i \to \infty$. Since $T : C \to C$ is a uniformly L-Lipschitzian and $(\lambda, \{k_n\})$-strict asymptotically pseudocontraction, from Lemma 2.3, we know that T is demiclosed. Hence we have $v \in \mathcal{F}$.

From the weakly lower semicontinuity of the norm and (3.4), it follows that

$$
\begin{aligned}
\|p - x\| &\leq \|v - x\| \leq \liminf_{i \to \infty} \|x_{n_i} - x\| \\
&\leq \limsup_{i \to \infty} \|x_{n_i} - x\| \leq \|p - x\|.
\end{aligned}
\tag{3.13}
$$

This shows $p = v$ and hence $x_{n_i} \rightharpoonup p$ as $i \to \infty$. By the arbitrariness of $\{x_{n_i}\} \subset \{x_n\}$, we obtain $x_n \rightharpoonup p$. Further, it follows from (3.13) that

$$\lim_{n \to \infty} \|x_n - x\| = \|p - x\|. \tag{3.14}$$

Since E is uniformly convex, it has the Kadec-Klee property. Hence, we have $x_n - x \to p - x$, that is, $x_n \to p$. This completes the proof. $\qquad\square$

Acknowledgment

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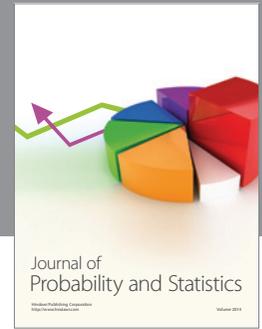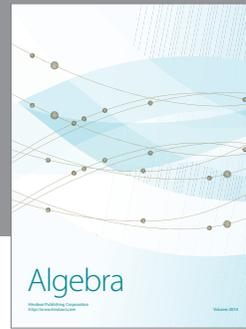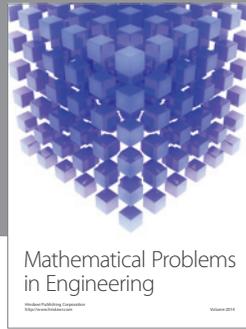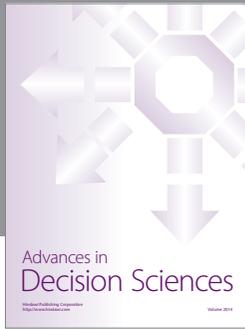
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