Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 684248, 24 pages
doi:10.1155/2012/684248

Research Article

An Approximation of Semigroups Method for Stochastic Parabolic Equations

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Received 25 May 2012; Accepted 8 June 2012

Academic Editor: Ravshan Ashurov

A single-step difference scheme for the numerical solution of the nonlocal-boundary value problem for stochastic parabolic equations is presented. The convergence estimate for the solution of the difference scheme is established. In application, the convergence estimates for the solution of the difference scheme are obtained for two nonlocal-boundary value problems. The theoretical statements for the solution of this difference scheme are supported by numerical examples.

1. Introduction

It is known that most problems in heat flow, fusion process, model financial instruments like options, bonds, and interest rates, and other areas which are involved with uncertainty lead to stochastic differential equation with parabolic type. These equations can be derived as models of indeterministic systems and considered as methods for solving boundary value problems.

The method of operators as a tool for investigation of the solution to stochastic partial differential equations in Hilbert and Banach spaces has been systematically developed by several authors (see [1–4] and the references therein). Finite difference method for the solution of initial boundary value problem for stochastic differential equations has been studied extensively by many researchers (see [5–15] and the references therein). However, multipoint nonlocal-boundary value problems were not well investigated.

In the present paper the multipoint nonlocal-boundary value problem

$$dv(t) = -Av(t)dt + f(t)dw_t, \quad 0 < t < T$$
$$v(0) = \sum_{j=1}^{J} \alpha_j v(\lambda_j) + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}),$$
$$\sum_{j=1}^{J} |\alpha_j| \le 1, \quad 0 < \lambda_1 < \cdots < \lambda_J \le T, \quad 0 \le t \le T \tag{1.1}$$

for stochastic parabolic differential equations in a Hilbert space H with a self-adjoint positive definite operator A is considered. Here

(i) w_t is a standard Wiener process given on the probability space (Ω, F, P).

(ii) For any $z \in [0,T]$, $f(z)$ is an element of space $M_w^2([0,T], H_1)$, where H_1 is subspace of H.

(iii) $\varphi(w_{\lambda_1}, \dots, w_{\lambda_J})$ is element of space $M_w^2([0,T], H_2)$ of H_2-valued measurable processes, where H_2 is a subspace of H.

Here, $M_w^2([0,T], H)$[20] denote the space of H-valued measurable processes which satisfy

(a) $\phi(t)$ is F_t measurable, a.e. in t,

(b) $E\int_0^T \|\phi(t)\|_H dt < \infty$.

The main goal of this study is to construct and investigate the difference schemes for the multipoint nonlocal-boundary value problems (1.1). The outline of the paper is as follows. In Section 2, the exact single-step difference scheme for the solution of the problem (1.1) is presented. In Section 3, the 1/2-th order of accuracy Rothe difference scheme is constructed and investigated for the approximate solution of the problem (1.1). The estimate of convergence for the solution of this difference scheme is obtained. In applications, the convergence estimates for the solution of difference schemes for the numerical solution of two multipoint nonlocal-boundary value problems for stochastic parabolic equations are obtained. In Section 4, the numerical application for one-dimensional stochastic parabolic equation is presented.

2. The Exact Single-Step Difference Scheme

Now, let us give some lemmas we need in the sequel. Throughout this paper, let H be a Hilbert space, let A be a positive definite self-adjoint operator with $A \ge \delta I$, where $\delta > 0$.

Lemma 2.1. *The following estimate holds:*

$$\left\| e^{-tA} \right\|_{H \to H} \le e^{-\delta t} \quad (t \ge 0), \qquad \left\| Ae^{-tA} \right\|_{H \to H} \le \frac{1}{t} \quad (t > 0). \tag{2.1}$$

Lemma 2.2. *Suppose that assumption*

$$\sum_{k=1}^{J}|\alpha_k| \leq 1 \tag{2.2}$$

holds. Then, the operator

$$I - \sum_{k=1}^{J}\alpha_k e^{-\lambda_k A} \tag{2.3}$$

has an inverse

$$\Upsilon = \left(I - \sum_{k=1}^{J}\alpha_k e^{-\lambda_k A}\right)^{-1}, \tag{2.4}$$

and the following estimate is satisfied:

$$\|\Upsilon\|_{H\to H} \leq \frac{1}{1-e^{-\lambda_1\delta}} \leq C(\delta,\lambda_1). \tag{2.5}$$

Proof. The proof follows from the triangle inequality, assumption (2.2), and estimate

$$\left\|\left(I - \sum_{k=1}^{J}\alpha_k e^{-\lambda_k A}\right)^{-1}\right\|_{H\to H} \leq \sup_{\delta\leq\mu<\infty}\frac{1}{\left|1-\sum_{k=1}^{J}\alpha_k e^{-\lambda_k\mu}\right|}. \tag{2.6}$$

Let us now obtain the formula for the mild solution of problem (1.1). It is clear that under the assumptions (i)-(ii) and

$$E\|v(0)\|_{H_2}^2 < \infty, \quad H_2 \subset H, \tag{2.7}$$

the Cauchy problem

$$dv(t) = -Av(t)dt + f(t)dw_t, \quad 0 < t < T, \ v(0) \text{ is given} \tag{2.8}$$

and has a unique mild solution, which is represented by the following formula:

$$v(t) = e^{-At}v(0) + \int_0^t e^{-A(t-s)}f(s)dw_s. \tag{2.9}$$

Then from this formula and the multipoint nonlocal-boundary condition

$$v(0) = \sum_{j=1}^{J}\alpha_j v(\lambda_j) + \varphi(w_{\lambda_1},\ldots,w_{\lambda_J}), \tag{2.10}$$

we get

$$v(0) = \sum_{j=1}^{J} \alpha_j e^{-A\lambda_j} v(0) + \sum_{j=1}^{J} \alpha_j \int_0^{\lambda_j} e^{-A(\lambda_j - s)} f(s) dw_s + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}). \tag{2.11}$$

By Lemma 2.2 the operator $I - \sum_{j=1}^{J} \alpha_j e^{-A\lambda_j}$ has a bounded inverse $\Upsilon = (I - \sum_{j=1}^{J} \alpha_j e^{-A\lambda_j})^{-1}$. Then

$$v(0) = \Upsilon \left\{ \sum_{j=1}^{J} \alpha_j \int_0^{\lambda_j} e^{-A(\lambda_j - s)} f(s) dw_s + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}) \right\}. \tag{2.12}$$

Therefore, we have formulas (2.9) and (2.12) for the solution of problem (1.1). □

Now, we will consider the single-step exact difference scheme. On the segment $[0, T]$ we consider the uniform grid space

$$[0, T]_\tau = \{t_k = k\tau,\ k = 0, 1, \dots, N,\ N\tau = T\} \tag{2.13}$$

with step $\tau > 0$. Here N is a fixed positive integer.

Theorem 2.3. *Let $v(t_k)$ be the solution of (1.1) at the grid points $t = t_k$. Then $\{v(t_k)\}_0^N$ is the solution of the multipoint nonlocal-boundary value problem for the following difference equation (see [16]):*

$$v(t_k) - v(t_{k-1}) + \left(I - e^{-\tau A}\right) v(t_{k-1}) = \int_{t_{k-1}}^{t_k} e^{-(t_k - s)A} f(s) dw_s, \quad 1 \le k \le N, \tag{2.14}$$

$$v(0) = \Upsilon \left\{ \sum_{j=1}^{J} \alpha_j \int_0^{\lambda_j} e^{-A(\lambda_j - s)} f(s) dw_s + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}) \right\}. \tag{2.15}$$

Proof. Putting $t = t_k$ and $t = t_{k-1}$ into the formula (2.9), we can write

$$\begin{aligned} v(t_k) &= e^{-t_k A} v(0) + \int_0^{t_k} e^{-(t_k - s)A} f(s) dw_s, \\ v(t_{k-1}) &= e^{-t_{k-1} A} v(0) + \int_0^{t_{k-1}} e^{-(t_{k-1} - s)A} f(s) dw_s. \end{aligned} \tag{2.16}$$

Hence, we obtain the following relation between $v(t_k)$ and $v(t_{k-1})$:

$$v(t_k) = e^{-\tau A} v(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-(t_k - s)A} f(s) dw_s. \tag{2.17}$$

Last relation and equality (2.14) are equivalent. Theorem 2.3 is proved. □

Note that problem (2.14) is called the single-step exact difference scheme for the solution of the problem (1.1).

3. Rothe Difference Scheme

In this section, the 1/2-th order of accuracy Rothe difference scheme is constructed and investigated for the approximate solution of the problem (1.1). The estimate of convergence for the solution of this difference scheme is established. In applications, the convergence estimates for the solution of difference schemes for the numerical solution of two multipoint nonlocal-boundary value problems for stochastic parabolic equations are obtained.

3.1. 1/2-th Order-of-Accuracy Rothe Difference Scheme

Let us give some lemmas we need in the sequel.

Lemma 3.1. *The following estimates hold:*

$$\left\| A^{\alpha} R^{k} \right\|_{H \to H} \leq \frac{1}{(k\tau)^{\alpha}}, \quad 1 \leq k \leq N, \ 0 \leq \alpha \leq 1, \tag{3.1}$$

$$\left\| A^{-\alpha} \left(R^{k} - e^{-k\tau A} \right) \right\|_{H \to H} \leq \frac{2\tau^{\alpha}}{k^{1-\alpha}}, \quad 1 \leq k \leq N, \ 0 \leq \alpha \leq 2, \tag{3.2}$$

where $R = (I + \tau A)^{-1}$.

Lemma 3.2. *Suppose that assumption* (2.2) *holds. Then, the operator*

$$I - \sum_{j=1}^{J} \alpha_j R^{[\lambda_j/\tau]} \tag{3.3}$$

has a bounded inverse

$$\Upsilon_{\tau} = \left(I - \sum_{j=1}^{J} \alpha_j R^{[\lambda_j/\tau]} \right)^{-1}, \tag{3.4}$$

and the following estimate is satisfied:

$$\|\Upsilon_{\tau}\|_{H \to H} \leq C(\delta, \lambda_1). \tag{3.5}$$

Proof. The proof follows from the triangle inequality, assumption (2.2), and estimate

$$\left\| \left(I - \sum_{j=1}^{J} \alpha_j R^{[\lambda_j/\tau]} \right)^{-1} \right\|_{H \to H} \leq \sup_{\delta \leq \mu < \infty} \frac{1}{\left| 1 - \sum_{j=1}^{J} \alpha_k \left(1/(1+\mu\tau)^{[\lambda_k/\tau]} \right) \right|}. \tag{3.6}$$

From (2.14) it is clear that for the approximate solution of the multipoint nonlocal-boundary value problem (1.1) it is necessary to approximate the expressions

$$e^{-\tau A}, \quad \frac{1}{\tau}\int_{t_{k-1}}^{t_k} e^{-(t_k-s)A} f(s)dw, \tag{3.7}$$

and multipoint nonlocal-boundary condition

$$v(0) = \sum_{j=1}^{J}\alpha_j v(\lambda_j) + \varphi(w_{\lambda_1},\dots,w_{\lambda_J}). \tag{3.8}$$

It is possible under stronger assumption than (ii) for $f(t)$. Assume that

$$\max_{t\in[0,T]}\left\|A^{-1/2}f'(t)\right\|_H + \max_{t\in[0,T]}\left\|A^{1/2}f(t)\right\|_H \leq C. \tag{3.9}$$

Replacing the expressions $e^{-\tau A}$, $e^{-(t_k-s)A}$ with $R = (I+\tau A)^{-1}$, the expression $v(\lambda_j)$ with $v([\lambda_j/\tau]\tau)$, and the function $f(s)$ with $f(t_{k-1})$, we get the implicit Rothe difference scheme:

$$\begin{aligned} &u_k - u_{k-1} + \tau A u_k = f(t_{k-1})(w_{t_k} - w_{t_{k-1}}), \quad 1\leq k\leq N,\\ &u_0 = \sum_{j=1}^{J}\alpha_j u_{[\lambda_j/\tau]} + \varphi(w_{\lambda_1},\dots,w_{\lambda_J}). \end{aligned} \tag{3.10}$$

Let us now obtain the formula for the solution of problem (3.10). It is clear that the Rothe difference scheme

$$u_k - u_{k-1} + \tau A u_k = f(t_{k-1})(w_{t_k} - w_{t_{k-1}}), \quad 1\leq k\leq N, \ u_0 \text{ is given}, \tag{3.11}$$

for the solution of the Cauchy problem (2.8) has a unique solution, which is represented by the following formula:

$$u_k = R^k u_0 + \sum_{s=1}^{k} R^{k-s+1} f(t_{s-1})(w_{t_s} - w_{t_{s-1}}), \quad 1\leq k\leq N. \tag{3.12}$$

Then from this formula and the multipoint nonlocal-boundary condition

$$u_0 = \sum_{j=1}^{J}\alpha_j u_{[\lambda_j/\tau]} + \varphi(w_{\lambda_1},\dots,w_{\lambda_J}), \tag{3.13}$$

we get

$$u_0 = \sum_{j=1}^{J}\alpha_j R^{[\lambda_j/\tau]}u_0 + \sum_{j=1}^{J}\alpha_j \sum_{s=1}^{[\lambda_j/\tau]} R^{[\lambda_j/\tau]-s+1} f(t_{s-1})(w_{t_s} - w_{t_{s-1}}) + \varphi(w_{\lambda_1},\dots,w_{\lambda_J}). \tag{3.14}$$

By Lemma 3.2 the operator $I-\sum_{j=1}^{J}\alpha_j R^{[\lambda_j/\tau]}$ has a bounded inverse $\Upsilon_\tau = \left(I - \sum_{j=1}^{J}\alpha_j R^{[\lambda_j/\tau]}\right)^{-1}$. Then

$$u_0 = \Upsilon_\tau \left\{ \sum_{j=1}^{J}\alpha_j \sum_{s=1}^{[\lambda_j/\tau]} R^{[\lambda_j/\tau]-s+1} f(t_{s-1})(w_{t_s} - w_{t_{s-1}}) + \varphi(w_{\lambda_1},\dots,w_{\lambda_J}) \right\}. \tag{3.15}$$

Therefore, we have formulas (3.12) and (3.15) for the solution of problem (3.10). Now, we will study the convergence of difference scheme (3.10). □

Theorem 3.3. *Assume that*

$$E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \le C. \tag{3.16}$$

Then the estimate of convergence

$$\max_{0\le k\le N}\left(E\|v(t_k) - u_k\|_H^2\right)^{1/2} \le C_1(\delta,\lambda_1)\tau^{1/2} \tag{3.17}$$

holds. Here C *and* $C_1(\delta,\lambda_1)$ *do not depend on* τ.

Proof. Using formulas (2.12) and (3.15), we can write

$$\begin{aligned} v(0) - u_0 &= (\Upsilon - \Upsilon_\tau)\varphi(w_{\lambda_1},\dots,w_{\lambda_J}) + \Upsilon\sum_{j=1}^{J}\alpha_j \int_0^{\lambda_j} e^{-A(\lambda_j - s)} f(s)dw_s \\ &\quad - \Upsilon_\tau \sum_{j=1}^{J}\alpha_j \sum_{s=1}^{[\lambda_j/\tau]} R^{[\lambda_j/\tau]-s+1} f(t_{s-1})(w_{t_s} - w_{t_{s-1}}) \\ &= P_{1,J} + P_{2,J} + P_{3,J} + P_{4,J} + P_{5,J} + P_{6,J} + P_{7,J}, \end{aligned} \tag{3.18}$$

where

$$P_{1,J}=(\Upsilon-\Upsilon_\tau)\varphi(w_{\lambda_1},\ldots,w_{\lambda_J}), \tag{3.19}$$

$$P_{2,J}=(\Upsilon-\Upsilon_\tau)\sum_{j=1}^{J}\alpha_j\int_0^{\lambda_j}e^{-A(\lambda_j-s)}f(s)dw_s, \tag{3.20}$$

$$P_{3,J}=\Upsilon_\tau\sum_{j=1}^{J}\alpha_j\int_{[\lambda_j/\tau]\tau}^{\lambda_j}e^{-A(\lambda_j-s)}f(s)dw_s, \tag{3.21}$$

$$P_{4,J}=\Upsilon_\tau\sum_{j=1}^{J}\alpha_j\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}\left(e^{-(\lambda_j-s)A}-e^{-([\lambda_j/\tau]\tau-s)A}\right)f(s)dw_s, \tag{3.22}$$

$$P_{5,J}=\Upsilon_\tau\sum_{j=1}^{J}\alpha_j\sum_{p=1}^{[\lambda_j/\tau]}\left(e^{-([\lambda_j/\tau]\tau-p\tau)A}-R^{[\lambda_j/\tau]-p}\right)\int_{t_{p-1}}^{t_p}e^{-(t_p-s)A}f(s)dw_s, \tag{3.23}$$

$$P_{6,J}=\Upsilon_\tau\sum_{j=1}^{J}\alpha_j\sum_{p=1}^{[\lambda_j/\tau]}R^{[\lambda_j/\tau]-p}\left(\int_{t_{p-1}}^{t_p}e^{-(t_p-s)A}f(s)dw_s-\int_{t_{p-1}}^{t_p}e^{-\tau A}f(t_{p-1})dw_s\right), \tag{3.24}$$

$$P_{7,J}=\Upsilon_\tau\sum_{j=1}^{J}\alpha_j\sum_{p=1}^{[\lambda_j/\tau]}R^{[\lambda_j/\tau]-p}\left(e^{-\tau A}-R\right)f(t_{p-1})\Delta w_{t_p}. \tag{3.25}$$

Let us estimate $P_{k,J}$ for all $k=1,\ldots,7$, separately. We start with $P_{1,J}$. Using formulas (2.4) and (3.4), we obtain

$$\Upsilon-\Upsilon_\tau=\Upsilon\Upsilon_\tau\left(\sum_{j=1}^{J}\alpha_j\left(e^{-A\lambda_j}-R^{[\lambda_j/\tau]}\right)\right), \tag{3.26}$$

and also the expression in the above sum can be written in the following formula:

$$\begin{aligned}
e^{-A\lambda_j}-R^{[\lambda_j/\tau]}&=-\int_0^1 d\left(R^{[\lambda_j/\tau]}(x)e^{-(1-x)A\lambda_j}\right)\\
&=-\int_0^1\left\{\left[\frac{\lambda_j}{\tau}\right]R^{[\lambda_j/\tau]-1}(x)\frac{-\tau A}{(1+x\tau A)^2}+A\lambda_jR^{[\lambda_j/\tau]}(x)\right\}e^{-(1-x)A\lambda_j}dx\\
&=-\int_0^1 R^{[\lambda_j/\tau]+1}(x)e^{-(1-x)A\lambda_j}\left\{-\left[\frac{\lambda_j}{\tau}\right]\tau A+(1+x\tau A)A\lambda_j\right\}dx\\
&=-\int_0^1 R^{[\lambda_j/\tau]+1}(x)e^{-(1-x)A\lambda_j}\left\{\left(\lambda_j-\left[\frac{\lambda_j}{\tau}\right]\tau\right)A+x\tau A^2\lambda_j\right\}dx.
\end{aligned} \tag{3.27}$$

Here $R(x) = (I + \tau x A)^{-1}$. Using formulas (3.26), (3.27), and (3.19), we can write

$$P_{1,J} = \Upsilon\Upsilon_{\tau}\sum_{j=1}^{J}\alpha_j \times \left(-\int_0^1 R^{[\lambda_j/\tau]+1}(x)e^{-(1-x)A\lambda_j}\left\{\left(\lambda_j - \left[\frac{\lambda_j}{\tau}\right]\tau\right)A + x\tau A^2\lambda_j\right\}dx\right)\varphi(w_{\lambda_1},\dots,w_{\lambda_J}). \tag{3.28}$$

Let us estimate expected value of $P_{1,J}$. Since

$$\left(\lambda_j - \left[\frac{\lambda_j}{\tau}\right]\tau\right) \le \tau, \tag{3.29}$$

we have that

$$\left(E\|P_{1,J}\|_H^2\right)^{1/2} \le \|\Upsilon\|_{H\to H}\|\Upsilon_{\tau}\|_{H\to H} \times \left(E\left\|\sum_{j=1}^{J}\alpha_j\tau\left(\int_0^1 A^{1/2}R^{[\lambda_j/\tau]+1}(x)e^{-(1-x)A\lambda_j}A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})dx\right)\right\|_H^2 + \lambda_j\tau E\left\|\int_0^1 A^{3/2}R^{[\lambda_j/\tau]+1}(x)e^{-(1-x)A\lambda_j}A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})dx\right\|_H^2\right)^{1/2}. \tag{3.30}$$

In the same manner by using the triangle inequality and estimates (3.2) and (3.1), we get

$$\left(E\|P_{1,J}\|_H^2\right)^{1/2} \le C_1(\delta,\lambda_1)\left(\sum_{j=1}^{J}|\alpha_j|\int_0^1 \tau\frac{dx}{\sqrt{[\lambda_j/\tau+1]\tau x}} \times \left(E\left\|A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2\right)^{1/2} + \sum_{j=1}^{J}|\alpha_j|\lambda_j\int_0^{1/2}\tau\frac{dx}{(1-x)^{3/2}\lambda_j^{3/2}}\right.$$

$$
\begin{aligned}
&\times \left(E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \right)^{1/2} \\
&+ \sum_{j=1}^{J}|\alpha_j|\lambda_j\int_{1/2}^{1}\tau\frac{dx}{([\lambda_j/\tau]\tau x)^{3/2}} \\
&\times \left(E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \right)^{1/2} \Bigg) \\
&\leq C_1(\delta,\lambda_1)\left(E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \right)^{1/2}\left(\sum_{j=1}^{J}C_2\tau\frac{1}{\sqrt{\lambda_j}}+\sum_{j=1}^{J}C_3\tau\frac{1}{\sqrt{\lambda_j}}\right) \\
&\leq C_4(\delta,\lambda_1)\sum_{j=1}^{J}C_4\tau^{1/2}\frac{1}{\lambda_j^{1/2}}\left(E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \right)^{1/2} \\
&\leq C_5(\delta,\lambda_1)\tau^{1/2}\left(E\left\| A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2 \right)^{1/2}.
\end{aligned}
\tag{3.31}
$$

Now, let us estimate $P_{2,J}$. Using formula (3.20), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{2,J}\|_H^2\right)^{1/2} &\leq C_5(\delta,\lambda_1)\tau^{1/2}\left(\sum_{j=1}^{J}E\left\|\int_0^{\lambda_j}e^{-A(\lambda_j-s)}A^{1/2}f(s)dw_s\right\|_H^2\right)^{1/2} \\
&\leq C_5(\delta,\lambda_1)\tau^{1/2}\left(\sum_{j=1}^{J}\int_0^{\lambda_j}\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2} \\
&\leq C_6(\delta,\lambda_1)\tau^{1/2}\max_{0\leq s\leq T}\left\|A^{1/2}f(s)\right\|_H.
\end{aligned}
\tag{3.32}
$$

Let us estimate $P_{3,J}$. Using formula (3.21), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{3,J}\|_H^2\right)^{1/2} \leq \|\Upsilon_\tau\|_{H\to H}\sum_{j=1}^{J}|\alpha_j|\Bigg(&\left\|e^{-A(\lambda_j-s)}\right\|_{H\to H}^2\left\|A^{-1/2}\right\|_{H\to H}^2 \\
&\times\int_{[\lambda_j/\tau]\tau}^{\lambda_j}\left\|A^{1/2}f(s)\right\|_H^2 ds\Bigg)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_6(\delta,\lambda_1)\sum_{j=1}^{J}|\alpha_j|\left(\int_{[\lambda_j/\tau]\tau}^{\lambda_j}\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2},\\
&\leq C_6(\delta,\lambda_1)\sum_{j=1}^{J}|\alpha_j|\left(\lambda_j-\left[\frac{\lambda_j}{\tau}\right]\tau\right)^{1/2}\max_{0\leq s\leq T}\left(\left\|A^{1/2}f(s)\right\|_H^2\right)^{1/2}\\
&\leq C_7(\delta,\lambda_1)\tau^{1/2}\max_{0\leq s\leq T}\|A^{1/2}f(s)\|_H.
\end{aligned}
\tag{3.33}
$$

Next, let us estimate $P_{4,J}$. Using formula (3.22), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{4,J}\|_H^2\right)^{1/2} &\leq \|\Upsilon_\tau\|_{H\to H}\sum_{j=1}^{J}|\alpha_j|\\
&\quad\times\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}\left\|A^{-1/2}\left(e^{-(\lambda_j-s)A}-e^{-([\lambda_j/\tau]\tau-s)A}\right)\right\|_{H\to H}^2\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2}\\
&\leq C_1(\delta,\lambda_1)\left(\sum_{p=1}^{[\lambda_j/\tau]}\tau\int_{t_{p-1}}^{t_p}\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2}\\
&\leq C_1(\delta,\lambda_1)\tau^{1/2}\max_{0\leq s\leq T}\left\|A^{1/2}f(s)\right\|_H.
\end{aligned}
\tag{3.34}
$$

Next, let us estimate $P_{5,J}$. Using formula (3.23), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{5,J}\|_H^2\right)^{1/2} &\leq \|\Upsilon_\tau\|_{H\to H}\left(\sum_{j=1}^{J}|\alpha_j|\sum_{p=1}^{[\lambda_j/\tau]}\left\|A^{-1/2}\left(e^{-([\lambda_j/\tau]\tau-p\tau)A}-R^{[\lambda_j/\tau]-p}\right)\right\|_{H\to H}^2\right.\\
&\quad\left.\times E\int_{t_{p-1}}^{t_p}\left\|e^{-(t_p-s)A}\right\|_{H\to H}^2\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2}\\
&\leq C_2(\delta,\lambda_1)\tau^{1/2}\left(\int_0^T\left\|A^{1/2}f(s)\right\|_H^2 ds\right)^{1/2}\\
&\leq C_2(\delta,\lambda_1)\tau^{1/2}\max_{0\leq s\leq T}\left\|A^{1/2}f(s)\right\|_H.
\end{aligned}
\tag{3.35}
$$

Next, let us estimate $P_{6,J}$. Using formula (3.24), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$\begin{aligned}
\left(E\left\|P_{6,J}\right\|_H^2\right)^{1/2} &\leq \|\Upsilon_\tau\|_{H\to H}\sum_{j=1}^{J}\left|\alpha_j\right| \\
&\quad\times\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}\left\|R^{[\lambda_j/\tau]-p}\right\|_{H\to H}^2 E\left\|e^{-(t_p-s)A}f(s)-e^{-\tau A}f(t_{p-1})\right\|_H^2 ds\right)^{1/2} \\
&\leq C_1(\delta,\lambda_1)\sum_{j=1}^{J}\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}E\left\|e^{-(t_p-s)A}f(s)-e^{-(t_p-t_{p-1})A}f(t_{p-1})\right\|_H^2 ds\right)^{1/2} \\
&= C_1(\delta,\lambda_1)\sum_{j=1}^{J}\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}E\left\|\left(e^{-(t_p-s)A}-e^{-(t_p-t_{p-1})A}\right)f(s)\right.\right. \\
&\qquad\left.\left.+\ e^{-(t_p-t_{p-1})A}\left(f(s)-f(t_{p-1})\right)\right\|_H^2 ds\right)^{1/2} \\
&\leq C_2(\delta,\lambda_1)\sum_{j=1}^{J}\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}\left\|A^{-1/2}\left(e^{-(t_p-s)A}-e^{-(t_p-t_{p-1})A}\right)A^{1/2}f(s)\right\|_H^2\right. \\
&\qquad\left.+\left\|e^{-(t_p-t_{p-1})A}\left(f(s)-f(t_{p-1})\right)\right\|_H^2 ds\right)^{1/2} \\
&\leq C_3(\delta,\lambda_1)\sum_{j=1}^{J}\left(\sum_{p=1}^{[\lambda_j/\tau]}\int_{t_{p-1}}^{t_p}\left(\tau\left\|A^{1/2}f(s)\right\|_H^2+\left\|f(s)-f(t_{p-1})\right\|_H^2\right)ds\right)^{1/2} \\
&\leq C_4(\delta,\lambda_1)\sum_{j=1}^{J}\left(\sum_{p=1}^{[\lambda_j/\tau]}E\int_{t_{p-1}}^{t_p}\left(\tau\left\|A^{1/2}f(s)\right\|_H^2+\left\|f'(s)\tau\right\|_H^2\right)ds\right)^{1/2} \\
&\leq C_4(\delta,\lambda_1)\tau^{1/2}\left(\int_0^T\left\|A^{1/2}f(s)\right\|_H^2 ds+\int_0^T\left\|f'(s)\right\|_H^2 ds\right)^{1/2} \\
&\leq C_5(\delta,\lambda_1)\tau^{1/2}\left(\max_{0\leq s\leq T}\left\|A^{1/2}f(s)\right\|_H+\max_{0\leq s\leq T}\left\|f'(s)\right\|_H\right).
\end{aligned} \tag{3.36}$$

Finally, let us estimate $P_{7,J}$. Using formula (3.25), the triangle inequality, and estimates (3.5), (3.2), and (3.1), we get

$$\left(E\|P_{5,J}\|_H^2\right)^{1/2} \le \|\Upsilon_\tau\|_{H\to H}\sum_{j=1}^{J}|\alpha_j| \times \left(\sum_{p=1}^{[\lambda_j/\tau]}\left\|R^{[\lambda_j/\tau]-p}\right\|_{H\to H}^2\left\|A^{-1/2}\left(e^{-\tau A}-R\right)\right\|_{H\to H}^2\left\|A^{1/2}f(t_{p-1})\right\|_H^2 \times E\left\|\Delta w_{t_p}\right\|_H^2\right)^{1/2} \le C(\delta,\lambda_1)\left(\sum_{p=1}^{[\lambda_j/\tau]}\tau\left\|A^{1/2}f(t_{p-1})\right\|_H^2 E\left\|\Delta w_{t_p}\right\|_H^2\right)^{1/2}. \tag{3.37}$$

Since Δw_{t_p} is a Wiener process and

$$E\left\|\Delta w_{t_p}\right\|_H^2 \le \Delta t_p = \tau, \tag{3.38}$$

we have that

$$\left(E\|P_{5,J}\|_H^2\right)^{1/2} \le C(\delta,\lambda_1)\tau^{1/2}\left(\sum_{p=1}^{[\lambda_j/\tau]}\left\|A^{1/2}f(t_{p-1})\right\|_H^2\tau\right)^{1/2} \le C_1(\delta,\lambda_1)\tau^{1/2}\max_{0\le s\le T}\left\|A^{1/2}f(s)\right\|_H. \tag{3.39}$$

Applying estimates for $P_{k,J}$, $k=1,\ldots,7$, we get the estimate:

$$\left(E\|v(t_0)-u_0\|_H^2\right)^{1/2} \le C_4(\delta,\lambda_1)\tau^{1/2}. \tag{3.40}$$

To prove the Theorem 3.3 it suffices to establish the following estimate:

$$\max_{1\le k\le N}\left(E\|v(t_k)-u_k\|_H^2\right)^{1/2} \le C_2(\delta,\lambda_1)\tau^{1/2}. \tag{3.41}$$

Using formulas (2.9) and (3.12), we can write

$$
\begin{aligned}
v(t_k) - u_k = e^{-k\tau A}v(0) + \sum_{s=1}^{k} e^{-(k-s)\tau A}\int_{t_{s-1}}^{t_s} e^{-A(t_s-p)}f(p)dw_p \\
-R^k u_0 - \sum_{s=1}^{k} R^{k-s+1} f(t_s)(w_{t_s} - w_{t_{s-1}}) = P_{1,k} + P_{2,k} + P_{3,k} + P_{4,k} + P_{5,k},
\end{aligned} \tag{3.42}
$$

where

$$
\begin{gathered}
P_{1,k} = \left(e^{-k\tau A} - R^k\right)\Upsilon\left\{\sum_{j=1}^{J}\alpha_j\int_0^{\lambda_j} e^{-A(\lambda_j-s)}f(s)dw_s + \varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\}, \\
P_{2,k} = R^k(v(0) - u_0), \\
P_{3,k} = \sum_{s=1}^{k-1}\left[e^{-(k-s)\tau A} - R^{k-s}\right]\int_{t_{s-1}}^{t_s} e^{-A(t_s-p)}f(p)dw_p, \\
P_{4,k} = \sum_{s=1}^{k} R^{k-s}\int_{t_{s-1}}^{t_s} e^{-A(t_s-p)}f(p)dw_p - e^{-\tau A}f(t_{s-1})(w_{t_s} - w_{t_{s-1}}), \\
P_{5,k} = \sum_{s=1}^{k} R^{k-s}\left[e^{-\tau A} - R\right]f(t_{s-1})(w_{t_s} - w_{t_{s-1}}).
\end{gathered} \tag{3.43}
$$

Let us estimate $P_{m,k}$ for all $m = 1,\dots,5$, separately. We start with $P_{1,k}$. Using the triangle inequality and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{1,k}\|_H^2\right)^{1/2} \le \Bigg(& \left\|\left(e^{-k\tau A} - R^k\right)A^{-1/2}\right\|_{H\to H}^2 \\
& \times E\left\|A^{1/2}\Upsilon\left\{\sum_{j=1}^{J}\alpha_j\int_0^{\lambda_j} e^{-A(\lambda_j-s)}f(s)dw_s + \varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\}\right\|_H^2\Bigg)^{1/2} \\
\le C_1(\delta,\lambda_1)\tau^{1/2}\Bigg(& E\Bigg\|\Upsilon\sum_{j=1}^{J}\alpha_j\int_0^{\lambda_j} e^{-A(\lambda_j-s)}A^{1/2}f(s)dw_s \\
& + A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\Bigg\|_H^2\Bigg)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_2(\delta,\lambda_1)\tau^{1/2}\|\Upsilon\|_{H\to H}\left(\left(\sum_{j=1}^{J}|\alpha_j|\right)^2\int_0^{\lambda_j}\left\|e^{-A(\lambda_j-s)}\right\|_{H\to H}^2\left\|A^{1/2}f(s)\right\|_H^2 ds\right.\\
&\qquad\left.+\left\|A^{1/2}\varphi(w_{\lambda_1},\dots,w_{\lambda_J})\right\|_H^2\right)^{1/2}\\
&\le C_3(\delta,\lambda_1)\tau^{1/2}\left(\int_0^T\left\|A^{1/2}f(s)\right\|_H^2 ds+E\left\|A^{1/2}\varphi\right\|_H^2\right)^{1/2}\\
&\le C_4(\delta,\lambda_1)\tau^{1/2}\left(\max_{0\le s\le T}\left\|A^{1/2}f(s)\right\|_H+\left(E\left\|A^{1/2}\varphi\right\|_H^2\right)^{1/2}\right).
\end{aligned}
\tag{3.44}
$$

Now, we estimate $P_{2,k}$. Using estimate (3.1), we get

$$
\left(E\|P_{2,k}\|_H^2\right)^{1/2}\le\left(\left\|R^k\right\|_{H\to H}^2E\|v(0)-u_0\|_H^2\right)^{1/2}\le\left(E\|v(0)-u_0\|_H^2\right)^{1/2}. \tag{3.45}
$$

Applying the estimate (3.40), we obtain

$$
\left(E\|P_{2,k}\|_H^2\right)^{1/2}\le C(\delta,\lambda_1)\tau^{1/2}. \tag{3.46}
$$

Now, we estimate $P_{3,k}$. Using the triangle inequality and estimates (3.5), (3.2), and (3.1), we get

$$
\begin{aligned}
\left(E\|P_{3,k}\|_H^2\right)^{1/2}&\le C(\delta,\lambda_1)\left(\sum_{s=1}^{k-1}\left\|A^{-1/2}\left[e^{-(k-s)\tau A}-R^{k-s}\right]\right\|_{H\to H}^2\right.\\
&\qquad\left.\times\left\|e^{-A(t_s-p)}\right\|_{H\to H}^2\int_{t_{s-1}}^{t_s}\left\|A^{1/2}f(p)\right\|_H^2 dp\right)^{1/2}\\
&\le C(\delta,\lambda_1)\left(\sum_{s=1}^{k-1}\tau\int_{t_{s-1}}^{t_s}\left\|A^{1/2}f(p)\right\|_H^2 dp\right)^{1/2}\\
&\le C(\delta,\lambda_1)\tau^{1/2}\left(\int_0^T\left\|A^{1/2}f(p)\right\|_H^2 dp\right)^{1/2}\\
&\le C(\delta,\lambda_1)\tau^{1/2}\max_{0\le s\le T}\left\|A^{1/2}f(s)\right\|_H.
\end{aligned}
\tag{3.47}
$$

Now, we estimate $P_{4,k}$. We denote that

$$b_j = \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{s} \left(e^{-\tau A} A^{-1/2} f'(z) + A^{1/2} e^{-(t_j - z)A} f(s)\right) dz\, dw_s,$$
$$b_j^* = \begin{cases} b_j, & 1 \le j \le k-1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.48}$$

Then

$$\begin{aligned} P_{4,k} &= \sum_{s=1}^{k} R^{k-s} \int_{t_{s-1}}^{t_s} \left(e^{-A(t_s - p)} f(p) - e^{-\tau A} f(t_{s-1})\right) dw_p \\ &= \sum_{s=1}^{k} R^{k-s} \int_{t_{s-1}}^{t_s} \left(\left(e^{-A(t_s - p)} - e^{-\tau A}\right) f(p) + e^{-\tau A}\left(f(p) - f(t_{s-1})\right)\right) dw_p \\ &= \sum_{s=1}^{k} A^{1/2} R^{k-s} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{s} \left(e^{-\tau A} A^{-1/2} f'(z) + A^{1/2} e^{-(t_j - z)A} f(s)\right) dz\, dw_s \\ &= \sum_{i=1}^{N} R^i A^{1/2} b_{k-i}^*, \end{aligned}$$
$$\left(E\|P_{4,k}\|_H^2\right)^{1/2} = \left(E\left\|\sum_{i=1}^{N} R^i A^{1/2} b_{k-i}^*\right\|_H^2\right)^{1/2}. \tag{3.49}$$

Using the triangle inequality and estimates (3.5), (3.2), and (3.1), we get

$$\begin{aligned} \left(E\|P_{4,k}\|^2\right)^{1/2} &\le \left(\sum_{i=1}^{N} E\left\|R^i A^{1/2} b_{k-i}^*\right\|_H^2\right)^{1/2} \\ &\le \left(\sum_{i=1}^{N} \|A^{1/2} R^i\|_{H \to H} E\|b_{k-i}^*\|_H^2\right)^{1/2}. \end{aligned} \tag{3.50}$$

Since

$$\begin{aligned} \left(E\left\|b_j^*\right\|_H^2\right)^{1/2} &= \left(E\left\|\int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{s} \left(e^{-\tau A} A^{-1/2} f'(z) + A^{1/2} e^{-(t_j - z)A} f(s)\right) dz\, dw_s\right\|_H^2\right)^{1/2} \\ &\le \int_{t_{j-1}}^{t_j} \left(\int_{t_{j-1}}^{s} \left\|\left(e^{-\tau A} A^{-1/2} f'(z) + A^{1/2} e^{-(\mathrm{t}_j - zA)} f(s)\right)\right\|_H^2 dz\right)^{1/2} ds \end{aligned}$$

$$\leq \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{s} \left\| \left(e^{-\tau A} A^{-1/2} f'(z) + A^{1/2} e^{-(t_j - z)A} f(s) \right) \right\|_H^2 dz\, ds$$
$$\leq C(\delta, \lambda_1) \tau^{3/2} \left(\max_{0 \leq s \leq T} \| A^{1/2} f(s) \|_H + \max_{0 \leq s \leq T} \| A^{-1/2} f'(s) \|_H \right), \tag{3.51}$$

we have that

$$\begin{aligned}
\left(E \| P_{4,k} \|_H^2 \right)^{1/2} &\leq \sum_{i=1}^{N} \frac{C_1}{\sqrt{i\tau}} \left(E \| b_{k-i}^* \|_H^2 \right)^{1/2} \\
&\leq \sum_{i=1}^{N} \frac{C_1}{\sqrt{i\tau}} C(\delta, \lambda_1) \tau^{3/2} \left(\max_{0 \leq s \leq T} \left\| A^{1/2} f(s) \right\|_H + \max_{0 \leq s \leq T} \left\| A^{-1/2} f'(s) \right\|_H \right) \\
&\leq C_1(\delta, \lambda_1) \tau^{1/2} \left(\max_{0 \leq s \leq T} \| A^{1/2} f(s) \|_H + \max_{0 \leq s \leq T} \| A^{-1/2} f'(s) \|_H \right).
\end{aligned} \tag{3.52}$$

Finally, we estimate $P_{5,k}$. We denote that

$$\begin{aligned}
q_j &= A^{1/2} f(t_{j-1}) \Delta w_{t_j}, \\
a_j^* &= \begin{cases} q_j, & 1 \leq j \leq k-1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.53}$$

Therefore,

$$P_{5,k} = \sum_{i=1}^{N} A^{1/2} R^i A^{-1} \left(e^{-\tau A} - R \right) q_{k-i}^*. \tag{3.54}$$

Using the triangle inequality and estimates (3.5), (3.2), and (3.1), we get

$$\begin{aligned}
\left(E \| P_{5,k} \|_H^2 \right)^{1/2} &\leq \sum_{i=1}^{N} \left\| A^{1/2} R^i \right\|_{H \to H} \left\| A^{-1} \left(e^{-\tau A} - R \right) \right\|_{H \to H} \left(E \| q_{k-i}^* \|_H^2 \right)^{1/2} \\
&\leq \sum_{i=1}^{N} \frac{2\tau}{\sqrt{i\tau}} \left(E \| q_{k-i}^* \|_H^2 \right)^{1/2} \leq C \max_{1 \leq j \leq N} \left(E \| q_j \|_H^2 \right)^{1/2}.
\end{aligned} \tag{3.55}$$

Since

$$\left(E \| q_j \|_H^2 \right)^{1/2} \leq \left(E \left\| A^{1/2} f(t_{j-1}) \Delta w_{t_j} \right\|_H^2 \right)^{1/2} \leq C_1(\delta, \lambda_1) \tau^{1/2} \max_{0 \leq s \leq T} \left\| A^{1/2} f(s) \right\|_H, \tag{3.56}$$

we have that

$$\left(E\|P_{5,k}\|_H^2\right)^{1/2} \leq C_2(\delta,\lambda_1)\tau^{1/2}\max_{0\leq s\leq T}\left\|A^{1/2}f(s)\right\|_H. \tag{3.57}$$

Combining estimates $P_{1,k}$, $P_{2,k}$, $P_{3,k}$, $P_{4,k}$, and $P_{5,k}$, we obtain (3.41). Theorem 3.3 is proved. □

3.2. Applications

Now, we consider applications of Theorem 3.3. First, let us consider the nonlocal-boundary value problem for one-dimensional stochastic parabolic equation:

$$\begin{gathered} du(t,x) - (a(x)u_x)_x dt + \delta u(t,x)dt = f(t,x)dw_t, \quad 0 < t < T,\ 0 < x < 1, \\ u(0,x) = \sum_{j=1}^{J}\alpha_j u(\lambda_j,x) + \varphi(w_{\lambda_1},\ldots,w_{\lambda_J},x), \quad 0 \leq x \leq 1, \\ \sum_{j=1}^{J}|\alpha_j| \leq 1, \quad 0 < \lambda_1 < \cdots < \lambda_J \leq T, \quad w_t = \sqrt{t}\xi, \quad \xi \in N(0,1), \quad 0 \leq t \leq T, \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad 0 \leq t \leq T, \end{gathered} \tag{3.58}$$

where $\delta > 0$, $a(x) \geq a > 0$ $(x \in (0,1))$, $\varphi(w_{\lambda_1},\ldots,w_{\lambda_J},x)$ $(x \in [0,1])$ and $f(t,x)$ $(t,x \in [0,1])$ are smooth functions with respect to x.

The discretization of problem (3.58) is carried out in two steps. In the first step, we define the grid space

$$[0,1]_h = \{x = x_n : x_n = nh,\ 0 \leq n \leq M,\ Mh = 1\}. \tag{3.59}$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,1]_h}|\varphi(x)|^2 h\right)^{1/2}. \tag{3.60}$$

To the differential operator A generated by problem (3.58), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \left\{-\left(a(x)\varphi_{\bar{x}}\right)_{x,n} + \delta\varphi_n\right\}_1^{M-1} \tag{3.61}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal-boundary value problem:

$$\begin{gathered} du^h(t,x) + A_h^x u^h(t,x)dt = f^h(t,x)dw_t, \quad 0 < t < T, \ x \in [0,1]_h, \\ u^h(0,x) = \sum_{j=1}^{J} \alpha_j u^h(\lambda_j, x) + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}, x), \quad x \in [0,1]_h. \end{gathered} \tag{3.62}$$

In the second step, we replace (3.62) with the difference scheme (3.10):

$$\begin{gathered} u_k^h(x) - u_{k-1}^h(x) + \tau A_h^x u_k^h(x) = f_{k-1}^h(x)(w_{t_k} - w_{t_{k-1}}), \quad 1 \le k \le N, \\ f_{k-1}^h(x) = f^h(t_{k-1}, x), \quad t_k = k\tau, \ 1 \le k \le N, \ x \in [0,1]_h, \\ u_0^h(x) = \sum_{j=1}^{J} \alpha_j u_{[\lambda_j/\tau]}^h(x) + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}, x), \quad x \in [0,1]_h. \end{gathered} \tag{3.63}$$

Theorem 3.4. *Let τ and h be sufficiently small positive numbers. Then, the solutions of difference scheme* (3.63) *satisfy the following convergence estimate:*

$$\max_{0 \le k \le N} \left(E \left\| u(t_k) - u_k^h \right\|_{L_{2h}}^2 \right)^{1/2} \le C(\delta, \lambda_1)\left(\tau^{1/2} + h\right), \tag{3.64}$$

where $C(\delta, \lambda_1)$ do not depend on τ and h. Here, one puts $u(t_k) = \{u(t_k, x_n)\}_0^M$ as the grid function of exact solution of problem (3.58) *at the grid points $t = t_k$, $0 \le k \le N$ and $x = x_n$, $0 \le n \le M$.*

Proof. Let us introduce the Banach space $C([0,1], H)$ of abstract mesh functions $u_k = u_k^h(x)$ defined on $[0,1]_\tau$ with values in $H = L_{2h}$. Then, difference scheme (3.63) can be reduced to the abstract difference scheme:

$$\begin{gathered} (u_k - u_{k-1}) + \tau A u_k = f_k, \\ f_k = f(t_{k-1}), \quad t_k = k\tau, \ 1 \le k \le N, \\ u_0 = \sum_{j=1}^{J} \alpha_j u_{[\lambda_j/\tau]} + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}), \end{gathered} \tag{3.65}$$

in a Hilbert space L_{2h} with the operator $A = A_h^x$ by formula (3.62). It is clear that $A = A^*$ and $A \ge \delta I$ in $H = L_{2h}$. Hence, A_h^x is a self-adjoint positive definite operator in L_{2h}. Therefore, Theorem 3.3 applies to this case, and Theorem 3.4 is proved. □

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : 0 < x_i < 1, i = 1, \ldots, n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, the nonlocal boundary value problem for the multidimensional parabolic equation

$$\begin{gathered} du(t,x) - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} dt = f(t,x) dw_t, \quad 0 < t < T, \\ x = (x_1, \ldots, x_n) \in \Omega, \\ u(0,x) = \sum_{j=1}^{J} \alpha_j u(\lambda_j, x) + \varphi(w_{\lambda_1}, \ldots, w_{\lambda_J}, x), \quad x \in \overline{\Omega}, \\ \sum_{j=1}^{J} |\alpha_j| \leq 1, \quad 0 < \lambda_1 < \cdots < \lambda_J \leq T, \quad w_t = \sqrt{t}\xi, \quad \xi \in N(0,1), \ 0 \leq t \leq T, \\ u(t,x) = 0, \quad x \in S, \ 0 \leq t \leq T \end{gathered} \tag{3.66}$$

with the Dirichlet condition is considered. Here $a_r(x)$, $(x \in \Omega)$, $\varphi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions with respect to x and $a_r(x) \geq a > 0$.

The discretization of problem (3.66) is carried out in two steps. In the first step, define the grid space $\widetilde{\Omega}_h = \{x = x_m = (h_1 m_1, \ldots, h_n m_n);\ m = (m_1, \ldots, m_n),\ 0 \leq m_r \leq N_r,\ h_r N_r = 1, r = 1, \ldots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega,\ S_h = \widetilde{\Omega}_h \cap S$.

Let L_{2h} denote the Hilbert space

$$L_{2h} = L_2\left(\widetilde{\Omega}_h\right) = \left\{ \varphi^h(x) : \left(\sum_{x \in \widetilde{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n \right)^{1/2} < \infty \right\}. \tag{3.67}$$

The differential operator A in (3.66) is replaced with

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(a_r(x) u_{\overline{x}_r}^h \right)_{x_r, j_r}, \tag{3.68}$$

where the difference operator A_h^x is defined on those grid functions $u^h(x) = 0$, for all $x \in S_h$. It is well known that A_h^x is a self-adjoint positive definite operator in L_{2h}.

Using (3.66) and (3.68), we get

$$\begin{gathered} du^h(t,x) + A_h^x u^h(t,x) dt = f^h(t,x) dw_t, \quad 0 < t < T,\ x \in \Omega_h, \\ u^h(0,x) = \sum_{j=1}^{J} \alpha_j u^h(\lambda_j, x) + \varphi(w_{\lambda_1}, \ldots, w_{\lambda_J}, x), \quad x \in \widetilde{\Omega}_h. \end{gathered} \tag{3.69}$$

In the second step, we replace (3.69) with the difference scheme (3.10):

$$u_k^h(x) - u_{k-1}^h(x) + \tau A_h^x u_k^h(x) = f_{k-1}^h(x)(w_{t_k} - w_{t_{k-1}}), \quad 1 \le k \le N,$$
$$f_{k-1}^h(x) = f^h(t_{k-1}, x), \quad t_k = k\tau,\ 1 \le k \le N,\ x \in \Omega_h, \tag{3.70}$$
$$u_0^h(x) = \sum_{j=1}^{J} \alpha_j u_{[\lambda_j/\tau]}^h(x) + \varphi(w_{\lambda_1}, \dots, w_{\lambda_J}, x), \quad x \in \widetilde{\Omega}_h.$$

Theorem 3.5. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, the solution of difference scheme* (3.70) *satisfies the following convergence estimate:*

$$\max_{0 \le k \le N} \left(E \left\| u(t_k) - u_k^h \right\|_{L_{2h}}^2 \right)^{1/2} \le C(\delta, \lambda_1) \left(\tau^{1/2} + |h|^2 \right), \tag{3.71}$$

where $C(\delta, \lambda_1)$ do not depend on τ and $|h|$. Here, one puts $u(t_k) = u(t_k, x)|_{x \in \widetilde{\Omega}_h}$ as the grid function of exact solution of problem (3.66) *at the grid points $t = t_k$, $0 \le k \le N$ and $x \in \widetilde{\Omega}_h$.*

The proof of Theorem 3.5 is based on the abstract Theorem 3.3 and the symmetry properties of the difference operator A_h^x defined by formula (3.68).

4. Numerical Application

Now, we consider the numerical application of nonlocal boundary value problem:

$$dv - v_{xx} dt = e^{-t} \sin x\, dw_t, \quad 0 < t < 1,\ 0 < x < \pi,$$
$$v(0, x, 0) = v(1, x, w_1) + \sin x - e^{-1} \sin x w_1 - e^{-1} \sin x, \quad 0 \le x \le \pi, \tag{4.1}$$
$$v(t, 0, w_t) = v(t, \pi, w_t) = 0, \quad 0 \le t \le 1,$$
$$w_t = \sqrt{t}\xi, \quad \xi \in N(0, 1), \quad 0 \le t \le 1,$$

for one-dimensional stochastic parabolic equation. For numerical solution of (4.1), we consider the difference scheme 1/2-th order of accuracy in t and second order of accuracy in x for the approximate solution of the nonlocal boundary value problem (4.1):

$$u_n^k - u_n^{k-1} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \tau = f(t_k, x_n) \tau \left(\sqrt{k\tau} - \sqrt{(k-1)\tau} \right) \xi,$$
$$f(t_k, x_n) = e^{-t_k} \sin x_n, \quad t_k = k\tau,\ x_n = nh,\ 1 \le k \le N,\ 1 \le n \le M - 1, \tag{4.2}$$
$$u_0^k = u_M^k = 0, \quad 0 \le k \le N,$$
$$u_n^0 = u_n^N + \sin x_n - e^{-1} \sin x_n w_1 - e^{-1} \sin x_n, \quad 0 \le n \le M.$$

We will write it in the matrix form

$$\begin{gathered} Au_{n+1} + Bu_n + Cu_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\ U_0 = \vec{0}, \qquad U_M = \vec{0}. \end{gathered} \tag{4.3}$$

Here

$$\begin{gathered}
\varphi_n = \begin{bmatrix} 0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 0 \end{bmatrix}_{(N+1)\times 1}, \\
\varphi_n^k = f(t_k, x_n)\tau\left(\sqrt{k\tau} - \sqrt{(k-1)\tau}\right)\xi, \quad 1 \le k \le N,\ 1 \le n \le M-1, \\
A = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 \\ 0 & a & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & a & 0 \\ 0 & 0 & \cdot & 0 & a \end{bmatrix}_{(N+1)\times(N+1)}, \qquad B = \begin{bmatrix} 1 & 0 & \cdot & 0 & -1 \\ b & c & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & c & 0 \\ 0 & 0 & \cdot & b & c \end{bmatrix}_{(N+1)\times(N+1)}, \\
a = \left(-\frac{\tau}{h^2}\right), \quad b = (-1), \quad c = \left(1 + \frac{2\tau}{h^2}\right), \quad C = A, \\
D = \begin{bmatrix} 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 & 0 \\ 0 & 0 & \cdot & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \qquad u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times(1)}, \\
s = n-1, n, n+1.
\end{gathered} \tag{4.4}$$

For the solution of the last matrix equation, we use the modified Gauss elimination method (see [17]). We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1}u_{n+1} + \beta_{n+1}, \quad n = M-1, \dots, 1,\ u_M = \vec{0}, \tag{4.5}$$

where α_j, are $(N+1) \times (N+1)$ square matrices and β_j, are $(N+1) \times 1$ column matrices and $(j = 1, \dots, M-1)$ defined by formulas

$$\begin{gathered} \alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, \dots, M-1. \end{gathered} \tag{4.6}$$

Table 1: Error analysis.

N/M	10/30	20/60	40/120
Difference scheme (4.2)	0.0929	0.0401	0.0187

Here

$$\alpha_1 = \begin{bmatrix} 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & . & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \quad \beta_1 = \vec{0}. \tag{4.7}$$

The error between the exact solution and the solutions derived by difference schemes is shown in Table 1. To obtain the results we simulated the 1,000 sample paths of Brownian motion for each level of discretization. The estimate (3.71) in Theorem 3.5 suggests that if we double the number of nodes, then the error should be decreased by a factor of $1/\sqrt{2}$. The theoretical statement for the solution of this difference scheme is supported by the results of the numerical experiment. In fact, we double N and M; the error is even less than half of the previous error.

Acknowledgment

The authors wish to thank Professor A. Lukashev (Istanbul, Turkey), for his valuable suggestions which helped us to improve the present paper.

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