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Research Article

The $H^1(R)$ Space Global Weak Solutions to the Weakly Dissipative Camassa-Holm Equation

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The existence of global weak solutions to the Cauchy problem for a generalized Camassa-Holm equation with a dissipative term is investigated in the space $C([0,\infty)\times \mathrm{R})\cap L^\infty([0,\infty); H^1(\mathrm{R}))$ provided that its initial value $u_0(x)$ belongs to the space $H^1(R)$. A one-sided super bound estimate and a space-time higher-norm estimate on the first-order derivatives of the solution with respect to the space variable are derived.

1. Introduction

In [1], the author investigated the following weakly dissipative Camassa-Holm model

$$u_t - u_{xxt} + 3uu_x + \lambda(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where $\lambda \geq 0$. When $\lambda = 0$, (1.1) becomes the classical Camassa-Holm equation [2]. The authors in [1] obtained the local well-posedness of the solution for the model by using the Kato theorem. A necessary and sufficient condition of the blow-up of the solution and some criteria guaranteeing the blow-up of the solution are derived. The blow-up rate of the solution is discussed. It is also shown in [1] that the equation has global strong solutions, and these strong solutions decay to zero as time goes to infinite provided the potentials associated to their initial data are of one sign. However, the existence of global weak solutions in the space $H^1(R)$ is not discussed in paper [1]. This will constitute the objective of this work.

More relevant for the present paper, here we state several works on the global weak solution for the Camassa-Holm and Degasperis-Procesi equations. The existence and uniqueness results for global weak solutions to the Camassa-Holm equation have been proved by Constantin and Escher [3], Constantin and Molinet [4], and Danchin [5, 6]. Xin and Zhang [7] proved that the global existence of the weak solution for the Camassa-Holm equation in the energy space $H^1(R)$ without any sign conditions on the initial value, and the uniqueness of this weak solution is obtained under certain conditions on the solution [8]. Coclite et al. [9] investigated the global weak solutions for a generalized hyperelastic-rod wave equation or a generalized Camassa-Holm equation. The existence of a strongly continuous semigroup of global weak solutions for the generalized hyperelastic-rod equation with any initial value in the space $H^1(R)$ was established in [9]. Under the sign condition imposed on the initial value, Yin et al. [10] proved the existence and uniqueness results of global weak solution for a nonlinear shallow water equation, which includes the famous Camassa-Holm and Degasperis-Procesi equations as special cases. For other dynamic properties about various generalized Camassa-Holm models and other partial differential equations, the reader is referred to [11–16].

The aim of this work is to study the existence of global weak solutions for (1.1) in the space $C([0,\infty)\times R)\bigcap L^\infty([0,\infty);H^1(R))$ under the assumption $u_0(x)\in H^1(R)$. The limits of viscous approximations for the equation are used to establish the existence of the global weak solution. Here, we should mention that up to now, there have been no global existence results for weak solutions to the generalized Camassa-Holm equation (1.1).

The rest of this paper is as follows. The main result is given in Section 2. In Section 3, we state the viscous problem and give a corresponding well-posedness result. An upper bound, higher integrability estimate, and basic compactness properties for the viscous approximations are also established in Section 3. Strong compactness of the derivative of the viscous approximations is obtained in Section 4, where the main result for the existence of (1.1) is proved.

2. Main Result

Consider the Cauchy problem for (1.1)

$$\begin{gathered} u_t - u_{txx} + \left(\frac{3u^2}{2}\right)_x + \lambda(u-u_{xx}) = 2u_xu_{xx} + uu_{xxx}, \\ u(0,x) = u_0(x), \end{gathered} \tag{2.1}$$

which is equivalent to

$$\begin{gathered} u_t + uu_x + \frac{\partial P}{\partial x} = 0, \\ \frac{\partial P(t,x)}{\partial x} = \Lambda^{-2}\partial_x\left[u^2+\frac{1}{2}u_x^2\right] + \lambda u, \\ u(0,x) = u_0(x), \end{gathered} \tag{2.2}$$

where operator $\Lambda^2 = 1-(\partial^2/\partial x^2)$. For a fixed $1\le p_0<\infty$, it has

$$\Lambda^{-2}g(x) = \frac{1}{2}\int_R e^{-|x-y|}g(y)dy \quad \text{for } g(x)\in L^{p_0}(R),\ 1<p_0<\infty. \tag{2.3}$$

In fact, problem (2.1) satisfies the following conservation law:

$$\int_R \left(u^2 + u_x^2\right) dx + 2\lambda \int_0^t \int_R \left(u^2 + u_x^2\right) dx dt = \int_R \left(u_0^2 + u_{0x}^2\right) dx. \tag{2.4}$$

Now, we introduce the definition of a weak solution to the Cauchy problem (2.1) or (2.2).

Definition 2.1. A continuous function $u : [0,\infty) \times R \to R$ is said to be a global weak solution to the Cauchy problem (2.2) if

(i) $u \in C([0,\infty) \times R) \bigcap L^\infty([0,\infty); H^1(R))$;

(ii) $\|u(t,.)\|_{H^1(R)} \le \|u_0\|_{H^1(R)}$;

(iii) $u = u(t,x)$ satisfies (2.2) in the sense of distributions and takes on the initial value pointwise.

The main result of this paper is stated as follows.

Theorem 2.2. *Assume $u_0(x) \in H^1(R)$. Then, the Cauchy problem* (2.1) *or* (2.2) *has a global weak solution $u(t,x)$ in the sense of Definition 2.1. Furthermore, this weak solution satisfies the following properties.*

(a) *There exists a positive constant c_0 depending on $\|u_0\|_{H^1(R)}$ and λ such that the following one-sided L^∞ norm estimate on the first order spatial derivative holds*

$$\frac{\partial u(t,x)}{\partial x} \le \frac{4}{t} + c_0, \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{2.5}$$

(b) *Let $0 < \gamma < 1$, $T > 0$, and $a, b \in R, a < b$. Then, there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b and λ such that the space higher integrability estimate holds*

$$\int_0^t \int_a^b \left| \frac{\partial u(t,x)}{\partial x} \right|^{2+\gamma} dx \le c_1. \tag{2.6}$$

3. Viscous Approximations

Defining

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \tag{3.1}$$

and setting the mollifier $\phi_\varepsilon(x) = \varepsilon^{-(1/4)}\phi(\varepsilon^{-(1/4)}x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon,0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon,0} \in C^\infty$ for any $u_0 \in H^s, s > 0$ (see [10]). In fact, suitably choosing the mollifier, we have

$$\|u_{\varepsilon,0}\|_{H^1(R)} \le \|u_0\|_{H^1(R)}, \qquad u_{\varepsilon,0} \longrightarrow u_0 \text{ in } H^1(R). \tag{3.2}$$

The existence of a weak solution to the Cauchy problem (2.2) will be established by proving compactness of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problem:

$$\frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial P_\varepsilon}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2},$$
$$\frac{\partial P_\varepsilon(t,x)}{\partial x} = \Lambda^{-2}\partial_x\left[u_\varepsilon^2 + \frac{1}{2}\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right] + \lambda u_\varepsilon, \tag{3.3}$$
$$u_\varepsilon(0,x) = u_{\varepsilon,0}(x).$$

The beginning point of our analysis is the following well-posedness result for problem (3.3).

Lemma 3.1. *Provided that $u_0 \in H^1(R)$, then for any $\sigma \geq 3$, there exists a unique solution $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ to the Cauchy problem* (3.3). *Moreover, for any $t > 0$, it holds that*

$$\int_R \left(u_\varepsilon^2 + \left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx + 2\lambda \int_0^t \int_R \left(u_\varepsilon^2 + \left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx\,dt$$
$$+ 2\varepsilon \int_0^t \int_R \left(\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2}\right)^2\right)(s,x)\,dx\,ds = \|u_{\varepsilon,0}\|^2_{H^1(R)}, \tag{3.4}$$

or

$$\|u_\varepsilon(t,.)\|^2_{H^1(R)} + 2\varepsilon \int_0^t \left\|\frac{\partial u_\varepsilon}{\partial x}(s,.)\right\|^2_{H^1(R)} ds$$
$$+ 2\lambda \int_0^t \|u_\varepsilon(t,.)\|^2_{H^1(R)}\,dt = \|u_{\varepsilon,0}\|^2_{H^1(R)}. \tag{3.5}$$

Proof. For any $\sigma \geq 3$ and $u_0 \in H^1(R)$, we have $u_{\varepsilon,0} \in C([0,\infty); H^\sigma(R))$. From Theorem 2.1 in [9] or Theorem 2.3 in [11], we conclude that problem (3.3) has a unique solution $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ for an arbitrary $\sigma > 3$.

We know that the first equation in system (3.3) is equivalent to the form

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\partial^3 u_\varepsilon}{\partial t x^2} + \frac{3}{2}\frac{\partial u_\varepsilon^2}{\partial x} + \lambda\left(u_\varepsilon - \frac{\partial^2 u_\varepsilon}{\partial x^2}\right)$$
$$= 2\frac{\partial u_\varepsilon}{\partial x}\frac{\partial^2 u_\varepsilon}{\partial x^2} + u_\varepsilon \frac{\partial^3 u_\varepsilon}{\partial x^3} \tag{3.6}$$
$$+ \varepsilon\left(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^4 u_\varepsilon}{\partial x^4}\right),$$

from which we derive that

$$\frac{1}{2}\frac{d}{dt}\int_R\left(u_\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx+\lambda\int_R\left(u_\varepsilon^2+\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx \\ +\varepsilon\int_R\left(\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2+\left(\frac{\partial^2 u_\varepsilon}{\partial x^2}\right)^2\right)dx=0, \tag{3.7}$$

which completes the proof. □

From Lemma 3.1, we have

$$\|u_\varepsilon\|_{L^\infty(R)}\le\|u_\varepsilon\|_{H^1(R)}\le\|u_{\varepsilon,0}\|_{H^1(R)}\le\|u_0\|_{H^1(R)}. \tag{3.8}$$

Differentiating the first equation of problem (3.3) with respect to the variable x and writing $\partial u_\varepsilon/\partial x=q_\varepsilon$, we obtain

$$\frac{\partial q_\varepsilon}{\partial t}+u_\varepsilon\frac{\partial q_\varepsilon}{\partial x}-\varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2}+\frac{1}{2}q_\varepsilon^2+\lambda u_x=u_\varepsilon^2-\Lambda^{-2}\left[u_\varepsilon^2+\frac{1}{2}q_\varepsilon^2\right]=Q_\varepsilon(t,x). \tag{3.9}$$

Lemma 3.2. *Let $0<\gamma<1$, $T>0$, and $a,b\in R, a<b$. Then, there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b and λ, but independent of ε, such that the space higher integrability estimate holds*

$$\int_0^T\int_a^b\left|\frac{\partial u_\varepsilon(t,x)}{\partial x}\right|^{2+\gamma}dx\le c_1, \tag{3.10}$$

where $u_\varepsilon=u_\varepsilon(t,x)$ is the unique solution of problem (3.3).

Proof. The proof is a variant of the proof presented by Xin and Zhang [7] (also see Coclite et al. [9]). Let $\chi\in C^\infty(R)$ be a cut-off function such that $0<\chi<1$ and

$$\phi(x)=\begin{cases}1, & \text{if } x\in[a,b],\\ 0, & \text{if } x\in(-\infty,a-1]\bigcup[b+1,\infty).\end{cases} \tag{3.11}$$

Consider the map $\theta(\xi):=\xi(1+|\xi|)^\gamma$, $\xi\in R, 0<\gamma<1$, and observe that

$$\begin{aligned}\theta'(\xi)&=(1+(1+\gamma)|\xi|)(1+|\xi|)^{\gamma-1},\\ \theta''(\xi)&=\gamma\operatorname{sign}(\xi)(1+|\xi|)^{\gamma-2}(2+(1+\gamma)|\xi|)\\ &=\gamma(1+\gamma)\operatorname{sign}(\xi)(1+|\xi|)^{\gamma-1}+(1-\gamma)\gamma\operatorname{sign}(\xi)(1+|\xi|)^{\gamma-2}.\end{aligned} \tag{3.12}$$

We have

$$|\theta(\xi)| \leq |\xi| + |\xi|^{1+\gamma}, \qquad \left|\theta'(\xi)\right| \leq 1 + (1+\gamma)|\xi|, \qquad \left|\theta''(\xi)\right| \leq 2\gamma, \tag{3.13}$$

$$\begin{aligned} \xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) &= \frac{1-\gamma}{2}\xi^2(1+|\xi|)^{\gamma} + \frac{\gamma}{2}\xi^2(1+|\xi|)^{\gamma-1} \\ &\geq \frac{1-\gamma}{2}\xi^2(1+|\xi|)^{\gamma}. \end{aligned} \tag{3.14}$$

Differentiating the first equation of problem (3.3) with respect to the variable x and writing $u = u_\varepsilon$ and $\partial u_\varepsilon/\partial x = q_\varepsilon = q$ for simplicity, we obtain

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 + \lambda u_x = u^2 - \Lambda^{-2}\left[u^2 + \frac{1}{2}u_x^2\right] = Q(t,x). \tag{3.15}$$

Multiplying (3.15) by $\chi\theta'(q)$, using the chain rule and integrating over $\Pi_T := [0,T] \times R$, we have

$$\begin{aligned} &\int_{\Pi_T} \chi(x)q\theta(q)\,dt\,dx - \frac{1}{2}\int_{\Pi_T} q^2\chi(x)\theta'(q)\,dt\,dx \\ &\quad = \int_R \chi(x)\big(\theta(q(t,x)) - \theta(q(0,x))\big)dx - \int_{\Pi_T} u\chi'(x)\theta(q)\,dt\,dx \\ &\qquad + \varepsilon\int_{\Pi_T} \frac{\partial q}{\partial x}\chi'(x)\theta'(q)\,dt\,dx + \varepsilon\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2\chi(x)\theta''(q)\,dt\,dx \\ &\qquad + \lambda\int_{\Pi_T} q\chi(x)\theta'(q)\,dt\,dx - \int_{\Pi_T} Q(t,x)\chi(x)\theta'(q)\,dt\,dx. \end{aligned} \tag{3.16}$$

From (3.14), we get

$$\begin{aligned} &\int_{\Pi_T} \chi(x)q\theta(q)\,dt\,dx - \frac{1}{2}\int_{\Pi_T} q^2\chi(x)\theta'(q)\,dt\,dx \\ &\quad = \int_{\Pi_T} \chi(x)\left(q\theta(q) - \frac{1}{2}q^2\theta'(q)\right)dt\,dx \\ &\quad \geq \frac{(1-\gamma)}{2}\int_{\Pi_T} \chi(x)q^2(1+|q|)^{\gamma}\,dt\,dx. \end{aligned} \tag{3.17}$$

Using the Hölder inequality, (3.8) and (3.13), yields

$$\begin{aligned}\left|\int_R \chi(x)\theta(q)dx\right| &\le \int_R \chi(x)\left(|q|^{1+\gamma}+|q|\right)dx\\&\le \|\chi\|_{L^{2/(1-\gamma)}(R)}\|q\|_{L^2(R)}^{1+\gamma}+\|\chi\|_{L^2(R)}\|q\|_{L^2(R)}\\&\le (b-a+2)^{(1-\gamma)/2}\|u_0\|_{H^1(R)}^{1+\gamma}+(b-a+2)^{1/2}\|u_0\|_{H^1(R)},\end{aligned}\tag{3.18}$$

$$\begin{aligned}\left|\int_{\Pi_T} u\chi'(x)\theta(q)dtdx\right| &\le \int_{\Pi_T}|u|\left|\chi'(x)\right|\left(|q|^{1+\gamma}+|q|\right)dtdx\\&\le \|u_0\|_{H^1(R)}\int_0^T\left(\|\chi'\|_{L^{2/(1-\gamma)}(R)}\|q\|_{L^2(R)}^{1+\gamma}+\|\chi'\|_{L^2(R)}\|q\|_{L^2(R)}\right)dt\\&\le T\|u_0\|_{H^1(R)}\left(\|\chi'\|_{L^{2/(1-\gamma)}(R)}\|u_0\|_{H^1(R)}^{1+\gamma}+\|\chi'\|_{L^2(R)}\|u_0\|_{H^1(R)}\right).\end{aligned}\tag{3.19}$$

Integration by parts gives rise to

$$\int_{\Pi_T}\frac{\partial q}{\partial x}\chi'(x)\theta'(q)dtdx=-\int_{\Pi_T}\theta(q)\chi''(x)dtdx.\tag{3.20}$$

From (3.13), (3.20), and the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\Pi_T}\frac{\partial q}{\partial x}\chi'(x)\theta'(q)dtdx\right| &\le \varepsilon\int_{\Pi_T}|\theta(q)||\chi''(x)|dtdx\\&\le \varepsilon\int_{\Pi_T}|\chi''(x)|\left(|q|^{1+\gamma}+|q|\right)dtdx\\&\le \varepsilon T\left(\|\chi''\|_{L^{2/(1-\gamma)}(R)}\|u_0\|_{H^1(R)}^{1+\gamma}+\|\chi''\|_{L^2(R)}\|u_0\|_{H^1(R)}\right).\end{aligned}\tag{3.21}$$

Using (3.13) and Lemma 3.1, we have

$$\varepsilon\left|\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2\chi(x)\theta''(q)dtdx\right|\le 2\gamma\varepsilon\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2dtdx\le\gamma\|u_0\|_{H^1(R)}^2.\tag{3.22}$$

From (3.13), it has

$$\begin{aligned}&\left|\lambda\int_{\Pi_T}q\chi(x)\theta'(q)dtdx\right|\\&\quad\le|\lambda|\int_{\Pi_T}|\chi(x)|\left((1+\gamma)q^2+|q|\right)dtdx\\&\quad\le|\lambda|\left((1+\gamma)\|\chi\|_{L^\infty}\|u_0\|_{H^1(R)}^2+\|\chi(x)\|_{L^2(R)}\|u_0\|_{H^1(R)}\right).\end{aligned}\tag{3.23}$$

Applying (3.8), the Hölder inequality, Lemma 3.1 and $\int_R e^{-|x-y|}dy = 2$, we have

$$\begin{aligned}|Q(t,x)| &= \left|u^2 - \Lambda^{-2}\left[u^2 + \frac{1}{2}q^2\right]\right| \\ &\leq \|u\|^2_{L^\infty(R)} + \frac{1}{2}\|u\|^2_{H^1(R)} \\ &\leq \frac{3}{2}\|u_0\|^2_{H^1(R)}.\end{aligned} \tag{3.24}$$

From (3.24), we obtain

$$\begin{aligned}\left|\int_{\Pi_T} Q(t,x)\chi(x)\theta'(q)dtdx\right| &\leq c\int_{\Pi_T}|\chi(x)|((1+\gamma)|q|+1)dtdx \\ &\leq \frac{3}{2}\|u_0\|^2_{H^1(R)}T\left((1+\gamma)\|\chi(x)\|_{L^2(R)}\|u_0\|_{H^1(R)} + \int_R|\chi(x)|dx\right).\end{aligned} \tag{3.25}$$

The inequalities (3.15)–(3.23) and (3.25) derive the desired result (3.10). □

Lemma 3.3. *There exists a positive constant C depending only on* $\|u_0\|_{H^1(R)}$ *and* λ *such that*

$$\|Q_\varepsilon(t,.)\|_{L^\infty(R)} \leq C, \tag{3.26}$$

$$\|Q_\varepsilon(t,.)\|_{L^1(R)} \leq C, \tag{3.27}$$

$$\|Q_\varepsilon(t,.)\|_{L^2(R)} \leq C, \tag{3.28}$$

$$\left\|\frac{\partial P_\varepsilon(t,.)}{\partial x}\right\|_{L^\infty(R)} \leq C, \tag{3.29}$$

$$\left\|\frac{\partial P_\varepsilon(t,.)}{\partial x}\right\|_{L^2(R)} \leq C, \tag{3.30}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ *is the unique solution of system* (3.3).

Proof. For simplicity, setting $u(t,x) = u_\varepsilon(t,x)$, we have

$$\begin{aligned}Q_\varepsilon(t,x) &= u^2 - \Lambda^{-2}\left[u^2 + \frac{1}{2}u_x^2\right], \\ \frac{\partial P_\varepsilon(t,x)}{\partial x} &= \Lambda^{-2}\partial_x\left[u^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] + \lambda u.\end{aligned} \tag{3.31}$$

The inequality (3.26) is proved in Lemma 3.2 (see (3.24)). Now, we prove (3.27). Using

$$\begin{aligned}
&\int_R \left|\Lambda^{-2}\left[u^2+\frac{1}{2}u_x^2\right]\right|dx \\
&\quad= \frac{1}{2}\int_R\left|\int_R e^{-|x-y|}\left(u^2+\frac{1}{2}u_x^2\right)dy\right|dx \\
&\quad\le \|u\|^2_{H^1(R)} \\
&\quad\le \|u_0\|^2_{H^1(R)},
\end{aligned} \tag{3.32}$$

and (3.8) result in (3.27).

Applying the Tonelli theorem, (3.26) and (3.27), we get

$$\|Q_\varepsilon(t,.)\|^2_{L^2(R)} \le \|Q_\varepsilon(t,.)\|_{L^\infty}\|Q_\varepsilon(t,.)\|_{L^1(R)} \le C. \tag{3.33}$$

Since

$$\begin{aligned}
&\left|\Lambda^{-2}\partial_x\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right]\right| \\
&\quad= \left|\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\partial_y\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\partial_y\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy\right| \\
&\quad= \left|-\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy + \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy\right| \\
&\quad\le \int_R e^{-|x-y|}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy \\
&\quad\le \|u\|^2_{H^1(R)} \\
&\quad\le \|u_0\|^2_{H^1(R)}, \\
&\int_R\left|\Lambda^{-2}\partial_x\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]\right|dx \\
&\quad= \int_R\left|-\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy + \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dy\right|dx \\
&\quad\le \int_R\int_R e^{-|x-y|}\left[u^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right]dydx \\
&\quad\le 2\|u\|^2_{H^1(R)} \\
&\quad\le 2\|u_0\|^2_{H^1(R)},
\end{aligned} \tag{3.34}$$

we have

$$
\begin{aligned}
&\int_R \left| \Lambda^{-2}\partial_x \left[u^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right] \right|^2 dx \\
&\quad \le \|u_0\|_{H^1(R)}^2 \int_R \left| \Lambda^{-2}\partial_x \left[u^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right] \right| dx \\
&\quad \le 2\|u_0\|_{H^1(R)}^4,
\end{aligned}
\tag{3.35}
$$

which, together with Lemma 3.1, we get (3.29) and (3.30). The proof of Lemma 3.3 is completed. □

Lemma 3.4. *Assume $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of problem* (3.3)*. There exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ and λ such that the following one-sided L^∞ norm estimate on the first order spatial derivative holds*

$$
\frac{\partial u_\varepsilon(t,x)}{\partial x} \le \frac{4}{t} + C, \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{3.36}
$$

Proof. From (3.9) and Lemma 3.3, we know that there exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ and λ such that $\|Q_\varepsilon(t,x)\|_{L^\infty(R)} \le C$. Therefore,

$$
\frac{\partial q_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial q_\varepsilon}{\partial x} + \frac{1}{2}q_\varepsilon^2 + \lambda q_\varepsilon = Q_\varepsilon(t,x) \le C. \tag{3.37}
$$

Let $f = f(t)$ be the solution of

$$
\frac{df}{dt} + \frac{1}{2}f^2 + \lambda f = C, \quad t > 0, \qquad f(0) = \left\| \frac{\partial u_{\varepsilon,0}}{\partial x} \right\|_{L^\infty(R)}. \tag{3.38}
$$

Letting $\sup_{x\in R} q_\varepsilon(t,x) = f(t)$, due to the comparison principle for parabolic equations, we get

$$
q_\varepsilon(t,x) \le f(t). \tag{3.39}
$$

Using $-\lambda f \le \rho^2 f^2 + (1/4\rho^2)\lambda^2$, we derive that

$$
\begin{aligned}
\frac{df}{dt} &= C - \frac{1}{2}f^2 - \lambda f \le C - \frac{1}{2}f^2 + \rho^2 f^2 + \frac{1}{4\rho^2}\lambda^2 \\
&\le C - \frac{1}{4}f^2 + \lambda^2,
\end{aligned}
\tag{3.40}
$$

where $(1/4\rho^2)\lambda^2 = \lambda^2$ and $\rho = 1/2$. Setting $M_0 = C + \lambda^2$, we obtain

$$
\frac{df}{dt} + \frac{1}{4}f^2 \le M_0. \tag{3.41}
$$

Letting $F(t) = (4/t) + 2\sqrt{M_0}$, we have $(dF(t)/dt) + (1/4)F^2(t) - M_0 = (4\sqrt{M_0}/t) > 0$. Due to the comparison principle for ordinary differential equations, we get $f(t) \leq F(t)$ for all $t > 0$. Therefore, by this and (3.39), the estimate (3.36) is proved. □

Lemma 3.5. *There exists a sequence* $\{\varepsilon_j\}_{j\in N}$ *tending to zero and a function* $u \in L^\infty([0,\infty); H^1(R)) \bigcap H^1([0,T] \times R)$, *such that*

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in } H^1([0,T] \times R), \quad \text{for each } T \geq 0, \tag{3.42}$$

$$u_{\varepsilon_j} \longrightarrow u \quad \text{in } L^\infty_{\text{loc}}([0,\infty) \times R), \tag{3.43}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ *is the unique solution of* (3.3).

Proof. For fixed $T > 0$, using Lemmas 3.1 and 3.3, and

$$\frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial P_\varepsilon}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}, \tag{3.44}$$

we obtain

$$\left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2([0,T]\times R)} \leq c\left(1 + \sqrt{\varepsilon}\|u_0\|_{H^1(R)}\right), \tag{3.45}$$

where c depends on T. Hence, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,\infty); H^1(R)) \bigcap H^1([0,T] \times R)$ and (3.42) follows.

Observe that, for each $0 \leq s,t \leq T$,

$$\begin{aligned} \|u_\varepsilon(t,.) - u_\varepsilon(s,.)\|^2_{L^2(R)} &= \int_R \left(\frac{\partial u_\varepsilon}{\partial t}(\tau,x)d\tau \right)^2 dx \\ &\leq \sqrt{|t-s|} \int_R \int_0^T \left(\frac{\partial u_\varepsilon}{\partial t}(\tau,x) \right)^2 d\tau dx. \end{aligned} \tag{3.46}$$

Moreover, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,T]; H^1(R))$ and $H^1(R) \subset L^\infty_{\text{loc}} \subset L^2_{\text{loc}}(R)$. Then, (3.43) is valid. □

Lemma 3.6. *There exists a sequence* $\{\varepsilon_j\}_{j\in N}$ *tending to zero and a function* $Q \in L^\infty([0,\infty); W^{1,\infty}(R))$ *such that for each* $1 < p < \infty$

$$Q_{\varepsilon_j} \longrightarrow Q \quad \text{strongly in } L^p_{\text{loc}}([0,\infty) \times R). \tag{3.47}$$

Proof. Using Lemma 3.3, we have the existence of pointwise convergence subsequence Q_{ε_j} which is uniformly bounded in $L^\infty([0,\infty) \times R)$. This implies (3.47). □

Throughout this paper we use overbars to denote weak limits (the space in which these weak limits are taken is $L^r_{\text{loc}}([0,\infty) \times R)$ with $1 < r < (3/2)$).

Lemma 3.7. *There exists a sequence $\{\varepsilon_j\}_{j\in N}$ tending to zero and two functions $q \in L^p_{\mathrm{loc}}([0,\infty)\times R)$, $\overline{q^2} \in L^r_{\mathrm{loc}}([0,\infty)\times R)$ such that*

$$q_{\varepsilon_j} \rightharpoonup q \quad \text{in } L^p_{\mathrm{loc}}([0,\infty)\times R), \qquad q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \text{in } L^\infty_{\mathrm{loc}}\Big([0,\infty); L^2(R)\Big), \tag{3.48}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in } L^r_{\mathrm{loc}}([0,\infty)\times R), \tag{3.49}$$

for each $1 < p < 3$ and $1 < r < (3/2)$. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x), \quad \text{for almost every } (t,x) \in [0,\infty)\times R \tag{3.50}$$

$$\frac{\partial u}{\partial x} = q \quad \text{in the sense of distributions on } [0,\infty)\times R. \tag{3.51}$$

Proof. (3.48) and (3.49) are direct consequence of Lemmas 3.1 and 3.2. Inequality (3.50) is valid because of the weak convergence in (3.49). Finally, (3.51) is a consequence of the definition of q_ε, Lemma 3.5 and (3.48). □

In the following, for notational convenience, we replace the sequence $\{u_{\varepsilon_j}\}_{j\in N}$, $\{q_{\varepsilon_j}\}_{j\in N}$ and $\{Q_{\varepsilon_j}\}_{j\in N}$ by $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$ and $\{Q_\varepsilon\}_{\varepsilon>0}$, separately.

Using (3.48), we conclude that for any convex function $\eta \in C^1(R)$ with η' bounded, Lipschitz continuous on R and any $1 < p < 3$ we get

$$\begin{aligned} \eta(q_\varepsilon) &\rightharpoonup \overline{\eta(q)} \quad \text{in } L^p_{\mathrm{loc}}([0,\infty)\times R), \\ \eta(q_\varepsilon) &\overset{\star}{\rightharpoonup} \overline{\eta(q)} \quad \text{in } L^\infty_{\mathrm{loc}}\Big([0,\infty); L^2(R)\Big). \end{aligned} \tag{3.52}$$

Multiplying (3.9) by $\eta'(q_\varepsilon)$ yields

$$\begin{aligned} &\frac{\partial}{\partial t}\eta(q_\varepsilon) + \frac{\partial}{\partial x}(u_\varepsilon \eta(q_\varepsilon)) - \varepsilon\frac{\partial^2}{\partial x^2}\eta(q_\varepsilon) + \varepsilon\eta''(q_\varepsilon)\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 \\ &\quad = q_\varepsilon\eta(q_\varepsilon) - \frac{1}{2}\eta'(q_\varepsilon)q_\varepsilon^2 - \lambda q_\varepsilon\eta'(q_\varepsilon) + Q_\varepsilon(t,x)\eta'(q_\varepsilon). \end{aligned} \tag{3.53}$$

Lemma 3.8. *For any convex $\eta \in C^1(R)$ with η' bounded, Lipschitz continuous on R, it holds that*

$$\overline{\frac{\partial\eta(q)}{\partial t}} + \frac{\partial}{\partial x}\Big(u\overline{\eta(q)}\Big) \le \overline{q\eta(q)} - \frac{1}{2}\overline{\eta'(q)q^2} - \lambda\overline{q\eta'(q)} + Q(t,x)\overline{\eta'(q)} \tag{3.54}$$

in the sense of distributions on $[0,\infty)\times R$. Here, $\overline{q\eta(q)}$ and $\overline{\eta'(q)q^2}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$ and $q_\varepsilon^2\eta'(q_\varepsilon)$ in $L^r_{\mathrm{loc}}([0,\infty)\times R)$, $1 < r < (3/2)$, respectively.

Proof. In (3.53), by the convexity of η, (3.8), Lemmas 3.5, 3.6, and 3.7, sending $\varepsilon \to 0$ gives rise to the desire result. □

Remark 3.9. From (3.48) and (3.49), we know that

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \qquad q^2 = (q_+)^2 + (q_-)^2, \qquad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2}, \tag{3.55}$$

almost everywhere in $[0,\infty) \times R$, where $\xi_+ := \xi_{\chi[0,+\infty)}(\xi), \xi_- := \xi_{\chi(-\infty,0]}(\xi)$ for $\xi \in R$. From Lemma 3.4 and (3.48), we have

$$q_\varepsilon(t,x), \qquad q(t,x) \le \frac{4}{t} + C, \quad \text{for } t > 0, x \in R, \tag{3.56}$$

where C is a constant depending only on $\|u_0\|_{H^1(R)}$ and λ.

Lemma 3.10. *In the sense of distributions on* $[0,\infty) \times R$, *it holds that*

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(uq) = \frac{1}{2}\overline{q^2} - \lambda q + Q(t,x). \tag{3.57}$$

Proof. Using (3.9), Lemmas 3.5 and 3.6, (3.48), (3.49), and (3.51), the conclusion (3.57) holds by sending $\varepsilon \to 0$ in (3.9). □

The next lemma contains a generalized formulation of (3.57).

Lemma 3.11. *For any* $\eta \in C^1(R)$ *with* $\eta \in L^\infty(R)$, *it has*

$$\begin{aligned} \frac{\partial \eta(q)}{\partial t} + \frac{\partial}{\partial x}(u\eta(q)) &= q\eta(q) + \left(\frac{1}{2}\overline{q^2} - q^2\right)\eta'(q) \\ &\quad - \lambda q\eta'(q) + Q(t,x)\eta'(q), \end{aligned} \tag{3.58}$$

in the sense of distributions on $[0,\infty) \times R$.

Proof. Let $\{\omega_\delta\}_\delta$ be a family of mollifiers defined on R. Denote $q_\delta(t,x) := (q(t,.) \star \omega_\delta)(x)$. The $\star$ is the convolution with respect to the x variable. Multiplying (3.57) by $\eta'(q_\delta)$, it has

$$\begin{aligned} \frac{\partial \eta(q_\delta)}{\partial t} = \eta'(q_\delta)\frac{\partial q_\delta}{\partial t} &= \eta'(q_\delta)\left[\frac{1}{2}\overline{q^2} \star \omega_\delta - \lambda q_\delta + Q(t,x) \star \omega_\delta \right. \\ &\qquad \left. - q^2 \star \omega_\delta - u\frac{\partial q}{\partial x} \star \omega_\delta\right], \\ \frac{\partial}{\partial x}(u\eta(q_\delta)) &= q\eta(q_\delta) + u\eta'(q_\delta)\left(\frac{\partial q_\delta}{\partial x}\right). \end{aligned} \tag{3.59}$$

Using the boundedness of η, η' and letting $\delta \to 0$ in the above two equations, we obtain (3.58). □

4. Strong Convergence of q_ε and Existence for (1.1)

Following the ideas in [7] or [9], in this section, we improve the weak convergence of q_ε in (3.48) to strong convergence, and then we have an existence result for problem (3.3). Generally speaking, we will derive a "transport equation" for the evolution of the defect measure $(\overline{q^2} - q^2) \geq 0$. Namely, we will prove that the measure is zero initially then it will continue to be zero at all later times $t > 0$.

Lemma 4.1. *Assume $u_0 \in H^1(R)$. It holds that*

$$\lim_{t\to 0}\int_R q^2(t,x)dx = \lim_{t\to 0}\int_R \overline{q^2}(t,x)dx = \int_R \left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{4.1}$$

Lemma 4.2. *If $u_0 \in H^1(R)$, for each $M > 0$, it has*

$$\lim_{t\to 0}\int_R \left(\overline{\eta_M^{\pm}(q)}(t,x) - \eta_M^{\pm}(q(t,x))\right)dx = 0, \tag{4.2}$$

where

$$\eta_M(\xi) := \begin{cases} \dfrac{1}{2}\xi^2, & \text{if } |\xi| \leq M, \\ M|\xi| - \dfrac{1}{2}M^2, & \text{if } |\xi| > M, \end{cases} \tag{4.3}$$

and $\eta_M^+(\xi) := \eta_M(\xi)\chi_{[0,+\infty)}(\xi)$, $\eta_M^-(\xi) := \eta_M(\xi)\chi_{(-\infty,0]}(\xi)$, $\xi \in R$.

The proof of Lemmas 4.1 and 4.2 is similar to those of Lemmas 6.1 and 6.2 in [9]. Here, we omit their proofs.

Lemma 4.3 (Coclite et al. [9]). *Let $M > 0$. Then, for each $\xi \in R$, it holds that*

$$\begin{aligned}
\eta_M(\xi) &= \frac{1}{2}\xi^2 - \frac{1}{2}(M - |\xi|)^2 \chi_{(-\infty,-M)\bigcap(M,\infty)}(\xi), \\
\eta'_M(\xi)\xi &= \xi + (M - |\xi|)\,\mathrm{sign}(\xi)\chi_{(-\infty,-M)\bigcap(M,\infty)}(\xi), \\
\eta_M^+(\xi) &= \frac{1}{2}(\xi_+)^2 - \frac{1}{2}(M-\xi)^2\chi_{(M,\infty)}(\xi), \\
(\eta_M^+)'(\xi) &= \xi_+ + (M-\xi)\chi_{(M,\infty)}(\xi), \\
\eta_M^-(\xi) &= \frac{1}{2}(\xi_-)^2 - \frac{1}{2}(M+\xi)^2\chi_{(-\infty,-M)}(\xi), \\
(\eta_M^-)'(\xi) &= \xi_- - (M+\xi)\chi_{(-\infty,-M)}(\xi).
\end{aligned} \tag{4.4}$$

Lemma 4.4. *Assume $u_0 \in H^1(R)$. Then, for almost all $t > 0$*

$$\frac{1}{2}\int_R \left(\overline{(q_+)^2} - q_+^2\right)(t,x)dx \leq \int_0^t \int_R Q(s,x)\left[\overline{q_+}(s,x) - q_+(s,x)\right]dsdx. \tag{4.5}$$

Proof. For an arbitrary $T > 0$ $(0 < t < T)$, we let M be sufficiently large (see Lemma 3.4). Using (3.54) minus (3.58), and the entropy η_M^+ (see Lemma 4.2) results in

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\overline{\eta_M^+(q)} - \eta_M^+(q)\right) + \frac{\partial}{\partial x}\left(u\left[\overline{\eta_M^+(q)} - \eta_M^+(q)\right]\right) \\
&\quad \le \left(\overline{q\eta_M^+(q)} - q\eta_M^+(q)\right) - \frac{1}{2}\left(\overline{q^2(\eta_M^+)'(q)} - q^2(\eta_M^+)'(q)\right) \\
&\qquad - \frac{1}{2}\left(\overline{q^2} - q^2\right)(\eta_M^+)'(q) - \lambda\left(\overline{q(\eta_M^+)'(q)} - q(\eta_M^+)'(q)\right) \\
&\qquad + Q(t,x)\left(\overline{(\eta_M^+)'(q)} - (\eta_M^+)'(q)\right).
\end{aligned} \tag{4.6}$$

By the increasing property of η_M^+ and the convexity of $q(\eta_M^+)'(q)$, from (3.50), we have

$$-\frac{1}{2}\left(\overline{q^2} - q^2\right)(\eta_M^+)'(q) \le 0, \qquad -\lambda\left(\overline{q(\eta_M^+)'(q)} - q(\eta_M^+)'(q)\right) \le 0. \tag{4.7}$$

It follows from Lemma 4.3 that

$$\begin{aligned}
q\eta_M^+(q) - \frac{1}{2}q^2(\eta_M^+)'(q) &= -\frac{M}{2}q(M-q)\chi_{(M,\infty)}(q), \\
\overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2(\eta_M^+)'(q)} &= -\frac{M}{2}\overline{q(M-q)\chi_{(M,\infty)}(q)}.
\end{aligned} \tag{4.8}$$

In view of Remark 3.9. Let $\Omega_M = (4/\alpha(M-C), \infty) \times R$. Applying (3.56) gives rise to

$$q\eta_M^+(q) - \frac{1}{2}q^2(\eta_M^+)'(q) = \overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2(\eta_M^+)'(q)} = 0, \quad \text{in } \Omega_M. \tag{4.9}$$

In Ω_M, it has

$$\eta_M^+ = \frac{1}{2}(q_+)^2, \qquad (\eta_M^+)'(q) = q_+, \qquad \overline{\eta_M^+(q)} = \frac{1}{2}\overline{(q_+)^2}, \qquad \overline{(\eta_M^+)'(q)} = \overline{q_+}. \tag{4.10}$$

From (4.6)–(4.10), we know that the following inequality holds in Ω_M

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\overline{\eta_M^+(q)} - \eta_M^+(q)\right) + \frac{\partial}{\partial x}\left(u\left[\overline{\eta_M^+(q)} - \eta_M^+(q)\right]\right) \\
&\quad \le Q(t,x)\left(\overline{(\eta_M^+)'(q)} - (\eta_M^+)'(q)\right).
\end{aligned} \tag{4.11}$$

Integrating the resultant inequality over $(4/(M-C),t)\times R$ yields

$$\begin{aligned}\frac{1}{2}\int_R\left(\overline{(q_+)}^2-q_+^2(t,x)\right)dx\le{}&\lim_{t\to 0}\int_R\left[\overline{\eta_M^+(q)}(t,x)-\eta_M^+(q)(t,x)\right]dx\\&+\int_{4/(\alpha(M-C))}^t\int_R Q(s,x)\left[\overline{q_+}(s,x)-q_+(s,x)\right]dsdx,\end{aligned}\tag{4.12}$$

for almost all $t>4/(M-C)$. Sending $M\to\infty$ and using Lemma 4.2, we complete the proof. □

Lemma 4.5. *For any $t>0$ and $M>0$, it holds that*

$$\begin{aligned}&\int_R\left(\overline{\eta_M^-(q)}-\eta_M^-(q)\right)(t,x)dx\\&\quad\le\frac{M^2}{2}\int_0^t\int_R\overline{u(M+q)\chi_{(-\infty,-M)}(q)}dsdx\\&\qquad-\frac{M^2}{2}\int_0^t\int_R u(M+q)\chi_{(-\infty,-M)}(q)dsdx\\&\qquad+M\int_0^t\int_R u\left[\overline{\eta_M^-(q)}-\eta_M^-(q)\right]dsdx\\&\qquad+\frac{M}{2}\int_0^t\int_R u\left(\overline{q_+^2}-q_+^2\right)dsdx+\int_0^t\int_R Q(t,x)\left(\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\right)dsdx.\end{aligned}\tag{4.13}$$

Proof. Let $M>0$. Subtracting (3.58) from (3.54) and using entropy η_M^-, we deduce

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\eta_M^-(q)}-\eta_M^-(q)\right)+\frac{\partial}{\partial x}\left(u\left[\overline{\eta_M^-(q)}-\eta_M^-(q)\right]\right)\\&\quad\le\left(\overline{q\eta_M^-(q)}-q\eta_M^-(q)\right)-\frac{1}{2}\left(\overline{q^2(\eta_M^-)'(q)}-q^2(\eta_M^-)'(q)\right)\\&\qquad-\frac{1}{2}\left(\overline{q^2}-q^2\right)(\eta_M^-)'(q)-\lambda\left(\overline{q(\eta_M^-)'(q)}-q(\eta_M^-)'(q)\right)\\&\qquad+Q(t,x)\left(\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\right).\end{aligned}\tag{4.14}$$

Since $-M\le(\eta_M^-)'\le 0$, we get

$$-\frac{1}{2}\left(\overline{q^2}-q^2\right)(\eta_M^-)'(q)\le\frac{M}{2}\left(\overline{q^2}-q^2\right).\tag{4.15}$$

By the convexity of $q(\eta_M^-)'(q)$, it has

$$-\lambda\left(\overline{q(\eta_M^-)'(q)} - q(\eta_M^-)'(q)\right) \leq 0. \tag{4.16}$$

Using Remark 3.9 and Lemma 4.3 yields

$$q\eta_M^-(q) - \frac{1}{2}q^2(\eta_M^-)'(q) = -\frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q), \tag{4.17}$$

$$\overline{q\eta_M^-(q)} - \frac{1}{2}\overline{q^2(\eta_M^-)'(q)} = -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)}. \tag{4.18}$$

Inserting the inequalities from (4.15) to (4.18) into (4.14) gives

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\overline{\eta_M^-(q)} - \eta_M^-(q)\right) + \frac{\partial}{\partial x}\left(u\left[\overline{\eta_M^-(q)} - \eta_M^-(q)\right]\right) \\
&\quad \leq -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)} + \frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q) \\
&\qquad + \frac{M}{2}\left(\overline{q^2} - q^2\right) + Q(t,x)\left(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right).
\end{aligned} \tag{4.19}$$

Integrating the above inequality over $(0,t)\times R$, we obtain

$$\begin{aligned}
&\int_R \left(\overline{\eta_M^-(q)} - \eta_M^-(q)\right)(t,x)dx \\
&\quad \leq -\frac{M}{2}\int_0^t\int_R \overline{q(M+q)\chi_{(-\infty,-M)}(q)}dsdx \\
&\qquad + \frac{M}{2}\int_0^t\int_R q(M+q)\chi_{(-\infty,-M)}(q)dsdx \\
&\qquad + \frac{M}{2}\int_0^t\int_R\left(\overline{q^2} - q^2\right)dsdx + \int_0^t\int_R Q(t,x)\left(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right)dsdx.
\end{aligned} \tag{4.20}$$

It follows from Lemma 4.3 that

$$\begin{aligned}
\overline{\eta_M^-(q)} - \eta_M^-(q) = {}& \frac{1}{2}\left(\overline{(q_-)^2} - (q_-)^2\right) + \frac{1}{2}(M+q)^2\chi_{(-\infty,-M)}(q) \\
&- \frac{1}{2}\overline{(M+q)^2\chi_{(-\infty,-M)}(q)}.
\end{aligned} \tag{4.21}$$

Using Remark 3.9 and (4.20), we have

$$\begin{aligned}
&\int_R \left(\overline{\eta_M^-(q)} - \eta_M^-(q)\right)(t,x)dx \\
&\quad \le -\frac{M}{2}\int_0^t\int_R \overline{q(M+q)\chi_{(-\infty,-M)}(q)}dsdx \\
&\quad\quad + \frac{M}{2}\int_0^t\int_R q(M+q)\chi_{(-\infty,-M)}(q)dsdx \\
&\quad\quad + M\int_0^t\int_R \left[\overline{\eta_M^-(q)} - \eta_M^-(q)\right]dsdx \\
&\quad\quad + \frac{M}{2}\int_0^t\int_R \overline{(M+q)^2\chi_{(-\infty,-M)}(q)}dsdx \\
&\quad\quad - \frac{M}{2}\int_0^t\int_R (M+q)^2 u\chi_{(-\infty,-M)}(q)dsdx + \frac{M}{2}\int_0^t\int_R \left(\overline{q_+^2} - q_+^2\right)dsdx \\
&\quad\quad + \int_0^t\int_R Q(t,x)\left(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right)dsdx.
\end{aligned} \tag{4.22}$$

Applying the identity $M(M+q)^2 - Mq(M+q) = M^2(M+q)$, we obtain (4.13). □

Lemma 4.6. *It holds that*

$$\overline{q^2} = q^2 \quad \textit{almost everywhere in} \quad [0,\infty)\times(-\infty,\infty). \tag{4.23}$$

Proof. Applying Lemmas 4.4 and 4.5 gives rise to

$$\begin{aligned}
&\int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)dx \\
&\quad \le \frac{M^2}{2}\left(\int_0^t\int_R \overline{(M+q)\chi_{(-\infty,-M)}(q)}dsdx - \frac{M^2}{2}\int_0^t\int_R (M+q)\chi_{(-\infty,-M)(q)}dsdx\right) \\
&\quad\quad + M\int_0^t\int_R \left[\overline{\eta_M^-} - \eta_M^-\right]dsdx + \frac{M}{2}\int_0^t\int_R \left[\overline{(q_+)^2} - (q_+)^2\right]dsdx \\
&\quad\quad + \int_0^t\int_R Q(s,x)\left(\left[\overline{q_+} - q_+\right] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right]\right)dsdx.
\end{aligned} \tag{4.24}$$

From Lemma 3.6, we know that there exists a constant $L > 0$, depending only on $\|u_0\|_{H^1(R)}$, such that

$$\|Q(t,x)\|_{L^\infty([0,\infty)\times R)} \le L. \tag{4.25}$$

By Remark 3.9 and Lemma 4.3, it has

$$\begin{aligned} q_+ + (\eta_M^-)'(q) &= q - (M+q)\chi_{(-\infty,-M)}, \\ \overline{q_+} + \overline{(\eta_M^-)'(q)} &= q - \overline{(M+q)\chi_{(-\infty,-M)}(q)}. \end{aligned} \tag{4.26}$$

Thus, by the convexity of the map $\xi \to \xi_+ + (\eta_M^-)'(\xi)$, we get

$$\begin{aligned} 0 &\le [\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right] \\ &= (M+q)\chi_{(-\infty,-M)} - \overline{(M+q)\chi_{(-\infty,-M)}(q)}. \end{aligned} \tag{4.27}$$

Using (4.25) derives

$$\begin{aligned} &Q(s,x)\left([\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right]\right) \\ &\quad \le -L\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right). \end{aligned} \tag{4.28}$$

Since $\xi \to (M+\xi)\chi_{(-\infty,-M)}$ is concave and choosing M large enough, we have

$$\begin{aligned} &\frac{M^2}{2}\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right) \\ &\qquad + Q(s,x)\left([\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right]\right) \\ &\quad \le \left(\frac{M^2}{2} - L\right)\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right) \le 0. \end{aligned} \tag{4.29}$$

Then, from (4.24) and (4.29), it has

$$\begin{aligned} 0 &\le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)dx \\ &\le cM\int_0^t\int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)dsdx. \end{aligned} \tag{4.30}$$

By making of the Gronwall inequality and Lemmas 4.1 and 4.2, for each $t > 0$, we conclude that

$$0 \le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)dx = 0. \tag{4.31}$$

By the Fatou lemma, Remark 3.9, and (3.50), sending $M \to \infty$, we obtain

$$0 \leq \int_R \left(\overline{q^2} - q^2\right)(t,x)\,dx = 0, \quad \text{for} \quad t > 0, \tag{4.32}$$

which completes the proof. □

Proof of the main result. Using (3.2), (3.4), and Lemma 3.5, we know that the Conditions (i) and (ii) in Definition 2.1 are satisfied. We have to verify (iii). Due to Lemma 4.6, we have

$$q_\varepsilon \longrightarrow q \quad \text{in} \quad L^2_{\text{loc}}([0,\infty) \times R). \tag{4.33}$$

From Lemma 3.5, (3.47), and (4.33), we know that u is a distributional solution to problem (2.2). In addition, inequalities (2.5) and (2.6) are deduced from Lemmas 3.2 and 3.4. The proof of the main result is completed. □

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