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Research Article

Application of Homotopy Perturbation and Variational Iteration Methods for Fredholm Integrodifferential Equation of Fractional Order

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This paper presents the application of homotopy perturbation and variational iteration methods as numerical methods for Fredholm integrodifferential equation of fractional order with initial-boundary conditions. The fractional derivatives are described in Caputo sense. Some illustrative examples are presented.

1. Introduction

Fractional differential equations have attracted much attention, recently, see for instance [1–4]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument for the description of many practical dynamical phenomena arising in engineering and scientific disciplines such as, physics, chemistry, biology, economy, viscoelasticity, electrochemistry, electromagnetic, control, porous media and many more, see for example, [5, 6].

During the past decades, the topic of fractional calculus has attracted many scientists and researchers due to its applications in many areas, see [4, 7–9]. Thus several researchers have investigated existence results for solutions to fractional differential equations, see [10, 11]. Further, many mathematical formulation of physical phenomena lead to integrodifferential equations, for example, mostly these type of equations arise in fluid dynamics, biological models and chemical kinetics, and continuum and statistical mechanics, for more details see [12–16]. Integrodifferential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. The homotopy perturbation method and variational iteration method which are proposed by He [17–26] are of the methods which have received much concern. These methods have been successfully applied by many authors, such as the works in [19, 27, 28].

In this work, we study the Integrodifferential equations which are combination of differential and Fredholm-Volterra equations that have the fractional order. In particular, we applied the HPM and VIM for fractional Fredholm Integrodifferential equations with constant coefficients

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(t) = g(t) + \lambda \int_0^a H(x,t) y(t) dt, \quad a \leq x,\ t \leq b, \tag{1.1}$$

under the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \tag{1.2}$$

$$D_*^{\alpha} y(0) = y'(a), \tag{1.3}$$

where a is constant, and $1 < \alpha < 2$, and D_*^{α} is the fractional derivative in the Caputo sense.

For the geometrical applications and physical understanding of the fractional Integrodifferential equations, see [14, 26]. Further, we also note that fractional integro-differential equations were associated with a certain class of phase angles and suggested a new way for understanding of Riemann's conjecture, see [29].

In present paper, we apply the HPM and VIM to solve the linear and nonlinear fractional Fredholm Integrodifferential equations of the form (1.1). The paper is organized as follows. In Section 2, some basic definitions and properties of fractional calculus theory are given. In Section 3, the basic idea of HPM exists. In Section 4, also is the basic idea of VIM. In Sections 5 and 6, analysis of HPM and VIM exsists, respectively. some examples are given in Section 7. Concluding remarks are listed in Section 8.

2. Preliminaries

In order to modeling the real world application the fractional differential equations are considered by using the fractional derivatives. Thus, in this section, we give some basic definitions and properties of fractional calculus theory which is used in this paper. There are many different starting points for the discussion of classical fractional calculus, see for example, [30]. One can begin with a generalization of repeated integration. If $f(t)$ is absolutely integrable on $[0,b)$, as in [31] then

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t2} f(t_1) dt_1 = \frac{1}{(n+1)!} \int_0^t (t-t_1)^{n-1} f(t_1) dt_1 = \frac{1}{(n+1)!} t^{n-1} * f(t), \tag{2.1}$$

where $n = 1, 2, \ldots$, and $0 \le t \le b$. On writing $\Gamma(n) = (n-1)!$, an immediate generalization in the form of the operation I^{α} defined for $\alpha > 0$ is

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - t_1)^{\alpha-1} f(t_1) dt_1 = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \quad 0 \le t < b, \tag{2.2}$$

where $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t) = \int_0^t f(t - t_1)^{\alpha-1}(t_1) dt_1$ is called the convolution product of $t^{\alpha-1}$ and $f(t)$. Equation (2.2) is called the Riemann-Liouville fractional integral of order α for the function $f(t)$. Then, we have the following definitions.

Definition 2.1. A real function $f(x)$, $x > 0$ is said to be in space $C\mu$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_{μ}^{n} if $f^n \in \mathbb{R}_{\mu}$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C\mu$, $\mu \ge -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \ t > 0. \tag{2.3}$$

In particular, $J^0 f(x) = f(x)$.

For $\beta \ge 0$ and $\gamma \ge -1$, some properties of the operator J^{α}:

(1) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$,

(2) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$,

(3) $J^{\alpha} x^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)) x^{\alpha+\gamma}$.

Definition 2.3. The Caputo fractional derivative of $f \in C_{-1}^{m}$, $m \in \mathbb{N}$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x f^m(t) dt, \quad m - 1 < \alpha \le m. \tag{2.4}$$

Lemma 2.4. *If $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, $f \in C_{\mu}^{m}$, $\mu > -1$ then the following two properties hold*

(1) $D^{\alpha}[J^{\alpha} f(x)] = f(x)$,

(2) $J^{\alpha}[D^{\alpha} f(x)] = f(x) - \sum_{k=1}^{m-1} f^k(0)(x^k/k!)$.

Now, if $f(x)$ is expanded to the block pulse functions, then the Riemann-Liouville fractional integral becomes

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x) \simeq \xi^T \frac{1}{\Gamma(\alpha)} \left\{ x^{\alpha-1} * \phi_m(x) \right\}. \tag{2.5}$$

Thus, if $x^{\alpha-1} * \phi_m(x)$ can be integrated, then expanded in block pulse functions, the Riemann-Liouville fractional integral is solved via the block pulse functions. Thus, one notes on that

Kronecker convolution product can be expanded in order to define the Riemann-Liouville fractional integrals for matrices by using the Block Pulse operational matrix as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-t_1)^{\alpha-1}\phi_m(t_1)dt_1 \simeq F_\alpha\phi_m(t), \tag{2.6}$$

where

$$F_\alpha = \left(\frac{b}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & 1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & 1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.7}$$

see [32].

3. Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{3.2}$$

where A is a general differential operator; B is a boundary operator; $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Equation (3.1) therefor, can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3.3}$$

By the homotopy technique [33–35], we construct a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \; r \in \Omega \tag{3.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of (3.1) which satisfies the boundary conditions. From (3.2) and (3.3) we have

$$H(v,0) = L(v) - L(u_0) = 0,$$
$$H(v,1) = A(v) - f(r) = 0 \tag{3.6}$$

the changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Now, assume that the solution of (3.2) and (3.3) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{3.7}$$

Setting $p = 1$ results in the approximate solution of (3.1).

Therefore,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{3.8}$$

4. The Variational Iteration Method

To illustrate the basic concepts of VIM, we consider the following differential equation

$$L(u) + N(u) = g(x), \tag{4.1}$$

where L is a linear operator; N is nonlinear operator, and $g(x)$ is an nonhomogeneous term. According to VIM, one constructs a correction functional as follows:

$$y_{n+1} = y_n + \int_0^x \lambda \left[L y_n(s) - N \widetilde{y_n}(s) \right] ds, \tag{4.2}$$

where λ is a general Lagrange multiplier, and $\widetilde{y_n}$ denotes restricted variation that is $\delta \widetilde{y_n} = 0$.

5. Analysis of Homotopy Perturbation Method

To illustrate the basic concepts of HPM for Fredholm Integrodifferential equation (1.1) with boundary conditions (1.2) and (1.3). We use the view of He in [19, 20], where the following homotopy was constructed for (1.1) as the following:

$$(1-p)\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) + p\left[\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) - g(t) - \lambda \int_a^b H(x,t) y(x) dx\right] = 0 \tag{5.1}$$

or

$$\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) = p\left[g(t) + \lambda \int_a^b H(x,t) y(x) dx\right], \tag{5.2}$$

where $p \in [0,1]$ is an embedding parameter. If $p = 0$, (5.2) becomes linear fractional differential equation

$$\sum_{k=0}^{\infty} P_k D_*{}^{\alpha} y(x) = 0, \tag{5.3}$$

and when $p = 1$, the (5.2) turn out to be the original equation. In view of basic assumption of HPM, solution of (1.1) can be expressed as a power series in p

$$y(x) = y_0(x) + p_1 y_1(x) + p_2 y_2(x) + \cdots, \tag{5.4}$$

when $p = 1$, we get the approximate solution of (5.4)

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots. \tag{5.5}$$

The convergence of series (5.5) has been proved in [21]. Substitution (5.4) into (5.2), and equating the terms with having identical power of p, we obtain the following series of equations:

$$\begin{aligned}
p^0 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_0 = 0, \\
p^1 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_1 = g(t) - \lambda \int_a^b H(x,t) y_0(x) dx, \\
p^2 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_2 = -\lambda \int_a^b H(x,t) y_1(x) dx, \\
p^3 &: \sum_{k=0}^{\infty} P_k D_*^{\alpha} y_3 = -\lambda \int_a^b H(x,t) y_2(x) dx, \\
&\vdots
\end{aligned} \tag{5.6}$$

with the initial-boundary conditions

$$D_*^{\alpha} y(a) = y(0), \qquad D_*^{\alpha} y(0) = y'(a). \tag{5.7}$$

The initial approximation can be chosen in the following manner.

$$y_0 = \sum_{j=0}^{1} \gamma_j \frac{x^j}{j!} = \gamma_0 + \gamma_1 x, \quad \text{where } \gamma_0 = D_*^{\alpha} y(a) \gamma_1 = D_*^{\alpha} y(0). \tag{5.8}$$

Note that the (5.6) can be solved by applying the operator J_*^α and by some computation, we approximate the series solution of HPM by the following N-term truncated series

$$\chi_n(x) = y_0(x) + y_1(x) + \cdots + y_{N-1}(x), \tag{5.9}$$

which is the approximate solution of (1.1)–(1.3).

6. Analysis of VIM

To solve the fractional Integrodifferential equation by using the variational iteration method, with boundary conditions (1.2) and (1.3) we construct the following correction functional:

$$y_{k+1}(x) = y_k(x) + J^\alpha\left[\mu\left(\sum_{k=0}^{\infty} P_k D_*^\alpha y(x) - \tilde{g}(x) - \lambda\int_0^a H(x,s)\tilde{y}_k(s)ds\right)\right] \tag{6.1}$$

or

$$y_{k+1}(x) = y_k(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\mu(s)\left[\sum_{k=0}^{\infty} P_k D_*^\alpha y(s) - \tilde{g}_k(s) - \lambda\int_0^a H(x,p)\tilde{y}_k(p)dp\right], \tag{6.2}$$

where μ is a general Lagrange multiplier, and $\tilde{g}_k(x)$ and $\tilde{y}_k(x)$ are considered as restricted variation, that is, $\delta\tilde{g}_k(x) = 0$ and $\delta\tilde{y}_k(x) = 0$.

Making the above correction functional stationary, the following condition can be obtained

$$\delta y_{k+1}(x) = \delta y_k(x) + \int_0^x (x-s)^{\alpha-1}\mu(s)\left[\sum_{k=0}^{\infty} P_k \delta D_*^\alpha y(s) - \delta\tilde{g}_k(s) - \lambda\int_0^a H(x,p)\delta\tilde{y}_k(p)dp\right]. \tag{6.3}$$

It's boundary condition can be obtained as follows:

$$1 - \mu'(s)\big|_{x=s} = 0, \qquad \mu(s)\big|_{x=s} = 1. \tag{6.4}$$

The Lagrange multipliers can be identified as follows:

$$\mu(s) = \frac{1}{2}(x-s). \tag{6.5}$$

We obtain the following iteration formula by substitution of (6.5) in (6.2):

$$y_{k+1}(x) = y_k(x) + \frac{1}{2\Gamma(\alpha-1)} \int_0^x (x-s)^{\alpha-2}(s-x) \left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s) - \tilde{g}_k(s) - \lambda \int_0^a H(x,p)\tilde{y}_k(p)dp \right] ds. \quad (6.6)$$

That is,

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(s) - \tilde{g}_k(s) - \lambda \int_0^a H(x,p)\tilde{y}_k(p)dp \right] ds. \quad (6.7)$$

This yields the following iteration formula:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2} J^{\alpha} \left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) - g_k(x) - \lambda \int_0^a H(x,s)\tilde{y}_k(s)ds \right]. \quad (6.8)$$

The initial approximation y_0 can be chosen by the following manner which satisfies initial-boundary conditions (1.2)-(1.3)

$$y_0 = \gamma_0 + \gamma_1 x, \quad \text{where } \gamma_0 = D_*^{\alpha} y(a) \ \gamma_1 = D_*^{\alpha} y(0). \quad (6.9)$$

We can obtain the following first-order approximation by substitution of (6.9) in (6.8)

$$y_1(x) = y_0(x) - \frac{(\alpha-1)}{2} J^{\alpha} \left[\sum_{k=0}^{\infty} P_k D_*^{\alpha} y(x) - g_0(x) - \lambda \int_0^a H(x,s)\tilde{y}_0(s)ds \right]. \quad (6.10)$$

Finally, by substituting the constant values of γ_0 and γ_1 in (6.10) we have the results as the approximate solutions of (1.1)–(1.3), see the further details in [36–40].

7. Applications

In this section, we have applied homotopy perturbation method and variational iteration method to fractional Fredholm Integrodifferential equations with known exact solution.

Example 7.1. Consider the following linear Fredholm Integrodifferential equation:

$$D^{\alpha} y(x) = \left(\frac{3}{2} + \frac{e^{2x}}{2} \right) + \int_0^x e^t y(t)dt \quad 0 \le x \le 1, \ 1 < \alpha \le 2, \quad (7.1)$$

with initial boundary conditions

$$y(0) = 1, \qquad y'(1) = e \tag{7.2}$$

the exact solution is $y(x) = e^x$. Now we construct

$$D^\alpha y(x) = p\left(\left(\frac{3}{2} + \frac{e^{2x}}{2}\right) + \int_0^x e^t y(t)dt\right). \tag{7.3}$$

Substitution of (5.4) in (7.3) and then equating the terms with same powers of p, we get the series

$$\begin{aligned} p^0 &: D^\alpha y_0(x) = 0, \\ p^1 &: D^\alpha y_1(x) = \left(\frac{3}{2} + \frac{2}{3}e^{2x}\right) + \int_0^x e^t y_0(t)dt, \\ p^2 &: D^\alpha y_2(x) = -\int_0^x e^t y_1(t)dt. \\ &\vdots \end{aligned} \tag{7.4}$$

Now applying the operator J_α to the equations (7.4) and using initial-boundary conditions yields

$$y_0(x) = 1, \tag{7.5}$$

$$y_1(x) = 1 + Ax + J^\alpha\left(\left(\frac{3}{2} + \frac{e^{2x}}{2}\right) + \int_0^x e^t y_0 dt\right), \tag{7.6}$$

$$y_2(x) = J^\alpha\left(\int_0^x e^t y_1 dt\right), \tag{7.7}$$

$$y_n(x) = J^\alpha\left(\int_0^x e^t y_{n-1} dt\right), \quad n = 2, 3, 4, \ldots. \tag{7.8}$$

Then by solving (7.5)–(7.8), we obtain $y_1, y_2, \ldots$ as

$$y_1(x) = 1 + Ax + \frac{5x^\alpha}{2\Gamma(\alpha+1)} + \frac{2x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{3x^{\alpha+2}}{2\Gamma(\alpha+3)} + \frac{5x^{\alpha+3}}{6\Gamma(\alpha+4)}, \tag{7.9}$$

$$\begin{aligned} y_2(x) = {}& \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + (A+1)\frac{x^{\alpha+2}}{2\Gamma(\alpha+3)} + \left(\frac{A}{3} + \frac{1}{2}\right)\frac{x^{\alpha+3}}{\Gamma(\alpha+4)} \\ & + \left(\frac{A}{8} + \frac{1}{12}\right)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{Ax^{\alpha+5}}{15\Gamma(\alpha+6)} + \frac{5x^{2\alpha+1}}{2\Gamma(2\alpha+2)} + \cdots. \end{aligned} \tag{7.10}$$

Table 1: Values of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	−2.33209843875457	−1.906444021198994	−0.88898224618462	−0.098915873901025

Table 2: Value of A for different values of α using (7.14).

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	1.23429062479478	0.73267858113358	0.66218167845861	0.54744784230252

Now, we can form the 2 term approximation as follows:

$$\begin{aligned}\phi_2(x) &= 1 + Ax + \frac{5x^{\alpha}}{2\Gamma(\alpha+1)} + \frac{3x^{\alpha+1}}{\Gamma(\alpha+2)} + (A+2)\frac{x^{\alpha+2}}{2\Gamma(\alpha+3)} \\ &\quad + (A+4)\frac{x^{\alpha+3}}{3\Gamma(\alpha+4)} + \left(\frac{A}{8}+\frac{1}{12}\right)\frac{x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{Ax^{\alpha+5}}{15\Gamma(\alpha+6)} + \frac{5x^{2\alpha+1}}{2\Gamma(2\alpha+2)} + \cdots,\end{aligned} \tag{7.11}$$

where A can be determined by imposing initial-boundary conditions (7.2) on ϕ_2. Table 1 shows the values of A for different values of α.

Now, we solve (7.1)-(7.2) by variational iteration method. According to variational iteration method, the formula (6.8) for (7.1) can be expressed in the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[D^{\alpha}y(x) - \left(\frac{3}{2}+\frac{e^{2x}}{2}\right) - \int_0^x e^t y(t)dt\right]. \tag{7.12}$$

Then, in order to avoid the complex and difficult fractional integration, we can consider the truncated Taylor expansions for exponential term in (7.6)–(7.8) for example, $e^x \sim 1 + x + x^2/2 + x^3/6$ and further, suppose that an initial approximation has the following form which satisfies the inial-boundary conditions

$$y_0(x) = 1 + Ax. \tag{7.13}$$

Now by iteration formula (7.12), the first approximation takes the following form:

$$\begin{aligned}y_1(x) &= y_0(x) - \frac{(\alpha-1)}{2}J^{\alpha}\left[D^{\alpha}y_0(x) - \left(\frac{3}{2}+\frac{e^{2x}}{2}\right) - \int_0^x e^t y_0(t)dt\right] \\ &= 1 + Ax + \frac{(\alpha-1)}{2}x^{\alpha}\left[\frac{5}{\Gamma(\alpha+1)} + \frac{2x}{\Gamma(\alpha+2)} + \frac{(A+3)x^2}{2\Gamma(\alpha+3)}\right. \\ &\qquad \left. + \frac{(5/2+A)x^3}{3\Gamma(\alpha+4)} + \frac{(A/2+1/6)x^4}{6\Gamma(\alpha+5)} - \frac{Ax^5}{30\Gamma(\alpha+6)}\right].\end{aligned} \tag{7.14}$$

By imposing initial-boundary conditions (7.2) on y_1, we can obtain the values of A for different α which we show in Table 2.

Example 7.2. Consider the following linear Fredholm Integrodifferential equation:

$$D^{\alpha}y(x) = \left(1 - \frac{x}{4}\right) + \int_0^x xty^2(t)dt \quad 0 \le x \le 1,\ 1 < \alpha \le 2 \tag{7.15}$$

with initial boundary conditions

$$y(0) = 0, \qquad y'(1) = 1. \tag{7.16}$$

then the exact solution is $y(x) = x$. By applying the HPM, we have

$$D^{\alpha}y(x) = p\left(\left(1 - \frac{x}{4}\right) + \int_0^x xty^2(t)dt\right). \tag{7.17}$$

Substitution of (5.4) in (7.15) and then equating the terms with same powers of p, we get the following series expressions:

$$\begin{aligned}
&p^0 : D^{\alpha}y_0(x) = 0,\\
&p^1 : D^{\alpha}y_1(x) = \left(1 - \frac{x}{4}\right) + \int_0^x xty_0{}^2(t)dt,\\
&p^2 : D^{\alpha}y_2(x) = 2\int_0^x xty_0(t)y_1(t)dt,\\
&p^3 : D^{\alpha}y_3(x) = \int_0^x xt\left(y_0(t)y_2(t) + y_1{}^2(t)\right)dt,\\
&p^4 : D^{\alpha}y_4(x) = \int_0^x xt\left(2y_0(t)y_4(t) + 2y_1y_3 + y_2{}^2(t)\right)dt,\\
&\ \vdots
\end{aligned} \tag{7.18}$$

Applying the operator J_{α} to (7.18) and using initial-boundary conditions, then we get

$$\begin{aligned}
&y_0(x) = 0,\\
&y_1(x) = Ax + J^{\alpha}\left(\left(1 - \frac{x}{4}\right) + \int_0^x xty_0{}^2(t)dt\right),\\
&y_2(x) = 0,\\
&y_3(x) = J^{\alpha}\left(\int_0^x xt\left(y_0(t)y_2(t) + y_1{}^2(t)\right)dt\right),\\
&y_4(x) = J^{\alpha}\left(\int_0^x xt\left(2y_0(t)y_4(t) + 2y_1y_3 + y_2{}^2(t)\right)dt\right),\\
&\quad \vdots
\end{aligned} \tag{7.19}$$

Table 3: Value of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	0.179304	0.153796	0.0673477	0.124989

Table 4: Value of A for different values of α.

	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
A	0.88967375	0.76492075	0.650263	0.5625

Thus, by solving (7.19), we obtain $y_1, y_2, y_3, \dots$

$$\begin{gathered} y_1(x) = Ax + \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)}, \\ y_2(x) = 0, \\ y_3(x) = \frac{A^2 x^{\alpha+5}}{4\Gamma(\alpha+6)} + \frac{2Ax^{2\alpha+4}}{(\alpha+3)\Gamma(\alpha+1)\Gamma(2\alpha+5)} + \frac{x^{3\alpha+3}}{(2\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+4)} + \cdots. \end{gathered} \tag{7.20}$$

Now, we can form the 3 term approximation

$$\begin{aligned} \phi_2(x) = {} & Ax + \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)} + \frac{A^2 x^{\alpha+5}}{4\Gamma(\alpha+6)} + \frac{2Ax^{2\alpha+4}}{(\alpha+3)\Gamma(\alpha+1)\Gamma(2\alpha+5)} \\ & + \frac{x^{3\alpha+3}}{(2\alpha+2)\Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(3\alpha+4)} + \cdots, \end{aligned} \tag{7.21}$$

where A can be determined by imposing initial-boundary conditions (7.16) on ϕ_2. Thus, we have Table 3.

Similarly, by variational iteration method we have the following form:

$$y_{k+1}(x) = y_k(x) - \frac{(\alpha-1)}{2} J^\alpha \left[D^\alpha y(x) - \left(1 - \frac{x}{4}\right) + \int_0^x xty^2(t)dt \right], \tag{7.22}$$

where we suppose that an initial approximation has the following form which satisfies the initial-boundary conditions $y_0(x) = Ax$. Now by using the iteration formula, the first approximation takes the following form:

$$\begin{aligned} y_1(x) &= y_0(x) - \frac{(\alpha-1)}{2} J^\alpha \left[D^\alpha y_0(x) - \left(1 - \frac{x}{4}\right) + \int_0^x xty_0{}^2(t)dt \right] \\ &= Ax + \frac{(\alpha-1)}{2} J^\alpha \left[\frac{-x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{4\Gamma(\alpha+2)} \right]. \end{aligned} \tag{7.23}$$

By imposing initial-boundary conditions, we can obtain the following Table 4.

8. Conclusion

In this work, homotopy perturbation method (HPM) and variational iteration method (VIM) have been applied to linear and nonlinear initial-boundary value problems for fractional Fredholm Integrodifferential equations. Two examples are presented in order to illustrate the accuracy of the present methods. Comparisons of HPM and VIM with exact solution have been given in the Tables 1–4.

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