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Research Article

Strong Convergence of an Implicit S-Iterative Process for Lipschitzian Hemicontractive Mappings

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We establish the strong convergence for the implicit S-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

1. Introduction

Let H be a Hilbert space and let $T: H \to H$ be a mapping. The mapping T is called *Lipshitzian* if there exists L > 0 such that

$$||Tx - Ty|| \le L||x - y||, \quad \forall x, y \in H.$$
 (1.1)

If L = 1, then T is called *nonexpansive* and if $0 \le L < 1$, then T is called *contractive*. The mapping T is said to be *pseudocontractive* ([1, 2]) if

$$||Tx - Ty||^2 \le ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \quad \forall x, y \in H,$$
 (1.2)

and the mapping T is said to be *strongly pseudocontractive* if there exists $k \in (0,1)$ such that

$$||Tx - Ty||^2 \le ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall x, y \in H.$$
(1.3)

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Let $F(T) := \{x \in H : Tx = x\}$ and the mapping T is called *hemicontractive* if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \le ||x - x^*||^2 + ||x - Tx||^2, \quad \forall x \in H, \ x^* \in F(T).$$
(1.4)

It is easy to see the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractive mappings. For the importance of fixed points of pseudocontractions the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

Theorem 1.1. Let K be a compact convex subset of a Hilbert space H and let $T: K \to K$ be a Lipschitzian pseudocontractive mapping.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \ge 1,$$
(1.5)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions:

- (i) $0 \le \alpha_n \le \beta_n \le 1$,
- (ii) $\lim_{n\to\infty}\beta_n=0$,
- (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Another iteration scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings.

In 2011, Sahu [4] and Sahu and Petruşel [5] introduced the S-iterative process as follows.

Let K be a nonempty convex subset of a normed space X and let $T: K \to K$ be a mapping. Then, for arbitrary $x_1 \in K$, the S-iterative process is defined by

$$x_{n+1} = Ty_n, y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \ge 1,$$
(1.6)

where $\{\beta_n\}$ is a real sequence in [0,1].

In this paper, we establish the strong convergence for the implicit *S*-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

2. Main Results

We need the follwing lemma.

Lemma 2.1 (see [6]). For all $x,y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$
(2.1)

Now we prove our main results.

Theorem 2.2. Let K be a compact convex subset of a real Hilbert space H and let $T: K \to K$ be a Lipschitzian hemicontractive mapping satisfying

$$||x - Ty|| \le ||Tx - Ty||, \quad \forall x, y \in K. \tag{C}$$

Let $\{\beta_n\}$ be a sequence in [0,1] satisfying

- (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$,
- (v) $\sum_{n=1}^{\infty} \beta_n^2 < \infty$.

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$x_n = Ty_n,$$

$$y_n = (1 - \beta_n)x_{n-1} + \beta_n Tx_n, \quad n \ge 1.$$
(2.2)

Then the sequence $\{x_n\}$ converges strongly to the fixed point x^* of T.

Proof. From Schauder's fixed point theorem, F(T) is nonempty since K is a convex compact set and T is continuous, let $x^* \in F(T)$. Using the fact that T is hemicontractive we obtain

$$||Tx_n - x^*||^2 \le ||x_n - x^*||^2 + ||x_n - Tx_n||^2, \tag{2.3}$$

$$||Ty_n - x^*||^2 \le ||y_n - x^*||^2 + ||y_n - Ty_n||^2.$$
 (2.4)

Now by (v), there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\beta_n \le \min\left\{\frac{1}{3}, \frac{1}{L^2}\right\},\tag{2.5}$$

which implies that

$$\frac{2\beta_n}{1-\beta_n} \le 1. \tag{2.6}$$

With the help of (2.2), (2.3), and Lemma 2.1, we obtain the following estimates:

$$||y_{n} - x^{*}||^{2} = ||(1 - \beta_{n})x_{n-1} + \beta_{n}Tx_{n} - x^{*}||^{2}$$

$$= ||(1 - \beta_{n})(x_{n-1} - x^{*}) + \beta_{n}(Tx_{n} - x^{*})||^{2}$$

$$= (1 - \beta_{n})||x_{n-1} - x^{*}||^{2} + \beta_{n}||Tx_{n} - x^{*}||^{2}$$

$$- \beta_{n}(1 - \beta_{n})||x_{n-1} - Tx_{n}||^{2}$$

$$\leq (1 - \beta_{n})||x_{n-1} - x^{*}||^{2} + \beta_{n}(||x_{n} - x^{*}||^{2} + ||x_{n} - Tx_{n}||^{2})$$

$$- \beta_{n}(1 - \beta_{n})||x_{n-1} - Tx_{n}||^{2},$$

$$||y_{n} - Ty_{n}||^{2} = ||(1 - \beta_{n})x_{n-1} + \beta_{n}Tx_{n} - Ty_{n}||^{2}$$

$$= ||(1 - \beta_{n})(x_{n-1} - Ty_{n}) + \beta_{n}(Tx_{n} - Ty_{n})||^{2}$$

$$= (1 - \beta_{n})||x_{n-1} - Ty_{n}||^{2} + \beta_{n}||Tx_{n} - Ty_{n}||^{2}$$

$$- \beta_{n}(1 - \beta_{n})||x_{n-1} - Tx_{n}||^{2}.$$
(2.7)

Substituting (2.7) in (2.4) we obtain

$$||Ty_{n} - x^{*}||^{2} \le (1 - \beta_{n})||x_{n-1} - x^{*}||^{2} + \beta_{n}(||x_{n} - x^{*}||^{2} + ||x_{n} - Tx_{n}||^{2})$$

$$+ (1 - \beta_{n})||x_{n-1} - Ty_{n}||^{2} + \beta_{n}||Tx_{n} - Ty_{n}||^{2}$$

$$- 2\beta_{n}(1 - \beta_{n})||x_{n-1} - Tx_{n}||^{2}.$$
(2.8)

Also with the help of condition (C) and (2.8), we have

$$||x_{n+1} - x^*||^2 = ||Ty_n - x^*||^2$$

$$\leq (1 - \beta_n)||x_{n-1} - x^*||^2 + \beta_n (||x_n - x^*||^2 + ||x_n - Tx_n||^2)$$

$$+ (1 - \beta_n)||x_{n-1} - Ty_n||^2 + \beta_n ||Tx_n - Ty_n||^2$$

$$- 2\beta_n (1 - \beta_n)||x_{n-1} - Tx_n||^2$$

$$\leq (1 - \beta_n)||x_{n-1} - x^*||^2 + \beta_n ||x_n - x^*||^2 + (1 - \beta_n)||x_{n-1} - Ty_n||^2$$

$$+ 2\beta_n ||Tx_n - Ty_n||^2 - 2\beta_n (1 - \beta_n)||x_{n-1} - Tx_n||^2,$$
(2.9)

which implies that

$$||x_{n+1} - x^*||^2 \le ||x_{n-1} - x^*||^2 + ||x_{n-1} - Ty_n||^2 + \frac{2\beta_n}{1 - \beta_n} ||Tx_n - Ty_n||^2 - 2\beta_n ||x_{n-1} - Tx_n||^2 \le ||x_{n-1} - x^*||^2 + ||x_{n-1} - Ty_n||^2 + ||Tx_n - Ty_n||^2 - 2\beta_n ||x_{n-1} - Tx_n||^2,$$
(2.10)

where

$$||x_{n-1} - Ty_{n}||^{2} \leq ||Tx_{n-1} - Ty_{n}||^{2}$$

$$\leq L^{2}||x_{n-1} - y_{n}||^{2} \qquad (2.11)$$

$$= L^{2}\beta_{n}^{2}||x_{n-1} - Tx_{n}||^{2},$$

$$||Tx_{n} - Ty_{n}||^{2} \leq L^{2}||x_{n} - y_{n}||^{2}$$

$$\leq L^{2}(||x_{n} - x_{n-1}|| + ||x_{n-1} - y_{n}||)^{2}$$

$$\leq L^{2}(||x_{n} - x_{n-1}|| + \beta_{n}||x_{n-1} - Tx_{n}||)^{2}$$

$$\leq L^{2}(||x_{n} - x_{n-1}|| + \beta_{n}M)^{2},$$

$$||x_{n} - x_{n-1}|| = ||x_{n-1} - Ty_{n}||$$

$$\leq ||Tx_{n-1} - Ty_{n}||$$

$$\leq L||x_{n-1} - y_{n}||$$

$$= L\beta_{n}||x_{n-1} - Tx_{n}||$$

$$\leq L\beta_{n}M$$

and consequently from (2.12), we obtain

$$||Tx_n - Ty_n||^2 \le L^2 (1+L)^2 M^2 \beta_n^2. \tag{2.13}$$

Hence by (2.5), (2.10), (2.11), and (2.13), we have

$$||x_{n} - x^{*}||^{2} \leq ||x_{n-1} - x^{*}||^{2} + L^{2}\beta_{n}^{2}||x_{n-1} - Tx_{n}||^{2}$$

$$+ L^{2}(1 + L)^{2}M^{2}\beta_{n}^{2} - 2\beta_{n}||x_{n-1} - Tx_{n}||^{2}$$

$$= ||x_{n-1} - x^{*}||^{2} + L^{2}(1 + L)^{2}M^{2}\beta_{n}^{2}$$

$$- \beta_{n}(2 - L^{2}\beta_{n})||x_{n-1} - Tx_{n}||^{2}$$

$$\leq ||x_{n-1} - x^{*}||^{2} + L^{2}(1 + L)^{2}M^{2}\beta_{n}^{2} - \beta_{n}||x_{n-1} - Tx_{n}||^{2},$$
(2.14)

which implies that

$$\beta_n \|x_{n-1} - Tx_n\|^2 \le \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + L^2(1+L)^2 M^2 \beta_{n}^2$$
(2.15)

so that

$$\frac{1}{2} \sum_{j=N}^{n} \beta_{j} \|x_{j-1} - Tx_{j}\|^{2} \le \|x_{N} - x^{*}\|^{2} - \|x_{n} - x^{*}\|^{2} + L^{2} (1 + L)^{2} M^{2} \sum_{j=N}^{n} \beta_{j}^{2}.$$
 (2.16)

Hence by conditions (iv) and (v), we get

$$\sum_{j=0}^{\infty} \|x_{j-1} - Tx_j\|^2 < \infty.$$
 (2.17)

It implies that

$$\lim_{n \to \infty} ||x_{n-1} - Tx_n|| = 0.$$
 (2.18)

Consider

$$||x_n - Tx_n|| \le ||x_n - x_{n-1}|| + ||x_{n-1} - Tx_n||, \tag{2.19}$$

which implies that

$$\lim_{n \to \infty} ||x_n - Tx_n|| = 0.$$
 (2.20)

The rest of the argument follows exactly as in the proof of Theorem of [3]. This completes the proof. \Box

Theorem 2.3. Let K be a compact convex subset of a real Hilbert space H and let $T: K \to K$ be a Lipschitzian hemicontractive mapping satisfying the condition (C). Let $\{\beta_n\}$ be a sequence in [0,1] satisfying the conditions (iv) and (v).

Assume that $P_K: H \to K$ be the projection operator of H onto K. Let $\{x_n\}$ be a sequence defined iteratively by

$$x_n = P_K(Ty_n),$$

$$y_n = P_K((1 - \beta_n)x_{n-1} + \beta_n Tx_n), \quad n \ge 1.$$
(2.21)

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The operator P_K is nonexpansive (see, e.g., [2]). K is a Chebyshev subset of H so that, P_K is a single-valued mapping. Hence, we have the following estimate:

$$||x_{n} - x^{*}||^{2} = ||P_{K}(Ty_{n}) - P_{K}x^{*}||^{2}$$

$$\leq ||Ty_{n} - x^{*}||^{2}$$

$$\leq ||x_{n-1} - x^{*}||^{2} + L^{2}(1 + L)^{2}M^{2}\beta_{n}^{2} - \beta_{n}||x_{n-1} - Tx_{n}||^{2}.$$

$$(2.22)$$

The set $K = K \cup T(K)$ is compact and so the sequence $\{||x_n - Tx_n||\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof.

Remark 2.4. In main results, the condition (*C*) is not new and it is due to Liu et al. [7].

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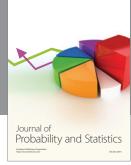
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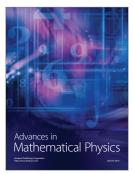




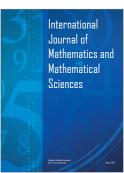


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