Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2012, Article ID 817436, 9 pages doi:10.1155/2012/817436

Research Article

Strong Convergence of a Modified Extragradient Method to the Minimum-Norm Solution of Variational Inequalities

Yonghong Yao,¹ Muhammad Aslam Noor,^{2,3} and Yeong-Cheng Liou⁴

- ¹ Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China
- ² Mathematics Department, COMSATS Institute of Information Technology, Islamabad 44000, Pakistan
- ³ Mathematics Department, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Yonghong Yao, yaoyonghong@yahoo.cn

Received 18 August 2011; Accepted 14 October 2011

Academic Editor: Khalida Inayat Noor

Copyright © 2012 Yonghong Yao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We suggest and analyze a modified extragradient method for solving variational inequalities, which is convergent strongly to the minimum-norm solution of some variational inequality in an infinite-dimensional Hilbert space.

1. Introduction

Let *C* be a closed convex subset of a real Hilbert space *H*. A mapping $A:C\to H$ is called α -inverse-strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \ge \alpha ||Au - Av||^2, \quad \forall u, v \in C.$$
 (1.1)

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \ge 0, \quad \forall v \in C.$$
 (1.2)

The set of solutions of the variational inequality problem is denoted by VI(C, A). It is well known that variational inequality theory has emerged as an important tool in studying a wide

⁴ Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan

class of obstacle, unilateral, and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems; see [1–36] and the references therein.

It is well known that variational inequalities are equivalent to the fixed point problem. This alternative formulation has been used to study the existence of a solution of the variational inequality as well as to develop several numerical methods. Using this equivalence, one can suggest the following iterative method.

Algorithm 1.1. For a given $u_0 \in C$, calculate the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_C[u_n - \lambda A u_n], \quad n = 0, 1, 2, \dots$$
 (1.3)

It is well known that the convergence of Algorithm 1.1 requires that the operator *A* must be both strongly monotone and Lipschitz continuous. These restrict conditions rules out its applications in several important problems. To overcome these drawbacks, Korpelevič suggested in [8] an algorithm of the form

$$y_n = P_C[x_n - \lambda A x_n],$$

$$x_{n+1} = P_C[x_n - \lambda A y_n], \quad n \ge 0.$$
(1.4)

Noor [2] further suggested and analyzed the following new iterative methods for solving the variational inequality (1.2).

Algorithm 1.2. For a given $u_0 \in C$, calculate the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_C[w_n - \lambda A w_n],$$

 $w_n = P_C[u_n - \lambda A u_n], \quad n = 0, 1, 2, ...,$
(1.5)

which is known as the modified extragradient method. For the convergence analysis of Algorithm 1.2, see Noor [1, 2] and the references therein. We would like to point out that Algorithm 1.2 is quite different from the method of Korpelevič [8]. However, Algorithm 1.2 fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces.

In this paper, we suggest and consider a very simple modified extragradient method which is convergent strongly to the minimum-norm solution of variational inequality (1.2) in an infinite-dimensional Hilbert space. This new method includes the method of Noor [2] as a special case.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a closed convex subset of H. It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$||u - u_0|| = \inf\{||u - x|| : x \in C\}.$$
(2.1)

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

- (i) $||P_C x P_C y|| \le ||x y||$ for all $x, y \in H$;
- (ii) $\langle x y, P_C x P_C y \rangle \ge ||P_C x P_C y||^2$ for every $x, y \in H$;
- (iii) $\langle x P_C x, y P_C x \rangle \le 0$ for all $x \in H$, $y \in C$.

We need the following lemma for proving our main results.

Lemma 2.1 (see [15]). Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n,\tag{2.2}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \gamma_n = \infty;$
- (2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

In this section we will state and prove our main result.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space E. Let $A: C \to E$ be an α -inverse-strongly monotone mapping. Suppose that $VI(C,A) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$y_n = P_C[(1 - \alpha_n)(x_n - \lambda A x_n)],$$

 $x_{n+1} = P_C(y_n - \lambda A y_n), \quad n \ge 0,$ (3.1)

where $\{\alpha_n\}$ is a sequence in (0,1) and $\lambda \in [a,b] \subset (0,2\alpha)$ is a constant. Assume the following conditions are satisfied:

- $(C1): \lim_{n\to\infty} \alpha_n = 0;$
- $(C2): \sum_{n=1}^{\infty} \alpha_n = \infty;$
- $(C3): \lim_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1.$

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $P_{VI(C,A)}(0)$ which is the minimum-norm element in VI(C,A).

We will divide our detailed proofs into several conclusions.

Proof. Take $x^* \in VI(C, A)$. First we need to use the following facts:

(1)
$$x^* = P_C(x^* - \lambda A x^*)$$
 for all $\lambda > 0$; in particular,

$$x^* = P_C[x^* - \lambda(1 - \alpha_n)Ax^*] = P_C[\alpha_n x^* + (1 - \alpha_n)(x^* - \lambda Ax^*)], \quad \forall n \ge 0;$$
 (3.2)

(2) $I - \lambda A$ is nonexpansive and for all $x, y \in C$

$$\|(I - \lambda A)x - (I - \lambda A)y\|^{2} \le \|x - y\|^{2} + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^{2}.$$
 (3.3)

From (3.1), we have

$$||y_{n} - x^{*}|| = ||P_{C}[(1 - \alpha_{n})(x_{n} - \lambda Ax_{n})] - P_{C}[\alpha_{n}x^{*} + (1 - \alpha_{n})(x^{*} - \lambda Ax^{*})]||$$

$$\leq ||\alpha_{n}(-x^{*}) + (1 - \alpha_{n})[(x_{n} - \lambda Ax_{n}) - (x^{*} - \lambda Ax^{*})]||$$

$$\leq \alpha_{n}||x^{*}|| + (1 - \alpha_{n})||(I - \lambda A)x_{n} - (I - \lambda A)x^{*}||$$

$$\leq \alpha_{n}||x^{*}|| + (1 - \alpha_{n})||x_{n} - x^{*}||.$$
(3.4)

Thus,

$$||x_{n+1} - x^*|| = ||P_C(y_n - \lambda A y_n) - P_C(x^* - \lambda A x^*)||$$

$$\leq ||(y_n - \lambda A y_n) - (x^* - \lambda A x^*)||$$

$$\leq ||y_n - x^*||$$

$$\leq \alpha_n ||x^*|| + (1 - \alpha_n) ||x_n - x^*||$$

$$\leq \max\{||x^*||, ||x_0 - x^*||\}.$$
(3.5)

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$. From (3.1), we have

$$||x_{n+1} - x_n|| = ||P_C(y_n - \lambda A y_n) - P_C(y_{n-1} - \lambda A y_{n-1})||$$

$$\leq ||(y_n - \lambda A y_n) - (y_{n-1} - \lambda A y_{n-1})||$$

$$\leq ||y_n - y_{n-1}||$$

$$= ||P_C[(1 - \alpha_n)(x_n - \lambda A x_n)] - P_C[(1 - \alpha_{n-1})(x_{n-1} - \lambda A x_{n-1})]||$$

$$\leq ||(1 - \alpha_n)[(I - \lambda A)x_n - (I - \lambda A)x_{n-1}] - (\alpha_n - \alpha_{n-1})(I - \lambda A)x_{n-1}||$$

$$\leq (1 - \alpha_n)||(I - \lambda A)x_n - (I - \lambda A)x_{n-1}|| + |\alpha_n - \alpha_{n-1}|||(I - \lambda A)x_{n-1}||$$

$$\leq (1 - \alpha_n)||x_n - x_{n-1}|| + |\alpha_n - \alpha_{n-1}|M,$$
(3.6)

where M > 0 is a constant such that $\sup_n \{ \|(I - \lambda A)x_n\|, \|(I - \lambda A)x_n\|(\|(I - \lambda A)x_n\| + 2\|x_n - x^*\|) \} \le M$. Hence, by Lemma 2.1, we obtain

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

From (3.4), (3.5) and the convexity of the norm, we deduce

$$||x_{n+1} - x^*||^2 \le ||\alpha_n(-x^*) + (1 - \alpha_n)[(x_n - \lambda Ax_n) - (x^* - \lambda Ax^*)]||^2$$

$$\le \alpha_n ||x^*||^2 + (1 - \alpha_n)||(I - \lambda A)x_n - (I - \lambda A)x^*||^2$$

$$\le \alpha_n ||x^*||^2 + (1 - \alpha_n)[||x_n - x^*||^2 + \lambda(\lambda - 2\alpha)||Ax_n - Ax^*||^2]$$

$$\le \alpha_n ||x^*||^2 + ||x_n - x^*||^2 + (1 - \alpha_n)a(b - 2\alpha)||Ax_n - Ax^*||^2.$$
(3.8)

Therefore, we have

$$(1 - \alpha_n)a(2\alpha - b)\|Ax_n - Ax^*\|^2 \le \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

$$\le \alpha_n\|x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times \|x_n - x_{n+1}\|.$$
(3.9)

Since $\alpha_n \to 0$ and $||x_n - x_{n+1}|| \to 0$ as $n \to \infty$, we obtain $||Ax_n - Ax^*|| \to 0$ as $n \to \infty$. By the property (ii) of the metric projection P_C , we have

$$\|y_{n} - x^{*}\|^{2} = \|P_{C}[(1 - \alpha)(x_{n} - \lambda Ax_{n})] - P_{C}(x^{*} - \lambda Ax^{*})\|^{2}$$

$$\leq \langle (1 - \alpha_{n})(x_{n} - \lambda Ax_{n}) - (x^{*} - \lambda Ax^{*}), y_{n} - x^{*} \rangle$$

$$= \frac{1}{2} \Big\{ \|(x_{n} - \lambda Ax_{n}) - (x^{*} - \lambda Ax^{*}) - \alpha_{n}(I - \lambda A)x_{n}\|^{2} + \|y_{n} - x^{*}\|^{2}$$

$$- \|(x_{n} - \lambda Ax_{n}) - (x^{*} - \lambda Ax^{*}) - (y_{n} - x^{*}) - \alpha_{n}(I - \lambda A)x_{n}\|^{2} \Big\}$$

$$\leq \frac{1}{2} \Big\{ \|(x_{n} - \lambda Ax_{n}) - (x^{*} - \lambda Ax^{*})\|^{2} + \alpha_{n}M + \|y_{n} - x^{*}\|^{2}$$

$$- \|(x_{n} - y_{n}) - \lambda (Ax_{n} - Ax^{*}) - \alpha_{n}(I - \lambda A)x_{n}\|^{2} \Big\}$$

$$\leq \frac{1}{2} \Big\{ \|x_{n} - x^{*}\|^{2} + \alpha_{n}M + \|y_{n} - x^{*}\|^{2} - \|x_{n} - y_{n}\|^{2}$$

$$+ 2\lambda \langle x_{n} - y_{n}, Ax_{n} - Ax^{*} \rangle + 2\alpha_{n} \langle (I - \lambda A)x_{n}, x_{n} - y_{n} \rangle$$

$$- \|\lambda (Ax_{n} - Ax^{*}) + \alpha_{n}(I - \lambda A)x_{n}\|^{2} \Big\}$$

$$\leq \frac{1}{2} \Big\{ \|x_{n} - x^{*}\|^{2} + \alpha_{n}M + \|y_{n} - x^{*}\|^{2} - \|x_{n} - y_{n}\|^{2}$$

$$+ 2\lambda \|x_{n} - y_{n}\| \|Ax_{n} - Ax^{*}\| + 2\alpha_{n}\| (I - \lambda A)x_{n}\| \|x_{n} - y_{n}\| \Big\}.$$

It follows that

$$||y_{n} - x^{*}||^{2} \le ||x_{n} - x^{*}||^{2} + \alpha_{n}M - ||x_{n} - y_{n}||^{2} + 2\lambda ||x_{n} - y_{n}|| ||Ax_{n} - Ax^{*}|| + 2\alpha_{n}||(I - \lambda A)x_{n}|| ||x_{n} - y_{n}||,$$
(3.11)

and hence

$$||x_{n+1} - x^*||^2 \le ||y_n - x^*||^2$$

$$\le ||x_n - x^*||^2 + \alpha_n M - ||x_n - y_n||^2 + 2\lambda ||x_n - y_n|| ||Ax_n - Ax^*||$$

$$+ 2\alpha_n ||(I - \lambda A)x_n|| ||x_n - y_n||$$
(3.12)

which implies that

$$||x_{n} - y_{n}||^{2} \le (||x_{n} - x^{*}|| + ||x_{n+1} - x^{*}||)||x_{n+1} - x_{n}|| + \alpha_{n}M + 2\lambda ||x_{n} - y_{n}|| ||Ax_{n} - Ax^{*}|| + 2\alpha_{n}||(I - \lambda A)x_{n}|| ||x_{n} - y_{n}||.$$

$$(3.13)$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, and $\|Ax_n - Ax^*\| \to 0$, we derive $\|x_n - y_n\| \to 0$. Next we show that

$$\limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle \le 0, \tag{3.14}$$

where $z_0 = P_{VI(C,A)}(0)$. To show it, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle = \lim_{i \to \infty} \langle z_0, z_0 - y_{n_i} \rangle. \tag{3.15}$$

As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ converges weakly to z. Next we show that $z \in VI(C, A)$. We define a mapping T by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$
 (3.16)

Then *T* is maximal monotone (see [16]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Av \rangle \ge 0$. On the other hand, from $y_n = P_C[(1 - \alpha_n)(x_n - \lambda Ax_n)]$, we have

$$\langle v - y_n, y_n - (1 - \alpha_n)(x_n - \lambda A x_n) \rangle \ge 0, \tag{3.17}$$

that is,

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda} + Ax_n + \frac{\alpha_n}{\lambda} (I - \lambda A) x_n \right\rangle \ge 0.$$
 (3.18)

Therefore, we have

$$\langle v - y_{n_{i}}, w \rangle \geq \langle v - y_{n_{i}}, Av \rangle$$

$$\geq \langle v - y_{n_{i}}, Av \rangle - \left\langle v - y_{n_{i}}, \frac{y_{n_{i}} - x_{n_{i}}}{\lambda} + Ax_{n_{i}} + \frac{\alpha_{n_{i}}}{\lambda} (I - \lambda A) x_{n_{i}} \right\rangle$$

$$= \left\langle v - y_{n_{i}}, Av - Ax_{n_{i}} - \frac{y_{n_{i}} - x_{n_{i}}}{\lambda} - \frac{\alpha_{n_{i}}}{\lambda} (I - \lambda A) x_{n_{i}} \right\rangle$$

$$= \left\langle v - y_{n_{i}}, Av - Ay_{n_{i}} \right\rangle + \left\langle v - y_{n_{i}}, Ay_{n_{i}} - Ax_{n_{i}} \right\rangle$$

$$- \left\langle v - y_{n_{i}}, \frac{y_{n_{i}} - x_{n_{i}}}{\lambda} + \frac{\alpha_{n_{i}}}{\lambda} (I - \lambda A) x_{n_{i}} \right\rangle$$

$$\geq \left\langle v - y_{n_{i}}, Ay_{n_{i}} - Ax_{n_{i}} \right\rangle - \left\langle v - y_{n_{i}}, \frac{y_{n_{i}} - x_{n_{i}}}{\lambda} + \frac{\alpha_{n_{i}}}{\lambda} (I - \lambda A) x_{n_{i}} \right\rangle.$$
(3.19)

Noting that $\alpha_{n_i} \to 0$, $||y_{n_i} - x_{n_i}|| \to 0$, and A is Lipschitz continuous, we obtain $\langle v - z, w \rangle \ge 0$. Since T is maximal monotone, we have $z \in T^{-1}(0)$, and hence $z \in VI(C, A)$. Therefore,

$$\limsup_{n \to \infty} \langle z_0, z_0 - y_n \rangle = \lim_{i \to \infty} \langle z_0, z_0 - y_{n_i} \rangle = \langle z_0, z_0 - z \rangle \le 0.$$
 (3.20)

Finally, we prove $x_n \to z_0$. By the property (ii) of metric projection P_C , we have

$$\|y_{n} - z_{0}\|^{2} = \|P_{C}[(1 - \alpha_{n})(x_{n} - \lambda Ax_{n})] - P_{C}[\alpha_{n}z_{0} + (1 - \alpha_{n})(z_{0} - \lambda Az_{0})]\|^{2}$$

$$\leq \langle \alpha_{n}(-z_{0}) + (1 - \alpha_{n})[(x_{n} - \lambda Ax_{n}) - (z_{0} - \lambda Az_{0})], y_{n} - z_{0} \rangle$$

$$\leq \alpha_{n}\langle z_{0}, z_{0} - y_{n} \rangle + (1 - \alpha_{n})\|(x_{n} - \lambda Ax_{n}) - (z_{0} - \lambda Az_{0})\|\|y_{n} - z_{0}\|$$

$$\leq \alpha_{n}\langle z_{0}, z_{0} - y_{n} \rangle + (1 - \alpha_{n})\|x_{n} - z_{0}\|\|y_{n} - z_{0}\|$$

$$\leq \alpha_{n}\langle z_{0}, z_{0} - y_{n} \rangle + \frac{1 - \alpha_{n}}{2}(\|x_{n} - z_{0}\|^{2} + \|y_{n} - z_{0}\|^{2}).$$

$$(3.21)$$

Hence,

$$||y_n - z_0||^2 \le (1 - \alpha_n)||x_n - z_0||^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle.$$
 (3.22)

Therefore,

$$||x_{n+1} - z_0||^2 \le ||y_n - z_0||^2 \le (1 - \alpha_n)||x_n - z_0||^2 + 2\alpha_n \langle z_0, z_0 - y_n \rangle.$$
(3.23)

We apply Lemma 2.1 to the last inequality to deduce that $x_0 \to z_0$. This completes the proof.

Remark 3.2. Our Algorithm (3.1) is similar to Noor's modified extragradient method; see [2]. However, our algorithm has strong convergence in the setting of infinite-dimensional Hilbert spaces.

Acknowledgments

Y. Yao was supported in part by Colleges and Universities Science and Technology Development Foundation (20091003) of Tianjin, NSFC 11071279 and NSFC 71161001-G0105. Y.-C. Liou was partially supported by the Program TH-1-3, Optimization Lean Cycle, of Sub-Projects TH-1 of Spindle Plan Four in Excellence Teaching and Learning Plan of Cheng Shiu University and was supported in part by NSC 100-2221-E-230-012.

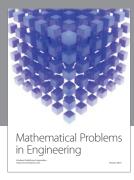
References

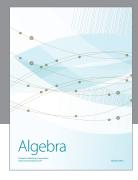
- [1] M. Aslam Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
- [2] M. A. Noor, "A class of new iterative methods for general mixed variational inequalities," *Mathematical and Computer Modelling*, vol. 31, no. 13, pp. 11–19, 2000.
- [3] R. E. Bruck,, "On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 61, no. 1, pp. 159–164, 1977.
- [4] J.-L. Lions and G. Stampacchia, "Variational inequalities," Communications on Pure and Applied Mathematics, vol. 20, pp. 493–519, 1967.
- [5] W. Takahashi, "Nonlinear complementarity problem and systems of convex inequalities," *Journal of Optimization Theory and Applications*, vol. 24, no. 3, pp. 499–506, 1978.
- [6] J. C. Yao, "Variational inequalities with generalized monotone operators," *Mathematics of Operations Research*, vol. 19, no. 3, pp. 691–705, 1994.
- [7] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.
- [8] G. M. Korpelevič, "An extragradient method for finding saddle points and for other problems," Èkonomika i Matematicheskie Metody, vol. 12, no. 4, pp. 747–756, 1976.
- [9] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.
- [10] L.-C. Ceng and J.-C. Yao, "An extragradient-like approximation method for variational inequality problems and fixed point problems," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 205– 215, 2007.
- [11] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.
- [12] A. Bnouhachem, M. Aslam Noor, and Z. Hao, "Some new extragradient iterative methods for variational inequalities," *Nonlinear Analysis*, vol. 70, no. 3, pp. 1321–1329, 2009.
- [13] M. A. Noor, "New extragradient-type methods for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 379–394, 2003.
- [14] B. S. He, Z. H. Yang, and X. M. Yuan, "An approximate proximal-extragradient type method for monotone variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 2, pp. 362–374, 2004.
- [15] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
- [16] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," SIAM Journal on Control and Optimization, vol. 14, no. 5, pp. 877–898, 1976.
- [17] Y. Censor, A. Motova, and A. Segal, "Perturbed projections and subgradient projections for the multiple-sets split feasibility problem," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1244–1256, 2007.
- [18] Y. Censor, A. Gibali, and S. Reich, "The subgradient extragradient method for solving variational inequalities in Hilbert space," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 318– 335, 2011.
- [19] Y. Yao and N. Shahzad, "Strong convergence of a proximal point algorithm with general errors," *Optimization Letters*. In press.
- [20] Y. Yao and N. Shahzad, "New methods with perturbations for non-expansive mappings in hilbert spaces," Fixed Point Theory and Applications, vol. 2011, article 79, 2011.

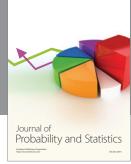
- [21] Y. Yao, R. Chen, and H.-K. Xu, "Schemes for finding minimum-norm solutions of variational inequalities," *Nonlinear Analysis*, vol. 72, no. 7-8, pp. 3447–3456, 2010.
- [22] M. Aslam Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
- [23] M. A. Noor, "Projection-proximal methods for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 53–62, 2006.
- [24] M. A. Noor, "Differentiable non-convex functions and general variational inequalities," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 623–630, 2008.
- [25] M. Aslam Noor and Z. Huang, "Wiener-Hopf equation technique for variational inequalities and nonexpansive mappings," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 504–510, 2007.
- [26] Y. Yao and M. A. Noor, "Convergence of three-step iterations for asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 883–892, 2007.
- [27] Y. Yao and M. A. Noor, "On viscosity iterative methods for variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 776–787, 2007.
- [28] Y. Yao and M. A. Noor, "On modified hybrid steepest-descent methods for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 1276–1289, 2007.
- [29] Y. Yao and M. A. Noor, "On convergence criteria of generalized proximal point algorithms," *Journal of Computational and Applied Mathematics*, vol. 217, no. 1, pp. 46–55, 2008.
- [30] M. A. Noor and Y. Yao, "Three-step iterations for variational inequalities and nonexpansive mappings," Applied Mathematics and Computation, vol. 190, no. 2, pp. 1312–1321, 2007.
- [31] Y. Yao and M. A. Noor, "On modified hybrid steepest-descent method for variational inequalities," *Carpathian Journal of Mathematics*, vol. 24, no. 1, pp. 139–148, 2008.
- [32] Y. Yao, M. A. Noor, R. Chen, and Y.-C. Liou, "Strong convergence of three-step relaxed hybrid steepest-descent methods for variational inequalities," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 175–183, 2008.
- [33] Y. Yao, M. A. Noor, and Y.-C. Liou, "A new hybrid iterative algorithm for variational inequalities," *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 822–829, 2010.
- [34] Y. Yao, M. Aslam Noor, K. Inayat Noor, Y.-C. Liou, and H. Yaqoob, "Modified extragradient methods for a system of variational inequalities in Banach spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1211–1224, 2010.
- [35] Y. Yao, M. A. Noor, K. I. Noor, and Y. -C. Liou, "On an iterative algorithm for variational inequalities in banach spaces," *Mathematical Communications*, vol. 16, no. 1, pp. 95–104, 2011.
- [36] M. A. Noor, E. Al-Said, K. I. Noor, and Y. Yao, "Extragradient methods for solving nonconvex variational inequalities," *Journal of Computational and Applied Mathematics*, vol. 235, no. 9, pp. 3104– 3108, 2011.











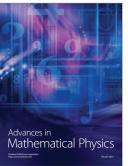


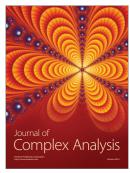




Submit your manuscripts at http://www.hindawi.com











Journal of Discrete Mathematics





