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Research Article

The Cauchy Problem to a Shallow Water Wave Equation with a Weakly Dissipative Term

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A shallow water wave equation with a weakly dissipative term, which includes the weakly dissipative Camassa-Holm and the weakly dissipative Degasperis-Procesi equations as special cases, is investigated. The sufficient conditions about the existence of the global strong solution are given. Provided that $(1-\partial_x^2)u_0 \in M^+(R)$, $u_0 \in H^1(R)$, and $u_0 \in L^1(R)$, the existence and uniqueness of the global weak solution to the equation are shown to be true.

1. Introduction

The Camassa-Holm equation (C-H equation)

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad t > 0, \ x \in R, \tag{1.1}$$

as a model for wave motion on shallow water, has a bi-Hamiltonian structure and is completely integrable. After the equation was derived by Camassa and Holm [1], a lot of works was devoted to its investigation of dynamical properties. The local well posedness of solution for initial data $u_0 \in H^s(R)$ with $s > 3/2$ was given by several authors [2–4]. Under certain assumptions, (1.1) has not only global strong solutions and blow-up solutions [2, 5–7] but also global weak solutions in $H^1(R)$ (see [8–10]). For other methods to handle the problems related to the Camassa-Holm equation and functional spaces, the reader is referred to [11–14] and the references therein.

To study the effect of the weakly dissipative term on the C-H equation, Guo [15] and Wu and Yin [16] discussed the weakly dissipative C-H equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \ x \in R. \tag{1.2}$$

The global existence of strong solutions and blow-up in finite time were presented in [16] provided that $y_0 = (1-\partial_x^2)u_0$ changes sign. The sufficient conditions on the infinite propagation speed for (1.2) are offered in [15]. It is found that the local well posedness and the blow-up phenomena of (1.2) are similar to the C-H equation in a finite interval of time. However, there are differences between (1.2) and the C-H equation in their long time behaviors. For example, the global strong solutions of (1.2) decay to zero as time tends to infinite under suitable assumptions, which implies that (1.2) has no traveling wave solutions (see [16]).

Degasperis and Procesi [17] derived the equation (D-P equation)

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \quad t > 0, \ x \in R, \tag{1.3}$$

as a model for shallow water dynamics. Although the D-P equation (1.3) has a similar form to the C-H equation (1.1), it should be addressed that they are truly different (see [18]). In fact, many researchers have paid their attention to the study of solutions to (1.3). Constantin et al. [19] developed an inverse scattering approach for smooth localized solutions to (1.3). Liu and Yin [20] and Yin [21, 22] investigated the global existence of strong solutions and global weak solutions to (1.3). Henry [23] showed that the smooth solutions to (1.3) have infinite speed of propagation. Coclite and Karlsen [24] obtained global existence results for entropy solutions in $L^1(R) \cap BV(R)$ and in $L^2(R) \cap L^4(R)$.

The weakly dissipative D-P equation

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \ x \in R \tag{1.4}$$

is investigated by several authors [25–27] to find out the effect of the weakly dissipative term on the D-P equation. The global existence, persistence properties, unique continuation and the infinite propagation speed of the strong solutions to (1.4) are studied in [26]. The blow-up solution modeling wave breaking and the decay of solution were discussed in [27]. The existence and uniqueness of the global weak solution in space $W^{1,\infty}_{\text{loc}}(R_+ \times R) \bigcap L^\infty_{\text{loc}}(R_+; H^1(R))$ were proved (see [25]).

In this paper, we will consider the Cauchy problem for the weakly dissipative shallow water wave equation

$$\begin{gathered} u_t - u_{txx} + (a+b)uu_x - au_x u_{xx} - buu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \ x \in R, \\ u(0,x) = u_0(x), \qquad x \in R, \end{gathered} \tag{1.5}$$

where $a > 0$, $b > 0$, and $\lambda \geq 0$ are arbitrary constants, u is the fluid velocity in the x direction, $\lambda(u - u_{xx})$ represents the weakly dissipative term. For $\lambda = 0$, we notice that (1.5) is a special case of the shallow water equation derived by Constantin and Lannes [28].

Since (1.5) is a generalization of the Camassa-Holm equation and the Degasperis-Procesi equation, (1.5) loses some important conservation laws that C-H equation and D-P equation possesses. It needs to be pointed out that Lai and Wu [12] studied global existence and blow-up criteria for (1.5) with $\lambda = 0$. To the best of our knowledge, the dynamical behaviors related to (1.5) with $\lambda = 0$, such as the global weak solution in space $W^{1,\infty}_{\text{loc}}(R_+ \times R) \bigcap L^\infty_{\text{loc}}(R_+; H^1(R))$, have not been yet investigated. The objective of this paper is to investigate several dynamical properties of solutions to (1.5). More precisely, we firstly use the Kato theorem [29] to establish the local well-posedness for (1.5) with initial value $u_0 \in H^s$ with $s > 3/2$. Then, we present a precise blow-up scenario for (1.5). Provided that $u_0 \in H^s(R) \bigcap L^1(R)$ and the potential $y_0 = (1-\partial_x^2)u_0$ does not change sign, the global existence of the strong solution is shown to be true. Finally, under suitable assumptions, the existence and uniqueness of global weak solution in $W^{1,\infty}(R_+ \times R) \bigcap L^\infty_{\text{loc}}(R_+; H^1(R))$ are proved. Our main ideas to prove the existence and uniqueness of the global weak solution come from those presented in Constantin and Molinet [8] and Yin [22].

2. Notations

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times R$ is denoted by C_0^∞. Let $1 \le p < +\infty$, and let $L^p = L^p(R)$ be the space of all measurable functions $h(t,x)$ such that $\|h\|_{L^p}^P = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R\setminus e} |h(t,x)|$. For any real number s, let $H^s = H^s(R)$ denote the Sobolev space with the norm defined by $\|h\|_{H^s} = (\int_R (1+|\xi|^2)^s |\widehat{h}(t,\xi)|^2 d\xi)^{1/2} < \infty$, where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

We denote by $*$ the convolution. Let $\|\cdot\|_X$ denote the norm of Banach space X and $\langle \cdot,\cdot \rangle$ the $H^1(R)$, $H^{-1}(R)$ duality bracket. Let $M(R)$ be the space of the Radon measures on R with bounded total variation and $M^+(R)$ the subset of positive measures. Finally, we write $BV(R)$ for the space of functions with bounded variation, $V(f)$ being the total variation of $f \in BV(R)$.

Note that if $G(x) := (1/2)e^{-|x|}$, $x \in R$. Then, $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(R)$ and $G * (u - u_{xx}) = u$. Using this identity, we rewrite problem (1.5) in the form

$$u_t + buu_x + \partial_x G * \left[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2\right] + \lambda u = 0, \quad t > 0,\ x \in R, \qquad u(0,x) = u_0(x), \quad x \in R, \tag{2.1}$$

which is equivalent to

$$y_t + buy_x + ayu_x + \lambda y = 0, \quad t > 0,\ x \in R, \qquad y = u - u_{xx}, \qquad u(0,x) = u_0(x). \tag{2.2}$$

3. Preliminaries

Throughout this paper, let $\{\rho_n\}_{n\ge1}$ denote the mollifiers

$$\rho_n(x) := \left(\int_R \rho(\xi)d\xi\right)^{-1} n\rho(nx), \quad x \in R,\ n \ge 1, \tag{3.1}$$

where $\rho \in C_c^\infty(R)$ is defined by

$$\rho(x) := \begin{cases} e^{1/(x^2-1)} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases} \tag{3.2}$$

Thus, we get

$$\int_R \rho_n(x)dx = 1, \quad \rho_n \geq 0,\ x \in R,\ n \geq 1. \tag{3.3}$$

Next, we give some useful results.

Lemma 3.1 (see [8]). *Let $f : R \to R$ be uniformly continuous and bounded. If $\mu \in M(R)$, then*

$$[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)]n \overset{\rightharpoonup}{\longrightarrow} \infty\ 0 \quad \text{in } L^1(R). \tag{3.4}$$

Lemma 3.2 (see [8]). *Let $f : R \to R$ be uniformly continuous and bounded. If $g \in L^\infty(R)$, then*

$$[\rho_n * (fg) - (\rho_n * f)(\rho_n * g)]n \overset{\rightharpoonup}{\longrightarrow} \infty\ 0 \quad \text{in } L^\infty(R). \tag{3.5}$$

Lemma 3.3 (see [30]). *Let $T > 0$. If $f, g \in L^2((0,T); H^1(R))$ and $df/dt, dg/dt \in L^2((0,T); H^{-1}(R))$, then f, g are a.e. equal to a function continuous from $[0,T]$ into $L^2(R)$ and*

$$\langle f(t), g(t)\rangle - \langle f(s), g(s)\rangle = \int_s^t \left\langle \frac{d(f(\tau))}{d\tau}, g(\tau)\right\rangle d\tau + \int_s^t \left\langle \frac{d(g(\tau))}{d\tau}, f(\tau)\right\rangle d\tau \tag{3.6}$$

for all $s, t \in [0,T]$.

Lemma 3.4 (see [8]). *Assume that $u(t,\cdot) \in W^{1,1}(R)$ is uniformly bounded in $W^{1,1}(R)$ for all $t \in R_+$. Then, for a.e. $t \in R_+$*

$$\begin{aligned} \frac{d}{dt}\int_R |\rho_n * u| dx &= \int_R (\rho_n * u_t)\, \operatorname{sgn}(\rho_n * u) dx, \\ \frac{d}{dt}\int_R |\rho_n * u_x| dx &= \int_R (\rho_n * u_{xt})\, \operatorname{sgn}(\rho_n * u_x) dx. \end{aligned} \tag{3.7}$$

4. Global Strong Solution

We firstly present the existence and uniqueness of the local solutions to the problem (2.1).

Theorem 4.1. *Let $u_0 \in H^s(R)$, $s > 3/2$. Then, the problem* (2.1) *has a unique solution u, such that*

$$u = u(t,x) \in C([0,T); H^s(R)) \cap C^1\Big([0,T); H^{s-1}(R)\Big), \tag{4.1}$$

where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.

Proof. The proof of Theorem 4.1 can be finished by using Kato's semigroup theory (see [29] or [4]). Here, we omit the detailed proof. □

Theorem 4.2. *Given $u_0 \in H^s$, $s > 3/2$, the solution $u = (\cdot, u_0)$ of problem* (2.1) *blows up in finite time $T < +\infty$ if and only if*

$$\lim_{t\to T} \inf\left\{ \inf_{x\in R}[u_x(t,x)] \right\} = -\infty. \tag{4.2}$$

Proof. Setting $y(t,x) = u(t,x) - u_{xx}(t,x)$, we get

$$\|y\|_{L^2}^2 = \int_R (u - u_{xx})^2 dx = \int_R \left(u^2 + 2u_x^2 + u_{xx}^2\right) dx, \tag{4.3}$$

which yields

$$\|u\|_{H^2}^2 \le \|y\|_{L^2}^2 \le 2\|u\|_{H^2}^2. \tag{4.4}$$

Using system (2.2), one has

$$\begin{aligned} \frac{d}{dt}\int_R y^2(t,x)dx &= 2\int_R yy_t dx \\ &= -2b\int_R uyy_x dx - 2a\int_R u_x y^2 dx - 2\lambda\int_R y^2 dx \\ &= -(2a-b)\int_R u_x y^2 dx - 2\lambda \int_R y^2 dx. \end{aligned} \tag{4.5}$$

Assume that there is a constant $M > 0$ such that

$$-u_x(t,x) \le M \quad \text{on } [0,T)\times R. \tag{4.6}$$

From (4.5), we get

$$\frac{d}{dt}\int_R y^2(t,x)dx \le |2a-b|M\int_R y^2 dx - 2\lambda\int_R y^2 dx. \tag{4.7}$$

Using Gronwall' inequality, we deduce the $\|u\|_{H^2}$ is bounded on $[0,T)$. On the other hand,

$$u(t,x) = \left(1-\partial_x^2\right)^{-1} y(t,x) = \int_R G(x-s)y(s)ds. \tag{4.8}$$

Therefore, using (4.4) leads to

$$\|u_x\|_{L^\infty} \le \left|\int_R G_x(x-s)y(s)ds\right| \le \|G_x\|_{L^2}\|y\|_{L^2} = \frac{1}{2}\|y\|_{L^2} \le \|u\|_{H^2}. \tag{4.9}$$

It shows that if $\|u\|_{H^2}$ is bounded, then $\|u_x\|_{L^\infty}$ is also bounded. This completes the proof. □

We consider the differential equation

$$q_t = bu(t,q), \quad t \in [0,T),\ x \in R,$$
$$q(0,x) = x, \quad x \in R, \tag{4.10}$$

where u solves (1.5) and $T > 0$.

Lemma 4.3. *Let $u_0 \in H^s(R)$ $(s > 3)$; T is the maximal existence time of the corresponding solution u to* (1.5). *Then, system* (4.10) *has a unique solution $q \in C^1([0,T) \times R; R)$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of R with*

$$q_x(t,x) = \exp\left(\int_0^t bu_x(s,q(s,x))ds\right) > 0, \quad \forall (t,x) \in [0,T) \times R,$$
$$y(t,q(t,x))q_x^2(t,x) = y_0(x)\exp\left(\int_0^t \left[-(a-2b)u_x(s,q(s,x)) - \lambda\right]ds\right). \tag{4.11}$$

Proof. From Theorem 4.1, we have $u \in C^1([0,T); H^{s-1}(R))$ and $H^{s-1} \in C^1(R)$. We conclude that both functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in space, and C^1 in time. Applying the existence and uniqueness theorem of ordinary differential equations implies that system (4.10) has a unique solution $q \in C^1([0,T) \times R, R)$.

Differentiating (4.10) with respect to x leads to

$$\frac{d}{dt}q_x = bu_x(t,q)q_x, \quad t \in [0,T),\ b > 0,$$
$$q_x(0,x) = 1, \quad x \in R, \tag{4.12}$$

which yields

$$q_x = \exp\left(\int_0^t bu_x(s,q(s,x))ds\right). \tag{4.13}$$

For every $T' < T$, using the Sobolev embedding theorem gives rise to

$$\sup_{(s,x)\in[0,T')\times R} |u_x(s,x)| < \infty. \tag{4.14}$$

It is inferred that there exists a constant $K_0 > 0$ such that $q_x \geq e^{-K_0 t}$ for $(t,x) \in [0,T) \times R$.

By computing directly, we derive

$$\frac{d}{dt}\left[y(t,q(t,x))q_x^2(t,x)\right] = -(a-2b)u_x(t,x)yq_x^2 - \lambda y q_x^2, \tag{4.15}$$

which results in

$$y(t,q(t,x))q_x^2(t,x) = y_0(x)\exp\left(\int_0^t \left[-(a-2b)u_x(s,q(s,x)) - \lambda\right]ds\right). \tag{4.16}$$

The proof of Lemma 4.3 is completed. □

Theorem 4.4. *Let $u_0 \in L^1(R) \cap H^s(R)$, $s > 3/2$, and $(1-\partial_x^2)u_0 \geq 0$ for all $x \in R$ (or equivalently $(1-\partial_x^2)u_0 \leq 0$ for all $x \in R$). Then, problem (2.1) has a global strong solution*

$$u = u(\cdot, u_0) \in C([0,\infty); H^s(R)) \cap C^1\left([0,\infty); H^{s-1}(R)\right). \tag{4.17}$$

Moreover, if $y(t,\cdot) = u - u_{xx}$, then one has for all $t \in R_+$,

(i) $y(t,\cdot) \geq 0$, $u(t,\cdot) \geq 0$, *and* $|u_x(t,\cdot)| \leq u(t,\cdot)$ *on* R,

(ii) $e^{-\lambda t}\|y_0\|_{L^1(R)} = \|y(t,\cdot)\|_{L^1(R)} = \|u(t,\cdot)\|_{L^1(R)} = e^{-\lambda t}\|u_0\|_{L^1(R)}$ *and* $\|u_x(t,\cdot)\|_{L^\infty(R)} \leq e^{-\lambda t}\|u_0\|_{L^1(R)}$,

(iii) $\|u\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 \exp[(|a-2b|/\lambda)(1-e^{-\lambda t})\|u_0\|_{L^1} - 2\lambda t]$.

Proof. Let $u_0 \in H^s$, $s > 3/2$, and let $T > 0$ be the maximal existence time of the solution u with initial date u_0 (cf. Theorem 4.1). If $y_0 \geq 0$, then Theorem 4.2 ensures that $y \geq 0$ for all $t \in [0,T)$.

Note that $u_0 = G * y_0$ and $y_0 = (u_0 - u_{0,xx}) \in L^1(R)$. By Young's inequality, we get

$$\|u_0\|_{L^1(R)} = \left\|G * y_0(t,\cdot)\right\|_{L^1(R)} \leq \|G\|_{L^1(R)} \leq \|G\|_{L^1(R)}\left\|y_0(t,\cdot)\right\|_{L^1(R)} \leq \left\|y_0\right\|_{L^1(R)}. \tag{4.18}$$

Integrating the first equation of problem (2.1) by parts, we get

$$\frac{d}{dt}\int_R u\,dx = -b\int_R uu_x dx - \int_R \partial_x\left(G * \left[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2\right]\right)dx + \lambda\int_R u\,dx = 0. \tag{4.19}$$

It follows that

$$\int_R u\,dx = e^{-\lambda t}\int_R u_0 dx. \tag{4.20}$$

Since $y = u - u_{xx}$, we have

$$\begin{aligned}\int_R y\,dx &= \int_R u\,dx - \int_R u_{xx}dx = \int_R u\,dx \\ &= e^{-\lambda t}\int_R u_0 dx = e^{-\lambda t}\left(\int_R u_0 dx - \int_R u_{0,xx}dx\right) = e^{-\lambda t}\int_R y_0 dx.\end{aligned} \tag{4.21}$$

Given $t \in [0,T)$, due to $u(t,x) \in H^s,\ s \geq 3/2$, from Theorem 4.1 and (4.21), we obtain

$$\begin{aligned} -u_x(t,x) + \int_{-\infty}^{x} u\,dx &= \int_{-\infty}^{x} u - u_{xx}dx = \int_{-\infty}^{x} y\,dx \\ &\leq \int_{-\infty}^{\infty} ydx = e^{-\lambda t}\int_R y_0dx = e^{-\lambda t}\int_R u_0dx. \end{aligned} \tag{4.22}$$

Note that $u = G * y,\ \ y \geq 0$ on $[0,T)$ and the positivity of G. Thus, we can infer that $u \geq 0$ on $[0,T)$. From (4.22) we have

$$u_x(t,x) \geq -e^{-\lambda t}\int_R u_0dx, \quad \forall (t,x) \in [0,T) \times R. \tag{4.23}$$

From Theorem 4.2 and (4.23), we find $T = \infty$. This implies that problem (2.1) has a unique solution

$$u = u(\cdot, u_0) \in C([0,\infty); H^s(R)) \bigcap C^1\left([0,\infty); H^{s-1}(R)\right), \quad s \geq \frac{3}{2}. \tag{4.24}$$

Due to $y(t,x) \geq 0$ and $u(t,x) \geq 0$ for all $t \geq 0$, it shows that

$$\begin{aligned} u(t,x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}y(\xi)d\xi + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}y(\xi)d\xi, \\ u_x(t,x) &= -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}y(\xi)d\xi + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\xi}y(\xi)d\xi. \end{aligned} \tag{4.25}$$

From the two identities above, we infer that $(u_x)^2 \leq u^2$ on R for all $t \geq 0$. This proves (i).

Due to $y \geq 0$, we obtain

$$u_x(t,x) - \int_{-\infty}^{x} udx = -\int_{-\infty}^{x} (u - u_{xx})dx = -\int_{-\infty}^{x} ydx \leq 0. \tag{4.26}$$

From $u \geq 0$ and the inequality above, we get

$$u_x(t,x) \leq \int_{-\infty}^{x} udx \leq \int_{-\infty}^{\infty} udx = e^{-\lambda t}\int_{-\infty}^{\infty} u_0dx = e^{-\lambda t}\|u_0\|_{L^1(R)}. \tag{4.27}$$

On the other hand, from (4.23), we have that $u_x(t,x) \geq -e^{-\lambda t}\|u_0\|_{L^1(R)}$. This proves (ii).

Multiplying the first equation of problem (2.1) by u and integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R\left(u^2+u_x^2\right)dx &= -(a+b)\int_R u^2u_x dx + a\int_R uu_xu_{xx}dx + b\int_R u^2u_{xxx}dx\\ &\quad -\lambda\int_R\left(u^2+u_x^2\right)dx\\ &= -\frac{a-2b}{2}\int_R u_x^3dx - \lambda\int_R\left(u^2+u_x^2\right)dx,\end{aligned}\tag{4.28}$$

which yields

$$\begin{aligned}\int_R\left(u^2+u_x^2\right)dx &= -(a-2b)\int_R u_x^3dx - 2\lambda\int_R\left(u^2+u_x^2\right)dx\\ &\le |a-2b|\int_R |u_x|^3dx - 2\lambda\int_R\left(u^2+u_x^2\right)dx\\ &\le |a-2b|\|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 - 2\lambda\int_R\left(u^2+u_x^2\right)dx\\ &\le |a-2b|\|u_x\|_{L^\infty}\|u\|_{H^1}^2 - 2\lambda\int_R\left(u^2+u_x^2\right)dx.\end{aligned}\tag{4.29}$$

From Gronwall's inequality, one has

$$\|u\|_{H^1}^2 \le \|u_0\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - 2\lambda t\right].\tag{4.30}$$

This proves (iii) and completes the proof of the theorem. □

5. Global Weak Solution

Theorem 5.1. *Let $u_0 \in H^1(R)\bigcap L^1(R)$ and $y_0 = (u_0 - u_{0xx}) \in M^+(R)$. Then equation* (1.5) *has a unique solution $u \in W^{1,\infty}(R_+\times R)\bigcap L^\infty_{\text{loc}}(R_+;H^1(R))$ with initial data $u(0) = u_0$ and such that $(u-u_{xx}) \in M^+$, a.e. $\in R_+$ is uniformly bounded on R.*

Proof. We split the proof of Theorem 5.1 in two parts.

Let $u_0 \in H^1(R)$ and $y_0 = u_0 - u_{0,xx} \in M^+(R)$. Note that $u_0 = G * y_0$. Thus, for $\varphi \in L^\infty(R)$, we have

$$\begin{aligned}\|u_0\|_{L^1(R)} = \left\|G*y_0\right\|_{L^1(R)} &= \sup_{\|\varphi\|_{L^\infty(R)}\le 1}\int_R \varphi(x)\left(G*y_0\right)(x)dx\\ &= \sup_{\|\varphi\|_{L^\infty(R)}\le 1}\int_R \varphi(x)\int_R G(x-\xi)dy_0(\xi)dx\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|\varphi\|_{L^\infty(R)}\le 1} \int_R (G*\varphi)(\xi)dy_0(\xi) \\
&= \sup_{\|\varphi\|_{L^\infty(R)}\le 1} \|G*\varphi\|_{L^\infty(R)}\|y_0\|_{M(R)} \\
&\le \sup_{\|\varphi\|_{L^\infty(R)}\le 1} \|G\|_{L^1(R)}\|\varphi\|_{L^\infty(R)}\|y_0\|_{M(R)} = \|y_0\|_{M(R)}.
\end{aligned}
\tag{5.1}
$$

Let us define $u_0^n := \rho_n * u_0 \in H^\infty(R)$ for $n \ge 1$. Obviously, we get

$$
\begin{gathered}
u_0^n \longrightarrow u_0 \quad \text{in } H^1(R) \text{ for } n \longrightarrow \infty, \\
\|u_0^n\|_{H^1(R)} = \|\rho_n * u_0\|_{H^1(R)} = \|\rho_n * u_0\|_{L^2} + \|\rho_n * u_0'\|_{L^2} \le \|u_0\|_{H^1(R)}, \\
\|u_0^n\|_{L^1(R)} = \|\rho_n * u_0\|_{L^1(R)} \le \|u_0\|_{L^1(R)}.
\end{gathered}
\tag{5.2}
$$

Note that, for all $n \ge 1$,

$$
y_0^n := u_0^n - u_{0,xx}^n = \rho_n * (y_0) \ge 0. \tag{5.3}
$$

Referring to the proof of (5.1), we have

$$
\|y_0^n\|_{L^1(R)} \le \|y_0\|_{M(R)}, \quad n \ge 1. \tag{5.4}
$$

From the Theorem 4.4, we know that there exists a global strong solution

$$
u^n = u^n(\cdot, u_0^n) \in C([0,\infty); H^s(R)) \cap C^1\Big([0,\infty); H^{s-1}(R)\Big), \quad s \ge \frac{3}{2}, \tag{5.5}
$$

and $u^n(t,x) - u_{xx}^n(t,x) \ge 0$ for all $(t,x) \in R_+ \times R$.

Note that for all $(t,x) \in R_+ \times R$

$$
(u^n)^2 = \int_{-\infty}^{x} 2u^n u_x^n d\xi \le \int_R \Big[(u^n)^2 + (u_x^n)^2\Big] d\xi = \|u^n\|_{H^1}^2. \tag{5.6}
$$

From Theorem 4.4 and (5.2), we obtain

$$
\begin{aligned}
\|u_x^n\|_{L^\infty(R)}^2 &\le \|u^n\|_{L^\infty(R)}^2 \le \|u^n\|_{H^1(R)}^2 \\
&\le \|u_0^n\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\Big(1-e^{-\lambda t}\Big)\|u_0^n\|_{L^1} - 2\lambda t\right] \\
&\le \|u_0\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\Big(1-e^{-\lambda t}\Big)\|u_0\|_{L^1} - 2\lambda t\right].
\end{aligned}
\tag{5.7}
$$

From the Hölder inequality, Theorem 4.4, and (5.2), for all $t \geq 0$ and $n \geq 1$, we have

$$\begin{aligned}\|bu^n(t)u_x^n(t)\|_{L^2(R)} &\leq b\|u^n(t)\|_{L^\infty(R)}\|u_x^n(t)\|_{L^2(R)} \leq b\|u^n\|_{H^1(R)}^2 \\ &\leq b\left\|u_0^n\right\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\left\|u_0^n\right\|_{L^1} - 2\lambda t\right] \\ &\leq b\|u_0\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - 2\lambda t\right].\end{aligned} \tag{5.8}$$

Using Young's inequality, we get

$$\begin{aligned}&\left\|\partial_x G * \left[\frac{a}{2}(u^n)^2 + \frac{3b-a}{2}(u_x^n)^2\right]\right\|_{L^2(R)} \\ &\quad\leq \frac{a}{2}\left\|\partial_x G * (u^n)^2\right\|_{L^2(R)} + \frac{3b+a}{2}\left\|\partial_x G * (u_x^n)^2\right\|_{L^2(R)} \\ &\quad\leq \frac{a}{2}\|\partial_x G\|_{L^2(R)}\left\|(u^n)^2\right\|_{L^1(R)} + \frac{3b+a}{2}\|\partial_x G\|_{L^2(R)}\left\|(u_x^n)^2\right\|_{L^1(R)} \\ &\quad\leq \frac{3b+a}{2}\|\partial_x G\|_{L^2(R)}\|u^n\|_{H^1(R)}^2 \\ &\quad\leq \frac{3b+a}{2}\|\partial_x G\|_{L^2(R)}\left\|u_0^n\right\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\left\|u_0^n\right\|_{L^1} - 2\lambda t\right] \\ &\quad\leq \frac{3b+a}{2}\|\partial_x G\|_{L^2(R)}\|u_0\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - 2\lambda t\right],\end{aligned} \tag{5.9}$$

where $\|\partial_x G\|_{L^2(R)}$ is bounded, and

$$\begin{aligned}\|\lambda u^n\|_{L^2} &\leq \lambda\|u^n\|_{H^1} \\ &\leq \lambda\|u_0\|_{H^1} \exp\left[\frac{|a-2b|}{2\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - \lambda t\right].\end{aligned} \tag{5.10}$$

Applying (5.8)–(5.10) and problem (2.1), we have

$$\begin{aligned}\left\|\frac{d}{dt}u^n\right\|_{L^2(R)} \leq{}& \left(b + \frac{3b+a}{2}\|\partial_x G\|_{L^2(R)}\right)\|u_0\|_{H^1}^2 \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - 2\lambda t\right] \\ &+ \lambda\|u_0\|_{H^1} \exp\left[\frac{|a-2b|}{2\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1} - \lambda t\right].\end{aligned} \tag{5.11}$$

For fixed $T > 0$, from (5.7) and (5.11), we deduce

$$\int_0^T \int_R \left([u^n(t,x)]^2 + [u_x^n(t,x)]^2 + \left[u_t^n(t,x)\right]^2\right) dx\,dt \leq M, \tag{5.12}$$

where M is a positive constant depending only on $\|G_x\|_{L^2(R)}$, $\|u_0\|_{H^1(R)}$, $\|u_0\|_{L^1(R)}$, and T. It follows that the sequence $\{u^n\}_{n\geq 1}$ is uniformly bounded in the space $H^1((0,T)\times R)$. Thus, we can extract a subsequence such that

$$u^{n_k} \rightharpoonup u, \quad \text{weakly in } H^1((0,T)\times R) \text{ for } n_k \longrightarrow \infty, \tag{5.13}$$

$$u^{n_k} \longrightarrow u, \quad \text{a.e. on } (0,T)\times R \text{ for } n_k \longrightarrow \infty, \tag{5.14}$$

for some $u \in H^1((0,T)\times R)$. From Theorem 4.4 and (5.2), for fixed $t \in (0,T)$, we have that the sequence $u_x^{n_k}(t,\cdot) \in BV(R)$ satisfies

$$\begin{aligned} V\left[u_x^{n_k}(t,x)\right] &= \left\|u_{xx}^{n_k}(t,\cdot)\right\|_{L^1(R)} \leq \|u^{n_k}(t,\cdot)\|_{L^1(R)} + \left\|y^{n_k}(t,\cdot)\right\|_{L^1(R)} \\ &\leq 2e^{-\lambda t}\left\|u_0^{n_k}(t,\cdot)\right\|_{L^1(R)} \leq 2e^{-\lambda t}\|u_0(t,\cdot)\|_{L^1(R)} \leq 2e^{-\lambda t}\left\|y_0(t,\cdot)\right\|_{M(R)}, \\ \left\|u_x^{n_k}(t,\cdot)\right\|_{L^\infty} &\leq e^{-\lambda t}\left\|u_0^{n_k}\right\|_{L^1(R)} \leq e^{-\lambda t}\|u_0\|_{L^1(R)} \leq e^{-\lambda t}\left\|y_0\right\|_{M(R)}. \end{aligned} \tag{5.15}$$

Applying Helly's theorem [31], we infer that there exists a subsequence, denoted again by $\{u_x^{n_k}(t,\cdot)\}$, which converges at every point to some function $v(t,\cdot)$ of finite variation with

$$V(v(t,\cdot)) \leq 2e^{-\lambda t}\left\|y_0\right\|_{M(R)}. \tag{5.16}$$

From (5.14), we get that for almost all $t \in (0,T)$, $u_x^{n_k}(t,\cdot) \to u_x(t,\cdot)$ in $D'(R)$, it follows that $v(t,\cdot) = u_x(t,\cdot)$ for a.e. $t \in (0,T)$. Therefore, we have

$$u_x^{n_k}(t,\cdot) \longrightarrow u_x(t,\cdot) \quad \text{a.e. on } (0,T)\times R \text{ for } n_k \longrightarrow \infty, \tag{5.17}$$

and, for a.e. $t \in (0,T)$,

$$V[u_x(t,\cdot)] = \|u_{xx}(t,\cdot)\|_{M(R)} = 2e^{-\lambda t}\|u_0\|_{L^1} \leq 2e^{-\lambda t}\left\|y_0\right\|_{M(R)}. \tag{5.18}$$

By Theorem 4.4 and (5.7), we have

$$\begin{aligned} &\left\|\frac{a}{2}(u^n)^2 + \frac{3b-a}{2}(u_x^n)^2\right\|_{L^2(R)} \\ &\quad \leq \frac{a}{2}\left\|(u^n)^2\right\|_{L^2(R)} + \frac{3b+a}{2}\left\|(u_x^n)^2\right\|_{L^2} \\ &\quad \leq \frac{a}{2}\|u^n\|_{L^\infty}\|u^n\|_{L^2(R)} + \frac{3b+a}{2}\|u_x^n\|_{L^\infty}\|u_x^n\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{a}{2}\|u^n\|^2_{H^1(R)} + \frac{3b+a}{2}\|u^n\|^2_{H^1(R)} \\
&= \left(a + \frac{3b}{2}\right)\|u^n\|^2_{H^1(R)} \\
&\leq \left(a + \frac{3b}{2}\right)\|u_0\|^2_{H^1} \exp\left[\frac{|a-2b|}{\lambda}\left(1-e^{-\lambda t}\right)\mathrm{x}\|u_0\|_{L^1} - 2\lambda t\right].
\end{aligned} \tag{5.19}$$

Note that for fixed $t \in (0,T)$, the sequence $\{(a/2)(u^n)^2 + ((3b-a)/2)(u_x^n)^2\}_{n\geq 1}$ is uniformly bounded in $L^2(R)$. Therefore, it has a subsequence $\{(a/2)(u^{n_k})^2 + ((3b-a)/2)(u_x^{n_k})^2\}_{n_k\geq 1}$, which converges weakly in $L^2(R)$. From (5.14), we infer that the weak $L^2(R)$-limit is $\{(a/2)(u)^2 + ((3b-a)/2)(u_x)^2\}$. It follows from $G_x \in L^2(R)$ that

$$\partial_x G * \left(\frac{a}{2}(u^{n_k})^2 + \frac{3b-a}{2}(u_x^{n_k})^2\right) \longrightarrow \partial_x G * \left(\frac{a}{2}(u)^2 + \frac{3b-a}{2}(u_x)^2\right) \quad \text{for } n_k \longrightarrow \infty. \tag{5.20}$$

From (5.14), (5.17), and(5.20), we have that u solves (2.1) in $D'((0,T)\times R)$.

For fixed $T > 0$, note that $u_t^{n_k}$ is uniformly bounded in $L^2(R)$ as $t \in [0,T)$ and $\|u^{n_k}(t)\|_{H^1(R)}$ is uniformly bounded for all $t \in [0,T)$ and $n \geq 1$, and we infer that the family $t \to u^{n_k} \in H^1(R)$ is weakly equicontinous on [0,T]. An application of the Arzela-Ascoli theorem yields that $\{u^{n_k}\}$ has a subsequence, denoted again $\{u^{n_k}\}$, which converges weakly in $H^1(R)$, uniformly in $t \in [0,T)$. The limit function is u. T being arbitrary, we have that u is locally and weakly continuous from $[0,\infty)$ into $H^1(R)$, that is, $u \in C_{w,\mathrm{loc}}(R_+; H^1(R))$.

Since, for a.e. $t \in R_+$, $u^{n_k}(t,\cdot) \rightharpoonup u(t,\cdot)$ weakly in $H^1(R)$, from Theorem 4.4, we get

$$\begin{aligned}
\|u(t,\cdot)\|_{L^\infty(R)} \leq \|u(t,\cdot)\|_{H^1(R)} &\leq \liminf_{n_k\to\infty}\|u^{n_k}(t,\cdot)\|_{H^1(R)} \\
&\leq \|u_0\|_{H^1} \exp\left[\frac{|a-2b|}{2\lambda}\left(1-e^{-\lambda t}\right)\|u_0\|_{L^1(R)} - \lambda t\right].
\end{aligned} \tag{5.21}$$

Inequality (5.21) shows that

$$u \in L^\infty_{\mathrm{loc}}(R_+\times R) \cap L^\infty_{\mathrm{loc}}\left(R_+; H^1(R)\right). \tag{5.22}$$

From Theorem 4.4, (5.1) and (5.2), for $t \in R_+$, we obtain

$$\|u_x^n(t,\cdot)\|_{L^\infty} \leq e^{-\lambda t}\left\|u_0^n\right\|_{L^1(R)} \leq e^{-\lambda t}\|u_0\|_{L^1(R)} \leq e^{-\lambda t}\left\|y_0\right\|_{M(R)}. \tag{5.23}$$

Combining with (5.14), we have

$$u_x \in L^\infty(R_+\times R). \tag{5.24}$$

Next, we will prove that $\int_R u(t,\cdot)dx = e^{-\lambda t}\int_R u(0,\cdot)dx$ by using a regularization approach.

Since u satisfies (2.1) in distribution sense, convoluting (2.1) with ρ_n, we have that, for a.e. $t \in R_+$,

$$\rho_n * u_t + \rho_n * (buu_x) + \rho_n * \partial_x p * \left[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2\right] + \lambda \rho_n * u = 0. \tag{5.25}$$

Integrating the above equation with respect to x on R, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_R \rho_n * u dx + \int_R \rho_n * (buu_x) dx \\ &\quad + \int_R \rho_n * \partial_x p * \left[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2\right] dx + \lambda \int_R \rho_n * u \, dx = 0. \end{aligned} \tag{5.26}$$

Integration by parts gives rise to

$$\frac{d}{dt}\int_R \rho_n * u dx = -\lambda \int_R \rho_n * u dx, \quad t \in R_+, \ n \geq 1. \tag{5.27}$$

Utilizing Lemma 3.3, we obtain that

$$\int_R \rho_n * u(t,\cdot) dx = e^{-\lambda t}\int_R \rho_n * u_0 dx. \tag{5.28}$$

Since

$$\lim_{n\to\infty} \left\|\rho_n * u(t,\cdot) - u(t,\cdot)\right\|_{L^1(R)} = \lim_{n\to\infty} \left\|\rho_n * u_0 - u_0\right\|_{L^1(R)} = 0. \tag{5.29}$$

it follows that, for a.e. $t \in R_+$,

$$\int_R u(t,\cdot) dx = \lim_{n\to\infty} \int_R \rho_n * u(t,\cdot) dx = \lim_{n\to\infty} e^{-\lambda t}\int_R \rho_n * u_0 dx = e^{-\lambda t}\int_R u_0 dx. \tag{5.30}$$

Finally, we prove that $(u(t,\cdot) - u_{xx}(t,\cdot)) \in M^+$ is uniformly bounded on R and $u(t,x) \in W^{1,\infty}(R_+ \times R)$.

Due to

$$L^1(R) \subset (L^\infty)^* \subset (C_0(R))^* = M(R), \tag{5.31}$$

from (5.18), we get that, for a.e. $t \in R_+$,

$$\begin{aligned} \|u(t,\cdot) - u_{xx}(t,\cdot)\|_{M(R)} &\leq \|u(t,\cdot)\|_{L^1(R)} + \|u_{xx}(t,\cdot)\|_{M(R)} \\ &\leq e^{-\lambda t}\|u_0\|_{L^1(R)} + 2e^{-\lambda t}\left\|y_0\right\|_{M(R)} \leq 3e^{-\lambda t}\left\|y_0\right\|_{M(R)}. \end{aligned} \tag{5.32}$$

The above inequality implies that, for a.e. $t \in R_+$, $(u(t,\cdot) - u_{xx}(t,\cdot)) \in M(R)$ is uniformly bounded on R. For fixed $T \geq 0$, applying (5.13) and (5.14), we have

$$\left[u^{n_k}(t,\cdot) - u^{n_k}_{xx}(t,\cdot)\right] \longrightarrow \left[u(t,\cdot) - u_{xx}(t,\cdot)\right] \quad \text{in } D'(R) \text{ for } n \longrightarrow \infty. \tag{5.33}$$

Since $(u^{n_k}(t,\cdot) - u^{n_k}_{xx}(t,\cdot)) \geq 0$ for all $(t,x) \in R_+ \times R$, we obtain that for a.e. $t \in R_+$, $(u(t,\cdot) - u_{xx}(t,\cdot)) \in M^+(R)$.

Note that $u(t,x) = G * (u(t,x) - u_{xx}(t,x))$. Then we get

$$\begin{aligned} |u(t,x)| = |G * (u(t,x) - u_{xx}(t,x))| &\leq \|G\|_{L^\infty(R)} \|u(t,x) - u_{xx}(t,x)\|_{M(R)} \\ &\leq 3e^{-\lambda t} \|y_0\|_{M(R)}. \end{aligned} \tag{5.34}$$

Combining with (5.24), it implies that $u(t,x) \in W^{1,\infty}(R_+ \times R)$.

This completes the proof of the existence of Theorem 5.1.

Next, we present the uniqueness proof of the Theorem 5.1.

Let $u, v \in W^{1,\infty}(R_+ \times R) \cap L^\infty_{loc}(R_+; H^1(R))$ be two global weak solutions of problem (2.1) with the same initial data u_0. Assume that $(u(t,\cdot) - u_{xx}(t,\cdot)) \in M^+(R)$ and $(v(t,\cdot) - v_{xx}(t,\cdot)) \in M^+(R)$ are uniformly bounded on R_+ and set

$$N := \sup_{t \in R_+} \left\{ \|u(t,\cdot) - u_{xx}(t,\cdot)\|_{M(R)} + \|v(t,\cdot) - v_{xx}(t,\cdot)\|_{M(R)} \right\}. \tag{5.35}$$

From assumption, we know that $N < \infty$. Then, for all $(t,x) \in R_+ \times R$,

$$\begin{aligned} |u(t,x)| &= |G * (u(t,x) - u_{xx}(t,x))| \\ &\leq \|G\|_{L^\infty(R)} \|u(t,x) - u_{xx}(t,x)\|_{M(R)} \leq \frac{N}{2}, \end{aligned} \tag{5.36}$$

$$\begin{aligned} |u_x(t,x)| &= |G_x * (u(t,x) - u_{xx}(t,x))| \\ &\leq \|G_x\|_{L^\infty(R)} \|u(t,x) - u_{xx}(t,x)\|_{M(R)} \leq \frac{N}{2}. \end{aligned} \tag{5.37}$$

Similarly,

$$|v(t,x)| \leq \frac{N}{2}, \quad |v_x(t,x)| \leq \frac{N}{2}, \quad (t,x) \in R_+ \times R. \tag{5.38}$$

Following the same procedure as in (5.1), we may also get that

$$\begin{aligned} \|u(t,x)\|_{L^1} &= \|G * (u(t,x) - u_{xx}(t,x))\|_{L^1(R)} \\ &\leq \|G\|_{L^1(R)} \|u(t,x) - u_{xx}(t,x)\|_{M(R)} \leq N, \\ \|u_x(t,x)\|_{L^1(R)} &= \|G_x * (u(t,x) - u_{xx}(t,x))\|_{L^1(R)} \\ &\leq \|G_x\|_{L^1(R)} \|u(t,x) - u_{xx}(t,x)\|_{M(R)} \leq N \end{aligned} \tag{5.39}$$

and, for all $(t,x) \in R_+ \times R$,

$$|v(t,x)| \le N, \quad |v_x(t,x)| \le N, \quad (t,x) \in R_+ \times R. \tag{5.40}$$

We define

$$w(t,x) := u(t,x) - v(t,x), \quad (t,x) \in R_+ \times R. \tag{5.41}$$

Convoluting (2.1) for u and v with ρ_n, we get that for a.e. $t \in R_+$ and all $n \ge 1$,

$$\rho_n * u_t + \rho_n * (buu_x) + \rho_n * \partial_x G * \left[\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2\right] + \lambda \rho_n * u = 0, \tag{5.42}$$

$$\rho_n * v_t + \rho_n * (bvv_x) + \rho_n * \partial_x G * \left[\frac{a}{2}v^2 + \frac{3b-a}{2}(v_x)^2\right] + \lambda \rho_n * v = 0. \tag{5.43}$$

Subtracting (5.43) from (5.42) and using Lemma 3.4, integration by parts shows that, for a.e. $t \in R_+$ and all $n \ge 1$

$$\begin{aligned}
\frac{d}{dt}\int_R |\rho_n * w| dx &= \int_R (\rho_n * w_t)\operatorname{sgn}(\rho_n * w) dx \\
&= -b\int_R (\rho_n * wu_x)\operatorname{sgn}(\rho_n * w) dx \\
&\quad - b\int_R (\rho_n * vw_x)\operatorname{sgn}(\rho_n * w) dx \\
&\quad - \frac{a}{2}\int_R (\rho_n * \partial_x G * w(u+v))\operatorname{sgn}(\rho_n * w) dx \\
&\quad - \frac{3b-a}{2}\int_R (\rho_n * \partial_x G * w_x(u_x+v_x))\operatorname{sgn}(\rho_n * w) dx \\
&\quad - \lambda\int_R (\rho_n * w)\operatorname{sgn}(\rho_n * w) dx.
\end{aligned} \tag{5.44}$$

Using (5.36)–(5.38) and Young's inequality to the first term on the right-hand side of (5.44) yields,

$$\begin{aligned}
&\left|\int_R (\rho_n * (wu_x))\operatorname{sgn}(\rho_n * w) dx\right| \\
&\quad \le \int_R |(\rho_n * (wu_x))| dx \\
&\quad \le \int_R |\rho_n * w||\rho_n * u_x| dx + \int_R |\rho_n * (wu_x) - (\rho_n * w)(\rho_n * u_x)| dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\rho_n * u_x\|_{L^\infty} \int_R |\rho_n * w| dx + \int_R |\rho_n * (wu_x) - (\rho_n * w)(\rho_n * u_x)| dx \\
&\leq \|\rho_n\|_{L^1} \|u_x\|_{L^\infty} \int_R |\rho_n * w| dx + \int_R |\rho_n * (wu_x) - (\rho_n * w)(\rho_n * u_x)| dx \\
&\leq \frac{N}{2} \int_R |\rho_n * w| dx + \int_R |\rho_n * (wu_x) - (\rho_n * w)(\rho_n * u_x)| dx.
\end{aligned}
\tag{5.45}
$$

Similarly, for the second term and the third term on the right-hand side of (5.44), we have

$$
\begin{aligned}
&\left| \int_R (\rho_n * (w_x v)) \operatorname{sgn}(\rho_n * w) dx \right| \\
&\quad \leq \int_R |(\rho_n * (w_x v))| \, dx \\
&\quad \leq \int_R |\rho_n * w_x| |\rho_n * v| dx + \int_R |\rho_n * (w_x v) - (\rho_n * w_x)(\rho_n * v)| dx \\
&\quad \leq \frac{N}{2} \int_R |\rho_n * w_x| dx + \int_R |\rho_n * (w_x v) - (\rho_n * w_x)(\rho_n * v)| dx, \\
&\left| \int_R (\rho_n * \partial_x G * [w(u+v)]) \operatorname{sgn}(\rho_n * w) dx \right| \\
&\quad \leq \int_R |\rho_n * G * [w_x(u+v)]| dx \\
&\qquad + \int_R |\rho_n * G * [w(u+v)_x]| dx \\
&\quad \leq \frac{1}{2} \int_R |\rho_n * [w_x(u+v)]| dx + \frac{1}{2} \int_R |\rho_n * [w(u_x + v_x)]| dx \\
&\quad \leq \frac{N}{2} \int_R |\rho_n * w_x| dx + \frac{N}{2} \int_R |\rho_n * w| dx \\
&\qquad + \int_R |\rho_n * (w_x(u+v)) - (\rho_n * w_x)[\rho_n * (u+v)]| dx \\
&\qquad + \int_R |\rho_n * [w(u+v)_x] - (\rho_n * w)[\rho_n * (u+v)_x]| dx.
\end{aligned}
\tag{5.46}
$$

For the last term on the right-hand side of (5.44), we have

$$
\begin{aligned}
&\left| \int_R (\rho_n * \partial_x G * [w_x(u+v)_x]) \operatorname{sgn}(\rho_n * w) dx \right| \\
&\quad \leq \int_R \rho_n * \partial_x G * [|w_x|(|u_x| + |v_x|)] dx
\end{aligned}
$$

$$
\begin{aligned}
&\le N\int_R \left|\rho_n * \partial_x G * |w_x|\right| dx \\
&\le N\|\partial_x G\|_{L^1(R)} \int_R \rho_n * |w_x| dx \\
&\le N\int_R \left|\rho_n * w_x\right| dx + N\left[\int_R \left(\rho_n * |w_x| - \left|\rho_n * w_x\right|\right) dx\right].
\end{aligned}
\tag{5.47}
$$

From (5.45)–(5.47), for a.e. $t \in R_+$ and all $n \ge 1$, we find

$$
\begin{aligned}
\frac{d}{dt}\int_R \left|\rho_n * w\right| dx \le \left(\frac{a+2b}{4}N + \lambda\right)\int_R \left|\rho_n * w\right| dx \\
+ (a+2b)N\int_R \left|\rho_n * w_x\right| dx + R_n(t),
\end{aligned}
\tag{5.48}
$$

where

$$
\begin{aligned}
&R_n(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \\
&|R_n(t)| \le K, \quad n \ge 1,\ t \in R_+,
\end{aligned}
\tag{5.49}
$$

where K is a positive constant depending on N and the $H^1(R)$-norms of $u(0)$ and $v(0)$.

In the same way, convoluting (2.1) for u and v with $\rho_{n,x}$ and using Lemma 3.4, we get that, for a.e. $t \in R_+$ and all $n \ge 1$,

$$
\begin{aligned}
\frac{d}{dt}\int_R \left|\rho_n * w_x\right| dx &= \int_R \left(\rho_n * w_{xt}\right) \operatorname{sgn}\left(\rho_{n,x} * w\right) dx \\
&= -b\int_R \left(\rho_n * w_x(u_x + v_x)\right) \operatorname{sgn}\left(\rho_{n,x} * w\right) dx \\
&\quad - b\int_R \left(\rho_n * v_{xx} w\right) \operatorname{sgn}\left(\rho_{n,x} * w\right) dx \\
&\quad - b\int_R \left(\rho_n * u w_{xx}\right) \operatorname{sgn}\left(\rho_{n,x} * w\right) dx \\
&\quad - \int_R \left(\rho_n * \partial_{xx} G * \left[\frac{a}{2}(u_2 - v_2) + \frac{3b-a}{2}\left(u_x^2 - v_x^2\right)\right] \operatorname{sgn}\left(\rho_{n,x} * w\right) dx \right.\\
&\quad - \lambda\int_R \left(\rho_n * w_x\right) \operatorname{sgn}\left(\rho_{n,x} * w\right) dx.
\end{aligned}
\tag{5.50}
$$

Using the identity $\partial_x^2(G*g)=G*g-g$ for $g\in L^2(R)$ and Young's inequality, we estimate the forth term of the right-hand side of (5.50)

$$\begin{aligned}
&\left|\int_R \left(\rho_n * \partial_{xx}G * \left[\frac{a}{2}\left(u^2-v^2\right)+\frac{3b-a}{2}\left(u_x^2-v_x^2\right)\right]dx\right|\\
&\quad\le \int_R\left|\left(\rho_n * G * \left[\frac{a}{2}\left(u^2-v^2\right)+\frac{3b-a}{2}\left(u_x^2-v_x^2\right)\right]\right|dx\\
&\qquad+\int_R\left|\left(\rho_n * \left[\frac{a}{2}\left(u^2-v^2\right)+\frac{3b-a}{2}\left(u_x^2-v_x^2\right)\right]\right|dx\\
&\quad\le \left(\|G\|_{L^1(R)}+1\right)\int_R\left|\left(\rho_n * \left[\frac{a}{2}w(u+v)+\frac{3b-a}{2}w_x(u_x+v_x)\right]\right|dx\\
&\quad\le a\int_R\left|\left(\rho_n*[w(u+v)]\right|dx+(3b+a)\int_R\left|\left(\rho_n*[w_x(u_x+v_x)]\right|dx\\
&\quad\le aN\int_R\left|\rho_n*w\right|dx+(3b+a)N\int_R\left|\rho_n*w_x\right|dx+R_n.
\end{aligned}\tag{5.51}$$

Using (5.36)–(5.38) and Young's inequality to the first term on the right-hand side of (5.50) gives rise to

$$\begin{aligned}
&-b\int_R\left(\rho_n*w_x(u_x+v_x)\right)\operatorname{sgn}(\rho_{n,x}*w)dx\\
&\quad\le b\int_R\left|\rho_n*w_x(u_x+v_x)\right|dx\\
&\quad\le b\int_R\left|\rho_n*w_x\right|\left|\rho_n*(u_x+v_x)\right|dx\\
&\qquad+b\int_R\left|\rho_n*w_x(u_x+v_x)-(\rho_n*w_x)(\rho_n*(u_x+v_x))\right|dx\\
&\quad\le bN\int_R\left|\rho_n*w_x\right|dx+R_n.
\end{aligned}\tag{5.52}$$

To treat the second term of the right-hand side of (5.50), we note that

$$\begin{aligned}
&\left|b\int_R\left(\rho_n*v_{xx}w\right)\operatorname{sgn}(\rho_{n,x}*w)dx\right|\\
&\quad\le b\int_R\left|(\rho_n*w)(\rho_n*v_{xx})\right|dx\\
&\qquad+b\int_R\left|(\rho_n*v_{xx}w)-(\rho_n*w)(\rho_n*v_{xx})\right|dx.
\end{aligned}\tag{5.53}$$

Applying Lemma 3.1, the second expression of the right-hand side of (5.53) can be estimated by a function $R_n(t)$ belonging to (5.49). Making use of the Hölder inequality and (5.1), for a.e. $t \in R_+$ and all $n \geq 1$, we have

$$\begin{aligned}\int_R \left|(\rho_n * w)(\rho_n * v_{xx})\right| dx &\leq \|\rho_n * w\|_{L^\infty(R)} \|\rho_n * v_{xx}\|_{L^1(R)} \\ &\leq \|\rho_n * w\|_{W^{1,1}(R)} \|v_{xx}\|_{M(R)}.\end{aligned} \tag{5.54}$$

It follows from (5.53) and (5.54) that

$$\begin{aligned}&\left| b \int_R (\rho_n * v_{xx} w) \operatorname{sgn}(\rho_{n,x} * w) dx \right| \\ &\quad\leq \mathrm{bN} \int_R |\rho_n * w| dx \\ &\qquad+ \mathrm{bN} \int_R |\rho_n * w_x| + R_n(t).\end{aligned} \tag{5.55}$$

Now, we deal with the third term on the right-hand side of (5.50)

$$\begin{aligned}&- b \int_R (\rho_n * u w_{xx}) \operatorname{sgn}(\rho_{n,x} * w) dx \\ &\quad= -b \int_R (\rho_n * u)(\rho_n * w_{xx}) \operatorname{sgn}(\rho_{n,x} * w) dx \\ &\qquad- b \int_R \left[(\rho_n * u w_{xx}) - (\rho_n * u)(\rho_n * w_{xx})\right] \operatorname{sgn}(\rho_{n,x} * w) dx \\ &\quad\leq -b \int_R (\rho_n * u) \frac{\partial}{\partial_x} |\rho_n * w_x| dx \\ &\qquad+ b \int_R \left|(\rho_n * u w_{xx}) - (\rho_n * u)(\rho_n * w_{xx})\right| dx \\ &\quad= b \int_R (\rho_n * u_x) |\rho_n * w_x| dx + R_n.\end{aligned} \tag{5.56}$$

Therefore, (5.56) implies that, for a.e. $t \in R_+$ and all $n \geq 1$,

$$\left| -b \int_R (\rho_n * u w_{xx}) \operatorname{sgn}(\rho_{n,x} * w) dx \right| \leq bN \int_R |\rho_n * w_x| dx + R_n. \tag{5.57}$$

From (5.51), (5.52), (5.55), and (5.57), for a.e. $t \in R_+$ and all $n \geq 1$, we deduce that

$$\frac{d}{dt}\int_R |\rho_n * w_x| dx \leq (a+b)N \int_R |\rho_n * w| dx + [(6b+a)N+\lambda] \int_R |\rho_n * w_x| dx + R_n. \tag{5.58}$$

Combining with (5.48) and (5.58), we find

$$\begin{aligned}\frac{d}{dt}\int_R (|\rho_n * w| + |\rho_n * w_x|) dx &\leq \left[\frac{5a+6b}{4}N+\lambda\right] \int_R |\rho_n * w| dx \\ &\quad + [(2a+8b)N+\lambda] \int_R |\rho_n * w_x| dx + R_n \\ &\leq [(2a+8b)N+\lambda] \int_R (|\rho_n * w| + |\rho_n * w_x|) dx + R_n.\end{aligned} \tag{5.59}$$

It follows from Gronwall' inequality that, for a.e. $t \in R_+$ and all $n \geq 1$,

$$\begin{aligned}&\int_R (|\rho_n * w| + |\rho_n * w_x|) dx \\ &\quad\leq \left[\int_0^t R_n(s) ds + \int_R (|\rho_n * w| + |\rho_n * w_x|)(0,x) dx\right] e^{[(2a+8b)N+\lambda]t}.\end{aligned} \tag{5.60}$$

Fix $t > 0$, and let $n \to \infty$ in (5.60). Since $w = u - v \in W^{1,1}(R)$ and relation (5.49) holds, making use of Lebesgue's dominated convergence theorem yields

$$\int_R (|w| + |w_x|) dx \leq \left[\int_R (|w| + |w_x|)(0,x) dx\right] e^{[(2a+8b)N+\lambda]t}. \tag{5.61}$$

Note that $w(0) = w_x(0) = 0$; therefore, we obtain $u(t,x) = v(t,x)$ for a.e. $(t,x) \in R_+ \times R$. This completes the proof of the theorem. □

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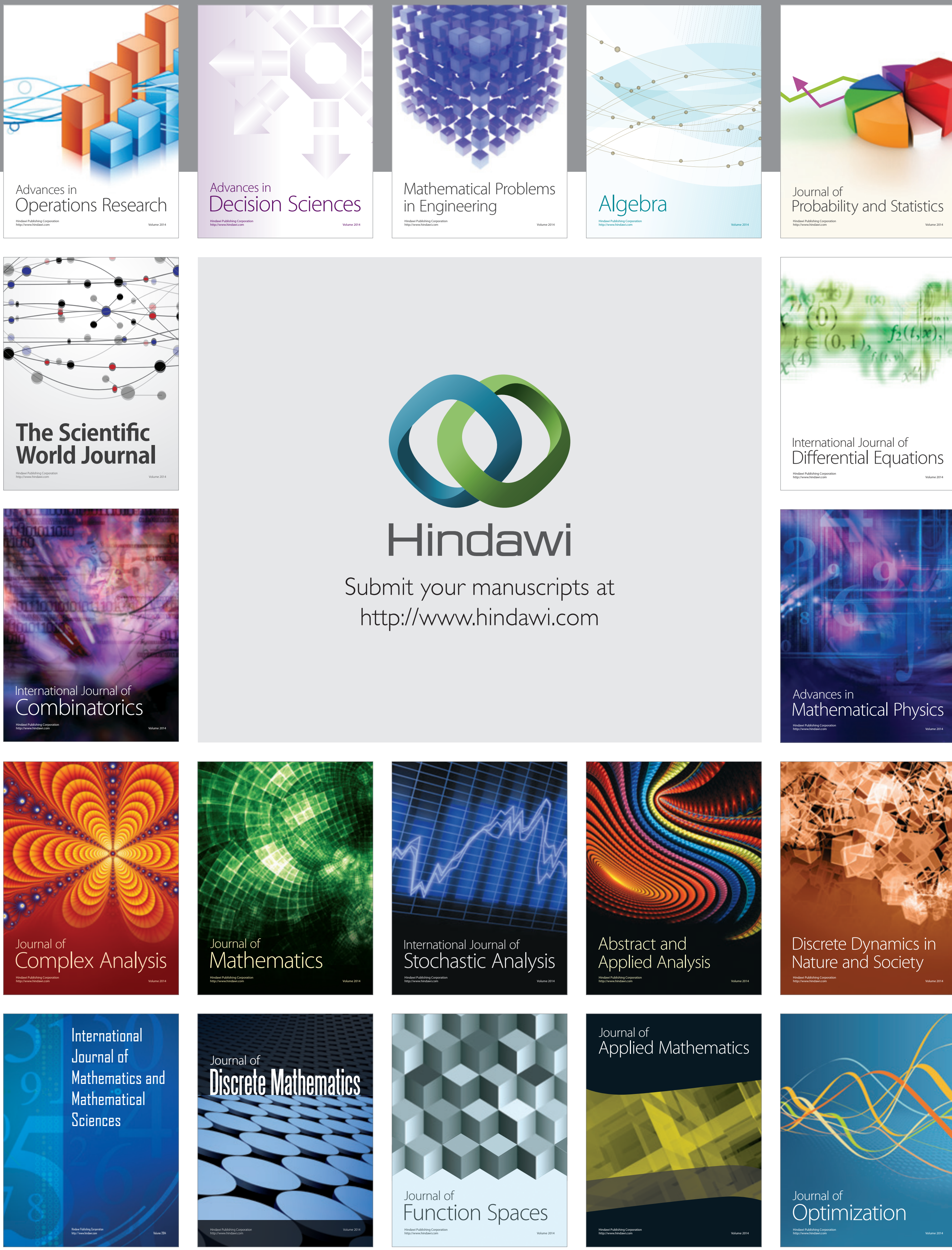
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