Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 872784, 12 pages
doi:10.1155/2012/872784

Research Article

On t-Derivations of BCI-Algebras

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Received 29 January 2012; Revised 19 April 2012; Accepted 20 April 2012

Academic Editor: Gerd Teschke

We introduce the notion of t-derivation of a BCI-algebra and investigate related properties. Moreover, we study t-derivations in a p-semisimple BCI-algebra and establish some results on t-derivations in a p-semisimple BCI-algebra.

1. Introduction

The notion of BCK-algebra was proposed by Imai and Iséki in 1966 [1]. In the same year, Iséki introduced the notion of a BCI-algebra [2], which is a generalization of a BCK-algebra. A series of interesting notions concerning BCI-algebras were introduced and studied, several papers have been written on various aspects of these algebras [3–5]. Recently, in the year 2004 [6], Jun and Xin have applied the notion of derivation in BCI-algebras which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [7]. In fact, the notion of derivation in ring theory is quite old and plays a significant role in analysis, algebraic geometry and algebra.

After the work of Jun and Xin (2004) [6], many research articles have appeared on the derivations of BCI-algebras in different aspects as follows: in 2005 [8], Zhan and Liu have given the notion of f-derivation of BCI-algebras and studied p-semisimple BCI-algebras by using the idea of regular f-derivation in BCI-algebras. In 2006 [9], Abujabal and Al-Shehri have extended the results of BCI-algebras. Further, in the next year 2007 [10], they defined and studied the notion of left derivation of BCI-algebras and investigated some properties of left derivation in p-semisimple BCI-algebras. In 2009 [11], Öztürk and Çeven have defined the notion of derivation and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [12], Öztürk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and explored some properties. In 2010 [13], Al-Shehri has applied the notion of left-right (resp., right-left) derivation in BCI-algebra to B-algebra and obtained some of its properties. In 2011 [14], Ilbira et al. have studied the notion of left-right (resp., right-left) symmetric biderivation in BCI-algebras.

Motivated by a lot of work done on derivations of BCI-algebras and on derivations of other related abstract algebraic structures, in this paper we introduce the notion of t-derivations on BCI-algebras and obtain some of its related properties. Further, we characterize the notion of p-semisimple BCI-algebra X by using the notion of t-derivation and show that if d_t and d'_t are t-derivations on X, then $d_t \circ d'_t$ is also a t-derivation and $d_t \circ d'_t = d'_t \circ d_t$. Finally, we prove that $d_t * d'_t = d'_t * d_t$, where d_t and d'_t are t-derivations on a p-semisimple BCI-algebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

Definition 2.1 (see [2]). Let X be a set with a binary operation "$*$" and a constant 0. Then $(X, *, 0)$ is called a BCI algebra if the following axioms are satisfied for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Define a binary relation $\leq$ on X by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra (see [1]).

In any BCI-algebra X for all $x, y \in X$, the following properties hold.

(1) $(x * y) * z = (x * z) * y$.

(2) $x * 0 = x$.

(3) $(x * z) * (y * z) \leq x * y$.

(4) $x * 0 = 0$ implies $x = 0$.

(5) $x \leq y \Leftrightarrow x * z \leq y * z$ and $z * y \leq z * x$. A BCI-algebra X is said to be associative if for all $x, y, z \in X$, the following holds:

(6) $(x*y)*z = x*(y*z)$ [4]. Let X be a BCI-algebra, we denote $X_+ = \{x \in X \mid 0 \leq x\}$, the BCK-part of X and by $G(X) = \{x \in X \mid 0 * x = x\}$, the BCI-G part of X. If $X_+ = \{0\}$, then X is called a p-semisimple BCI-algebra. In a p-semisimple BCI-algebra X, the following properties hold.

(7) $x * (x * y) = y$.

(8) $x * (0 * y) = y * (0 * x)$.

(9) $x * y = 0$ implies $x = y$.

(10) $(x * z) * (y * z) = x * y$.

(11) $x * a = x * b$ implies $a = b$ that is left cancelable.

(12) $a * x = b * x$ implies $a = b$ that is right cancelable.

Definition 2.2 (see [6]). A subset S of a BCI-algebra X is called subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

For a BCI-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$ [6]. For more details we refer to [3, 5, 6].

3. t-Derivations in a BCI-Algebra/p-Semisimple BCI-Algebra

The following definitions introduce the notion of t-derivation for a BCI-algebra.

Definition 3.1. Let X be a-BCI-algebra. Then for any $t \in X$, we define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$.

Definition 3.2. Let X be a BCI-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a left-right t-derivation or (l, r)-t-derivation of X if it satisfies the identity $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all $x, y \in X$.

Similarly, we get the following.

Definition 3.3. Let X be a BCI-algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a right-left t-derivation or (r, l)-t-derivation of X if it satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$.

Moreover, if d_t is both a (l, r)- and a (r, l)-t-derivation on X, we say that d_t is a t-derivation on X.

Example 3.4. Let $X = \{0, 1, 2\}$ be a BCI-algebra with the following Cayley table:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{array} \tag{3.1}$$

For any $t \in X$, define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$. Then it is easily checked that d_t is a t-derivation of X.

Proposition 3.5. *Let d_t be a self map of an associative BCI-algebra X. Then d_t is a (l, r)-t-derivation of X.*

Proof. Let X be an associative BCI-algebra, then we have

$$\begin{aligned} d_t(x * y) &= (x * y) * t \\ &= \{x * (y * t)\} * 0 \quad \text{by Property (6) and (2)} \\ &= \{x * (y * t)\} * [\{x * (y * t)\} * \{x * (y * t)\}] \quad \text{by Property (iii)} \\ &= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * y) * t\}] \quad \text{by Property (6)} \end{aligned}$$

$$\begin{aligned}
&= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * t) * y\}] \quad \text{by Property (1)} \\
&= ((x * t) * y) \wedge (x * (y * t)) \\
&= (d_t(x) * y) \wedge (x * d_t(y)).
\end{aligned} \tag{3.2}$$

□

Proposition 3.6. *Let d_t be a self map of an associative BCI-algebra X. Then, d_t is a (r, l)-t-derivation of X.*

Proof. Let X be an associative BCI-algebra, then we have

$$\begin{aligned}
d_t(x * y) &= (x * y) * t \\
&= \{(x * t) * y\} * 0 \quad \text{by Property (1) and (2)} \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * t) * y\}] \quad (\text{as } x * x = 0) \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * y) * t\}] \quad \text{by Property (1)} \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{x * (y * t)\}] \quad \text{by Property (6)} \\
&= (x * (y * t)) \wedge ((x * t) * y) \quad (\text{as } y * (y * x) = x \wedge y) \\
&= (x * d_t(y)) \wedge (d_t(x) * y).
\end{aligned} \tag{3.3}$$

Combining Propositions 3.5 and 3.6, we get the following Theorem. □

Theorem 3.7. *Let d_t be a self map of an associative BCI-algebra X. Then, d_t is a t-derivation of X.*

Definition 3.8. A self map d_t of a BCI-algebra X is said to be t-regular if $d_t(0) = 0$.

Example 3.9. Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(3.4)

(i) For any $t \in X$, define a self map $d_t : X \to X$ by

$$d_t(x) = x * t = \begin{cases} b & \text{if } x = 0, a \\ 0 & \text{if } x = b. \end{cases} \tag{3.5}$$

Then it is easily checked that d_t is (l, r) and (r, l)-t-derivations of X, which is not t-regular.

(ii) For any $t \in X$, define a self map $d_t' : X \to X$ by

$$d_t'(x) = x * t = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b. \end{cases} \tag{3.6}$$

Then it is easily checked that d_t' is (l, r) and (r, l)-t-derivations of X, which is t-regular.

Proposition 3.10. *Let d_t be a self map of a BCI-algebra X. Then*

(i) *If d_t is a (l, r)-t-derivation of X, then $d_t(x) = d_t(x) \wedge x$ for all $x \in X$.*

(ii) *If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if d_t is t-regular.*

Proof of (i). Let d_t be a (l, r)-t-derivation of X, then

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (d_t(x) * 0) \wedge (x * d_t(0)) \\ &= d_t(x) \wedge \{x * d_t(0)\} \\ &= \{x * d_t(0)\} * [\{x * d_t(0)\} * d_t(x)] \\ &= \{x * d_t(0)\} * [\{x * d_t(x)\} * d_t(0)] \\ &\leq x * \{x * d_t(x)\} \quad \text{by Property (3)} \\ &= d_t(x) \wedge x. \end{aligned} \tag{3.7}$$

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (i) holds. □

Proof of (ii). Let d_t be a (r, l)-t-derivation of X. If $d_t(x) = x \wedge d_t(x)$ then

$$\begin{aligned} d_t(0) &= 0 \wedge d_t(0) \\ &= d_t(0) * \{d_t(0) * 0\} \\ &= d_t(0) * d_t(0) \\ &= 0 \end{aligned} \tag{3.8}$$

thereby implying d_t is t-regular. Conversely, suppose that d_t is t-regular, that is $d_t(0) = 0$, then we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (x * d_t(0)) \wedge (d_t(x) * 0) \\ &= (x * 0) \wedge d_t(x) \\ &= x \wedge d_t(x). \end{aligned} \tag{3.9}$$

This completes the proof. □

Theorem 3.11. *Let d_t be a (l,r)-t-derivation of a p-semisimple BCI-algebra X. Then the following hold:*

(i) $d_t(0) = d_t(x) * x$ *for all* $x \in X$.

(ii) d_t *is one-one.*

(iii) *If* d_t *is t-regular, then it is an identity map.*

(iv) *if there is an element* $x \in X$ *such that* $d_t(x) = x$, *then* d_t *is identity map.*

(v) *if* $x \le y$, *then* $d_t(x) \le d_t(y)$ *for all* $x, y \in X$.

Proof of (i). Let d_t be a (l,r)-t-derivation of a p-semisimple BCI-algebra X. Then for all $x \in X$, we have $x * x = 0$ and so

$$\begin{aligned} d_t(0) &= d_t(x * x) \\ &= (d_t(x) * x) \wedge (x * d_t(x)) \\ &= \{x * d_t(x)\} * [\{x * d_t(x)\} * \{d_t(x) * x\}] \\ &= d_t(x) * x \quad \text{by property (7).} \end{aligned} \tag{3.10}$$

□

Proof of (ii). Let $d_t(x) = d_t(y) \Longrightarrow x * t = y * t$, then by property (12), we have $x = y$ and so d_t is one-one. □

Proof of (iii). Let d_t be t-regular and $x \in X$. Then, $0 = d_t(0)$ so by the above part (i), we have $0 = d_t(x) * x$ and hence by property (9), we obtain $d_t(x) = x$ for all $x \in X$. Therefore, d_t is the identity map. □

Proof of (iv). It is trivial and follows from the above part (iii). □

Proof of (v). Let $x \le y$ implying $x * y = 0$. Now,

$$\begin{aligned} d_t(x) * d_t(y) &= (x * t) * (y * t) \\ &= x * y \quad \text{by property (10)} \\ &= 0. \end{aligned} \tag{3.11}$$

Therefore, $d_t(x) \le d_t(y)$. This completes the proof. □

Definition 3.12. Let d_t be a t-derivation of a BCI-algebra X. Then, d_t is said to be an isotone t-derivation if $x \le y \Longrightarrow d_t(x) \le d_t(y)$ for all $x, y \in X$.

Example 3.13. In Example 3.9(ii), d'_t is an isotone t-derivation, while in Example 3.9(i), d_t is not an isotone t-derivation.

Proposition 3.14. *Let X be a BCI-algebra and* d_t *be a t-derivation on X. Then for all* $x, y \in X$, *the following hold:*

(i) *If* $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$, *then* d_t *is an isotone t-derivation.*

(ii) *If* $d_t(x * y) = d_t(x) * d_t(y)$, *then* d_t *is an isotone t-derivation.*

Proof of (i). Let $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$. If $x \leq y \Longrightarrow x \wedge y = x$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x \wedge y) \\ &= d_t(x) \wedge d_t(y) \\ &\leq d_t(y). \end{aligned} \tag{3.12}$$

Henceforth $d_t(x) \leq d_t(y)$ which implies that d_t is an isotone t-derivation. □

Proof of (ii). Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \leq y \Longrightarrow x * y = 0$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= d_t\{x * (x * y)\} \\ &= d_t(x) * d_t(x * y) \\ &= d_t(x) * \{d_t(x) * d_t(y)\} \\ &\leq d_t(y) \quad \text{by property (ii).} \end{aligned} \tag{3.13}$$

Thus, $d_t(x) \leq d_t(y)$. This completes the proof. □

Theorem 3.15. *Let d_t be a t-regular (r, l)-t-derivation of a BCI-algebra X. Then, the following hold:*

(i) $d_t(x) \leq x$ *for all* $x \in X$.

(ii) $d_t(x) * y \leq x * d_t(y)$ *for all* $x, y \in X$.

(iii) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ *for all* $x, y \in X$.

(iv) $\ker(d_t) := \{x \in X : d_t(x) = 0\}$ *is a subalgebra of* X.

Proof of (i). For any $x \in X$, we have $d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$. □

Proof of (ii). Since $d_t(x) \leq x$ for all $x \in X$, then $d_t(x) * y \leq x * y \leq x * d_t(y)$ and hence the proof follows. □

Proof of (iii). For any $x, y \in X$, we have

$$\begin{aligned} d_t(x * y) &= (x * d_t(y)) \wedge (d_t(x) * y) \\ &= \{d_t(x) * y\} * [\{d_t(x) * y\} * \{x * d_t(y)\}] \\ &= \{d_t(x) * y\} * 0 \\ &= d_t(x) * y \leq d_t(x) * d_t(y). \end{aligned} \tag{3.14}$$

□

Proof of (iv). Let $x, y \in \ker(d_t) \Longrightarrow d_t(x) = 0 = d_t(y)$. From (iii), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$. Therefore, $x * y \in \ker(d_t)$. Consequently $\ker(d_t)$ is a subalgebra of X. This completes the proof. □

Definition 3.16. Let X be a BCI-algebra and let d_t, d_t' be two self maps of X. Then we define $d_t \circ d_t' : X \to X$ by $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Example 3.17. Let $X = \{0, a, b\}$ be a BCI algebra which is given in Example 3.4. Let d_t and d_t' be two self maps on X as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map $d_t \circ d_t' : X \to X$ by

$$(d_t \circ d_t')(x) = \begin{cases} 0 & \text{if } x = a, b \\ b & \text{if } x = 0. \end{cases} \tag{3.15}$$

Then, it is easily checked that $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Proposition 3.18. *Let X be a p-semisimple BCI-algebra X and let d_t, d_t' be (l, r)-t-derivations of X. Then, $d_t \circ d_t'$ is also a (l, r)-t-derivation of X.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d_t' are (l, r)-t-derivations of X. Then for all $x, y \in X$, we get

$$\begin{aligned}
(d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) \\
&= d_t\left[(d_t'(x) * y) \ \wedge (x * d_t'(y))\right] \\
&= d_t\left[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}\right] \\
&= d_t(d_t'(x) * y) \quad \text{by property (7)} \\
&= \{x * d_t(d_t'(y))\} * \left[\{x * d_t(d_t'(y))\} * \{d_t(d_t'(x) * y)\}\right] \\
&= \{d_t(d_t'(x) * y)\} \wedge \{x * d_t(d_t'(y))\} \\
&= ((d_t \circ d_t')(x) * y) \wedge (x * (d_t \circ d_t')(y)).
\end{aligned} \tag{3.16}$$

Therefore, $(d_t \circ d_t')$ is a (l, r)-t-derivation of X.

Similarly, we can prove the following. □

Proposition 3.19. *Let X be a p-semisimple BCI-algebra and let d_t, d_t' be (r, l)-t-derivations of X. Then $d_t \circ d_t'$ is also a (r, l)-t-derivation of X.*

Combining Propositions 3.18 and 3.19, we get the following.

Theorem 3.20. *Let X be a p-semisimple BCI-algebra and let d_t, d_t' be t-derivations of X. Then, $d_t \circ d_t'$ is also a t-derivation of X.*

Now, we prove the following theorem.

Theorem 3.21. *Let X be a p-semisimple BCI-algebra and let d_t, d_t' be t-derivations of X. Then $d_t \circ d_t' = d_t' \circ d_t$.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d'_t, t-derivations of X. Suppose d'_t is a (l,r)-t-derivation, then for all $x, y \in X$, we have

$$\begin{aligned}(d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t[(d'_t(x) * y) \wedge (x * d'_t(y))] \\ &= d_t[(x * d'_t(y)) * \{(x * d'_t(y)) * (d'_t(x) * y)\}] \\ &= d_t(d'_t(x) * y) \quad \text{by property (7).}\end{aligned} \tag{3.17}$$

As d_t is a (r,l)-t-derivation, then

$$\begin{aligned}&= (d'_t(x) * d_t(y)) \wedge (d_t(d'_t(x)) * y) \\ &= d'_t(x) * d_t(y).\end{aligned} \tag{3.18}$$

Again, if d_t is a (r,l)-t-derivation, then we have

$$\begin{aligned}(d'_t \circ d_t)(x * y) &= d'_t[d_t(x * y)] \\ &= d'_t[(x * d_t(y)) \wedge (d_t(x) * y)\ ] \\ &= d'_t[x * d_t(y)] \quad \text{by property (7)}\end{aligned} \tag{3.19}$$

But d'_t is a (l,r)-t-derivation, then

$$\begin{aligned}&= (d'_t(x) * d_t(y)) \wedge (x * d'_t(d_t(y))) \\ &= d'_t(x) * d_t(y).\end{aligned} \tag{3.20}$$

Therefore from (3.18) and (3.20), we obtain

$$(d_t \circ d'_t)(x * y) = (d'_t \circ d_t)(x * y). \tag{3.21}$$

By putting $y = 0$, we get

$$(d_t \circ d'_t)(x) = (d'_t \circ d_t)(x) \quad \forall x \in X. \tag{3.22}$$

Hence, $d_t \circ d'_t = d'_t \circ d_t$. This completes the proof. □

Definition 3.22. Let X be a BCI-algebra and let d_t, d'_t be two self maps of X. Then we define $d_t * d'_t : X \to X$ by $(d_t * d'_t)(x) = d_t(x) * d'_t(x)$ for all $x \in X$.

Example 3.23. Let $X = \{0, a, b\}$ be a BCI algebra which is given in Example 3.4. Let d_t and d_t' be two self maps on X as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map $d_t * d_t' : X \to X$ by

$$(d_t * d_t')(x) = \begin{cases} 0 & \text{if } x = a, b \\ b & \text{if } x = 0. \end{cases} \tag{3.23}$$

Then, it is easily checked that $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all $x \in X$.

Theorem 3.24. *Let X be a p-semisimple BCI-algebra and let d_t, d_t' be t-derivations of X. Then $d_t * d_t' = d_t' * d_t$.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d_t', t-derivations of X.

Since d_t' is a (r, l)-t-derivation of X, then for all $x, y \in X$, we have

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\ &= d_t[(x * d_t'(y)) \wedge (d_t'(x) * y)] \\ &= d_t[x * d_t'(y)] \quad \text{by property (7)}. \end{aligned} \tag{3.24}$$

But d_t is a (l, r)-t-derivation, so

$$\begin{aligned} &= (d_t(x) * d_t'(y)) \wedge (x * d_t(d_t'(y))) \\ &= d_t(x) * d_t'(y). \end{aligned} \tag{3.25}$$

Again, if d_t' is a (l, r)-t-derivation of X, then for all $x, y \in X$, we have

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\ &= d_t[(d_t'(x) * y) \wedge (x * d_t'(y))] \\ &= d_t[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}] \\ &= d_t(d_t'(x) * y) \quad \text{by property (7)}. \end{aligned} \tag{3.26}$$

As d_t is a (r, l)-t-derivation, then

$$\begin{aligned} &= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y) \\ &= d_t'(x) * d_t(y). \end{aligned} \tag{3.27}$$

Henceforth from (3.25) and (3.27), we conclude

$$d_t(x) * d'_t(y) = d'_t(x) * d_t(y) \tag{3.28}$$

By putting $y = x$, we get

$$\begin{aligned} d_t(x) * d'_t(x) &= d'_t(x) * d_t(x) \\ (d_t * d'_t)(x) &= (d'_t * d_t)(x) \quad \forall x \in X. \end{aligned} \tag{3.29}$$

Hence, $d_t * d'_t = d'_t * d_t$. This completes the proof. □

4. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galois theory (namely, Suzuki [15] and Van der Put and Singer [16, 17]) and the theory of invariants. An extensive and deep theory has been developed for derivations in algebraic structures viz. BCI-algebras, C^*-algebras, commutative Banach algebras and Galois theory of linear differential equations (see, e.g., Jun and Xin [6], Ara and Mathieu [18], Bonsall and Duncan [19], Murphy [20] and Villena [21] where further references can be found). It plays a significant role in functional analysis; algebraic geometry; algebra and linear differential equations.

In the present paper, we have considered the notion of t-derivations in BCI-algebras and investigated the useful properties of the t-derivations in BCI-algebras. Finally, we investigated the notion of t-derivations in a p-semisimple BCI-algebra and established some results on t-derivations in a p-semisimple BCI-algebra. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras [11], B-algebras [13], MV-algebras [22], d-algebras, Q-algebras and so forth. In future we can study the notion of t-derivations on various algebraic structures which may have a lot of applications in different branches of theoretical physics, engineering and computer science. It is our hope that this work would serve as a foundation for the further study in the theory of derivations of BCK/BCI-algebras.

In our future study of t-derivations in BCI-algebras, may be the following topics should be considered:

(1) to find the generalized t-derivations of BCI-algebras,

(2) to find more results in t-derivations of BCI-algebras and its applications,

(3) to find the t-derivations of B-algebras, Q-algebras, subtraction algebras, d-algebra and so forth.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous referees for their valuable comments and several useful suggestions. This research is supported by the Deanship of Scientific Research, University of Tabuk, Tabuk, Saudi Arabia.

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