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Research Article

The Univalence Conditions of Some Integral Operators

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We introduce new integral operators of analytic functions f and g defined in the open unit disk \mathbb{U} . For these operators, we discuss some univalence conditions.

1. Introduction and Preliminaries

Let \mathcal{A} denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$
 (1.1)

which are analytic in the open unit disk

$$\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \tag{1.2}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0. (1.3)$$

Also, let \mathcal{S} denote the subclass of \mathcal{A} consisting of functions f, which are univalent in \mathbb{U} (see, for details [1]; see also [2, 3]).

In [4, 5], Pescar gave the following univalence conditions for the functions $f \in \mathcal{A}$.

Theorem 1.1 (see [4]). Let α be a complex number, $\text{Re } \alpha > 0$, and c a complex number, $|c| \le 1$, $c \ne -1$, and $f(z) = z + \cdots$ a regular function in \mathbb{U} . If

$$\left|c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)}\right| \le 1,\tag{1.4}$$

for all $z \in \mathbb{U}$, then the function

$$F_{\alpha}(z) = \left(\alpha \int_{0}^{z} t^{\alpha - 1} f'(t) dt\right)^{1/\alpha} = z + \cdots$$
 (1.5)

is regular and univalent in \mathbb{U} .

Theorem 1.2 (see [5]). Let α be a complex number, $\operatorname{Re} \alpha > 0$, and c a complex number, $|c| \le 1$, $c \ne -1$, and $f \in \mathcal{A}$. If

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \le 1-|c| \tag{1.6}$$

for all $z \in \mathbb{U}$, then for any complex number β , Re $\beta \ge \text{Re } \alpha$, the function

$$F_{\beta}(z) = \left(\beta \int_0^z t^{\beta - 1} f'(t) dt\right)^{1/\beta} \tag{1.7}$$

is in the class S.

On the other hand, for the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [6] proved another univalence condition asserted by Theorem 1.3.

Theorem 1.3 (see [6]). Let $f \in \mathcal{A}$ satisfy the condition

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1 \quad (z \in \mathbb{U}). \tag{1.8}$$

Then f is univalent in \mathbb{U} .

In the paper [7], Pescar determined some univalence conditions for the following integral operators.

Theorem 1.4 (see [7]). Let the function g satisfy (1.8), M a positive real number fixed, and c a complex number. If $\alpha \in [(2M+1)/(2M+2), (2M+1)/2M]$,

$$|c| \le 1 - \left| \frac{\alpha - 1}{\alpha} \right| (2M + 1), \quad c \ne -1,$$

$$|g(z)| \le M$$
(1.9)

for all $z \in \mathbb{U}$, then the function

$$G_{\alpha}(z) = \left(\alpha \int_{0}^{z} [g(t)]^{\alpha - 1} dt\right)^{1/\alpha}$$

$$\tag{1.10}$$

is in the class S.

Theorem 1.5 (see [7]). Let $g \in \mathcal{A}$, α a real number, $\alpha \geq 1$, and c a complex number, $|c| \leq 1/\alpha$, $c \neq -1$. If

$$\left| \frac{g''(z)}{g'(z)} \right| \le 1 \quad (z \in \mathbb{U}), \tag{1.11}$$

then the function

$$H_{\alpha}(z) = \left(\alpha \int_{0}^{z} \left[tg'(t)\right]^{\alpha-1} dt\right)^{1/\alpha} \tag{1.12}$$

is in the class S.

Theorem 1.6 (see [7]). Let $g \in \mathcal{A}$ satisfies (1.8), α a complex number, M > 1 fixed, $\operatorname{Re} \alpha > 0$, and c a complex number, |c| < 1. If $|g(z)| \le M$ for all $z \in \mathbb{U}$, then for any complex number β

$$\operatorname{Re} \beta \ge \operatorname{Re} \alpha \ge \frac{2M+1}{|\alpha|(1-|c|)} \tag{1.13}$$

the function

$$H_{\beta}(z) = \left(\beta \int_0^z t^{\beta - 1} \left(\frac{g(t)}{t}\right)^{1/\alpha} dt\right)^{1/\beta} \tag{1.14}$$

is in the class S.

In this paper, we introduce the following integral operators as follows:

$$F_1(f,g)(z) = \left(\alpha \int_0^z \left(f(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha} \quad (f,g \in \mathcal{A}; \alpha \in \mathbb{C}), \tag{1.15}$$

$$G_1(f,g)(z) = \left(\alpha \int_0^z \left(tf'(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha} \quad (f,g \in \mathcal{A}; \alpha \in \mathbb{C}), \tag{1.16}$$

$$H_1(f,g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{1/\alpha} dt\right)^{1/\beta} \quad (f,g \in \mathcal{A}; \alpha,\beta \in \mathbb{C} - \{0\}). \tag{1.17}$$

Remark 1.7. For $e^{g(z)} = 1$ and f(z) = g(z), the integral operators (1.15), (1.16), and (1.17) would reduce to the integral operators (1.10), (1.12), and (1.14).

In this paper, we generalize the integral operators given by Pescar [7], and we study the univalence conditions for the integral operators defined by (1.15), (1.16), and (1.17).

For this purpose, we need the following result.

Lemma 1.8 (General Schwarz Lemma [8]). Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with |f(z)| < M for fixed M. If f has one zero with multiplicity order bigger than m for z = 0, then

$$|f(z)| \le \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$
 (1.18)

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m, \tag{1.19}$$

where θ is constant.

2. Main Results

Theorem 2.1. Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), M_1 and M_2 are real positive numbers, and α a complex number, $\text{Re } \alpha > 0$. If

$$\left| \frac{f'(z)}{f(z)} \right| \le M_1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le M_2 \quad (z \in \mathbb{U}), \tag{2.1}$$

$$|c| \le 1 - \left| \frac{\alpha - 1}{\alpha} \right| \left(M_1 + 2M_2^2 + 1 \right), \quad c \in \mathbb{C}, \ c \ne -1,$$
 (2.2)

then the integral operator $F_1(f,g)(z)$ defined by (1.15) is in the class \mathcal{S} .

Proof. From (1.15), we have

$$F_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha}.$$
 (2.3)

Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha - 1} dt.$$
 (2.4)

The function h is regular in \mathbb{U} . From (2.4), we get

$$h'(z) = \left(\frac{f(z)}{z}e^{g(z)}\right)^{\alpha-1},$$

$$h''(z) = (\alpha - 1)\left(\frac{f(z)}{z}e^{g(z)}\right)^{\alpha-2}\left(\frac{zf'(z) - f(z)}{z^2}e^{g(z)} + \frac{f(z)}{z}g'(z)e^{g(z)}\right).$$
(2.5)

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1) \left[\left(\frac{zf'(z)}{f(z)} - 1 \right) + zg'(z) \right],\tag{2.6}$$

which readily shows that

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)} \right| = \left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{\alpha - 1}{\alpha} \left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z) \right) \right| \\
\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(\left(\left| \frac{zf'(z)}{f(z)} \right| + 1 \right) + \left| \frac{z^2 g'(z)}{[g(z)]^2} \right| \left| \frac{[g(z)]^2}{z} \right| \right) \quad (z \in \mathbb{U}).$$
(2.7)

From the hypothesis of Theorem 2.1, we have

$$\left| \frac{f'(z)}{f(z)} \right| \le M_1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le M_2 \quad (z \in \mathbb{U}), \tag{2.8}$$

then by *General Schwarz Lemma* for the function *g*, we obtain

$$|g(z)| \le M_2|z| \quad (z \in \mathbb{U}). \tag{2.9}$$

Using the inequality (2.7), we have

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)} \right| \le |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(M_1 + 1 + \left(\left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) M_2^2 \right). \tag{2.10}$$

From (2.10) and since g satisfies the condition (1.8), we have

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha} \right) \frac{zh''(z)}{\alpha h'(z)} \right| \le |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left(M_1 + 2M_2^2 + 1 \right), \tag{2.11}$$

from which, by (2.2), we get

$$\left|c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)}\right| \le 1 \quad (z \in \mathbb{U}). \tag{2.12}$$

Applying Theorem 1.1, we conclude that the integral operator $F_1(f,g)(z)$ defined by (1.15) is in the class S.

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 2.2. *Let* f, $g \in \mathcal{A}$, where g satisfies the condition (1.8) and α a complex number, $\operatorname{Re} \alpha > 0$. *If*

$$\left| \frac{f'(z)}{f(z)} \right| \le 1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le 1 \quad (z \in \mathbb{U}),$$

$$|c| \le 1 - \left| \frac{4(\alpha - 1)}{\alpha} \right|, \quad c \in \mathbb{C}, \ c \ne -1,$$

$$(2.13)$$

then the integral operator $F_1(f,g)(z)$ defined by (1.15) is in the class S.

Theorem 2.3. Let $f, g \in \mathcal{A}$, where g satisfies the inequality $|g(z)| \leq M$, $M \geq 1$. Also, let α be a real number, $\alpha \geq 1$, and c a complex number with

$$|c| \le 1 - \left| \frac{\alpha - 1}{\alpha} \right| (M + 1), \quad c \ne -1.$$
 (2.14)

If

$$\left| \frac{f''(z)}{f'(z)} \right| \le 1 \quad (z \in \mathbb{U}), \qquad \left| \frac{g'(z)}{g(z)} \right| \le 1 \quad (z \in \mathbb{U}), \tag{2.15}$$

then the integral operator $G_1(f,g)(z)$ defined by (1.16) is in the class \mathcal{S} .

Proof. We observe that

$$G_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(f'(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha}.$$
 (2.16)

Let us consider the function

$$h(z) = \int_0^z \left(f'(t)e^{g(t)} \right)^{\alpha - 1} dt.$$
 (2.17)

The function h is regular in \mathbb{U} . From (2.17), we have

$$h'(z) = \left(f'(z)e^{g(z)}\right)^{\alpha-1},$$

$$\frac{zh''(z)}{h'(z)} = (\alpha - 1)\left(\frac{zf''(z)}{f'(z)} + zg'(z)\right),$$
(2.18)

which readily shows that

$$\left|c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)}\right| \le |c| + \left|\frac{\alpha - 1}{\alpha}\right| \left(\left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{zg'(z)}{g(z)}\right| |g(z)|\right) \quad (z \in \mathbb{U}). \quad (2.19)$$

From (2.19) and the conditions of Theorem 2.3, we get

$$\left|c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zh''(z)}{\alpha h'(z)}\right| \le |c| + \left|\frac{\alpha - 1}{\alpha}\right| (1 + M) \le 1 \quad (z \in \mathbb{U}). \tag{2.20}$$

Applying Theorem 1.1, we conclude that the integral operator $G_1(f,g)(z)$ defined by (1.16) is in the class S.

Setting M = 1 in Theorem 2.3, we obtain the following consequence of Theorem 2.3.

Corollary 2.4. Let $f,g \in \mathcal{A}$, where g satisfies the condition $|g'(z)/g(z)| \le 1$, α a real number, $\alpha \ge 1$, and c a complex number with $|c| \le 1 - |2(\alpha - 1)/\alpha|$, $c \ne -1$. If

$$\left| \frac{f''(z)}{f'(z)} \right| \le 1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le 1 \quad (z \in \mathbb{U}), \tag{2.21}$$

then the integral operator $G_1(f,g)(z)$ defined by (1.16) is in the class \mathcal{S} .

Theorem 2.5. Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), α a complex number, $\operatorname{Re} \alpha > 0$, M_1 and M_2 are real positive numbers, and c a complex number, |c| < 1. If

$$\left| \frac{f'(z)}{f(z)} \right| \le M_1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le M_2 \quad (z \in \mathbb{U}), \tag{2.22}$$

then for any complex number β ,

$$\operatorname{Re} \beta \ge \operatorname{Re} \alpha \ge \frac{M_1 + 2M_2^2 + 1}{|\alpha|(1 - |c|)},$$
 (2.23)

the integral operator $H_1(f,g)(z)$ defined by (1.17) is in the class S.

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)}\right)^{1/\alpha} dt.$$
 (2.24)

The function h is regular in \mathbb{U} . From (2.24), we have

$$h'(z) = \left(\frac{f(z)}{z}e^{g(z)}\right)^{1/\alpha},$$

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right),$$
(2.25)

which readily shows that

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha} \left(\left| \frac{zf'(z)}{f(z)} \right| + 1 + \left| \frac{z^{2}g'(z)}{[g(z)]^{2}} \right| \left| \frac{[g(z)]^{2}}{z} \right| \right) \\
\leq \frac{1 - |z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha} \left(\left| \frac{zf'(z)}{f(z)} \right| + 1 + \left(\left| \frac{z^{2}g'(z)}{[g(z)]^{2}} - 1 \right| + 1 \right) \left| \frac{[g(z)]^{2}}{z} \right| \right).$$
(2.26)

By the *General Schwarz Lemma* for the function *g*, we obtain

$$|g(z)| \le M_2|z| \quad (z \in \mathbb{U}), \tag{2.27}$$

and using the inequality (2.26), we have

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \le \frac{1 - |z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha} \left(M_1 + 1 + \left(\left| \frac{z^2g'(z)}{[g(z)]^2} - 1 \right| + 1 \right) M_2^2 \right).$$
(2.28)

From (2.28) and since g satisfies the condition (1.8), we get

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \le \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \frac{(M_1 + 2M_2^2 + 1)}{|\alpha|} \\
\le \frac{M_1 + 2M_2^2 + 1}{|\alpha|\operatorname{Re}\alpha}.$$
(2.29)

From (2.23), we have

$$\frac{M_1 + 2M_2^2 + 1}{|\alpha| \operatorname{Re} \alpha} \le 1 - |c|, \tag{2.30}$$

and using (2.29), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \le 1-|c| \quad (z \in \mathbb{U}).$$
 (2.31)

Applying Theorem 1.2, we conclude that the integral operator $H_1(f,g)(z)$ defined by (1.17) is in the class S.

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. Let $f, g \in \mathcal{A}$, where g satisfies the condition (1.8), α a complex number, $\text{Re } \alpha > 0$, and c a complex number, |c| < 1. If

$$\left| \frac{f'(z)}{f(z)} \right| \le 1 \quad (z \in \mathbb{U}), \qquad \left| g(z) \right| \le 1 \quad (z \in \mathbb{U}), \tag{2.32}$$

then for any complex number β ,

$$\operatorname{Re} \beta \ge \operatorname{Re} \alpha \ge \frac{4}{|\alpha|(1-|c|)}$$
 (2.33)

the integral operator $H_1(f,g)(z)$ defined by (1.17) is in the class S.

Acknowledgments

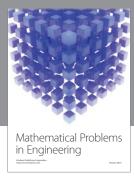
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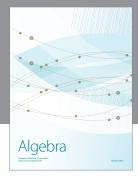
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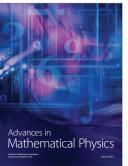


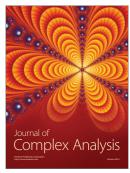


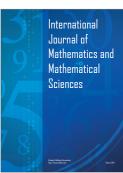


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