Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 929381, 14 pages
doi:10.1155/2012/929381

Research Article

Necessary and Sufficient Conditions for Boundedness of Commutators of the General Fractional Integral Operators on Weighted Morrey Spaces

Zengyan Si[1] and Fayou Zhao[2]

[1] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China*
[2] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

Correspondence should be addressed to Fayou Zhao, zhaofayou2008@yahoo.com.cn

Received 20 March 2012; Revised 5 July 2012; Accepted 20 July 2012

Academic Editor: Giovanni Galdi

We prove that b is in $\mathrm{Lip}_\beta(\omega)$ if and only if the commutator $[b, L^{-\alpha/2}]$ of the multiplication operator by b and the general fractional integral operator $L^{-\alpha/2}$ is bounded from the weighted Morrey space $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, where $0 < \beta < 1$, $0 < \alpha + \beta < n, 1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 \le k < p/q$, $\omega^{q/p} \in A_1$, and $r_\omega > (1 - k)/(p/(q - k))$, and here r_ω denotes the critical index of ω for the reverse Hölder condition.

1. Introduction and Main Results

Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$\left|p_t(x, y)\right| \le \frac{C}{t^{n/2}} e^{-c(|x-y|^2/t)} \tag{1.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$. Since we assume only upper bound on heat kernel $p_t(x, y)$ and no regularity on its space variables, this property (1.1) is satisfied by a class of differential operator, see [1] for details.

For $0 < \alpha < n$, the general fractional integral $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f \frac{dt}{t^{-\alpha/2+1}}(x). \tag{1.2}$$

Note that, if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then, $L^{-\alpha/2}$ is the classical fractional integral I_α which plays important roles in many fields. Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined by

$$\left[b, L^{-\alpha/2}\right] f(x) = b(x) L^{-\alpha/2} f(x) - L^{-\alpha/2}(bf)(x). \tag{1.3}$$

For the special case of $L = -\Delta$, many results have been produced. Paluszyński [2] obtained that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $1 < p < r < \infty,\ 0 < \beta < 1$ and $1/p - 1/r = (\alpha+\beta)/n$ with $p < n/(\alpha+\beta)$. Shirai [3] proved that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ is bounded from the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$ for $1 < p < q < \infty,\ 0 < \alpha,\ 0 < \beta < 1$, and $0 < \alpha+\beta = (1/p - 1/q)(n-\lambda) < n$ or $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ for $1 < p < q < \infty,\ 0 < \alpha,\ 0 < \beta < 1,\ 0 < \alpha+\beta = (1/p - 1/q) < n,\ 0 < \lambda < n - (\alpha+\beta)p$, and $\mu/q = \lambda/p$. Wang [4] established some weighted boundedness of properties of commutator $[b, I_\alpha]$ on the weighted Morrey spaces $L^{p,k}$ under appropriated conditions on the weight ω, where the symbol b belongs to (weighted) Lipschitz spaces. The weighted Morrey space was first introduced by Komori and Shirai [5]. For the general case, Wang [6] proved that if $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, then the commutator $[b, L^{-\alpha/2}]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$, where $0 < \beta < 1,\ 0 < \alpha+\beta < n, 1 < p < n/(\alpha+\beta), 1/p - 1/q = (\alpha+\beta)/n$, and $\omega^q \in A_1$.

The purpose of this paper is to give necessary and sufficient conditions for boundedness of commutators of the general fractional integrals with $b \in \mathrm{Lip}_\beta(\omega)$ (the weighted Lipschitz space). Our theorems are the following.

Theorem 1.1. *Let $0 < \beta < 1, 0 < \alpha+\beta < n, 1 < p < n/(\alpha+\beta)$, $1/q = 1/p - (\alpha+\beta)/n, 0 \le k < \min\{p/q, p\beta/n\}$, and $\omega^q \in A_1$. Then one has the following.*

(a) *If $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, then $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$;*

(b) *If $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$, then $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$.*

Theorem 1.2. *Let $0 < \beta < 1, 0 < \alpha+\beta < n, 1 < p < n/(\alpha+\beta)$, $1/q = 1/p - (\alpha+\beta)/n, 0 \le k < p/q, \omega^{q/p} \in A_1$, and $r_\omega > (1-k)/(p/(q-k))$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then one has the following.*

(a) *If $b \in \mathrm{Lip}_\beta(\omega)$, then $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$;*

(b) *If $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, then $b \in \mathrm{Lip}_\beta(\omega)$.*

Our results not only extend the results of [4] from $(-\Delta)$ to a general operator L, but also characterize the (weighted) Lipschitz spaces by means of the boundedness of $[b, L^{-\alpha/2}]$ on the weighted Morrey spaces, which extend the results of [4, 6]. The basic tool is based on a modification of sharp maximal function $M_L^{\#}$ introduced by [7].

Throughout this paper all notation is standard or will be defined as needed. Denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $\omega(B)$, where $\omega(B) = \int_B \omega(x)dx$. For a measurable set E, denote by χ_E the characteristic function of E. For a real number $p, 1 < p < \infty$, let p' be the dual of p such that $1/p + 1/p' = 1$. The letter C will be used for various constants, and may change from one occurrence to another.

2. Some Preliminaries

A nonnegative function ω defined on $\mathbb{R}^n$ is called weight if it is locally integral. A weight ω is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C, \tag{2.1}$$

for every ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined replacing the above inequality by

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) \le C \operatorname*{essinf}_{x\in B} \omega(x). \tag{2.2}$$

When $p = \infty$, $\omega \in A_\infty$, if there exist positive constants δ and C such that given a ball B and E is a measurable subset of B, then

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{\delta}. \tag{2.3}$$

A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$ if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'} dx\right)^{1/p'} \le C. \tag{2.4}$$

From the definition of $A_{p,q}$, we can get that

$$\omega \in A_{p,q}, \quad \text{iff } \omega^q \in A_{1+q/p'}. \tag{2.5}$$

Since $\omega^{q/p} \in A_1$, then by (2.5), we have $\omega^{1/p} \in A_{p,q}$.

A weight function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality,

$$\left(\frac{1}{|B|}\int_B \omega(x)^r dx\right)^{1/r} \le C\frac{1}{|B|}\int_B \omega(x)dx, \tag{2.6}$$

holds for every ball B in $\mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 \le p < \infty$, then there exists $r > 1$ such that $\omega \in \mathrm{RH}_r$. It follows from Hölder inequality that $\omega \in \mathrm{RH}_r$ implies $\omega \in \mathrm{RH}_s$ for all $1 < s < r$. Moreover, if $\omega \in \mathrm{RH}_r, r > 1$, then we have $\omega \in \mathrm{RH}_{r+\varepsilon}$ for some $\epsilon > 0$. We thus write $r_w = \sup\{r > 1 : \omega \in \mathrm{RH}_r\}$ to denote the critical index of ω for the reverse Hölder condition. For more details on Muckenhoupt class $A_{p,q}$, we refer the reader to [8–10].

Definition 2.1 (see [5]). Let $1 \le p < \infty$ and $0 \le k < 1$. Then for two weights μ and ν, the weighted Morrey space is defined by

$$L^{p,k}(\mu,\nu) = \left\{ f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p,k}(\mu,\nu)} < \infty \right\}, \tag{2.7}$$

where

$$\|f\|_{L^{p,k}(\mu,\nu)} = \sup_B \left(\frac{1}{\nu(B)^k} \int_B |f(x)|^p \mu(x) dx \right)^{1/p}, \tag{2.8}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

If $\mu = \nu$, then we have the classical Morrey space $L^{p,k}(\mu)$ with measure μ. When $k = 0$, then $L^{p,0}(\mu) = L^p(\mu)$ is the Lebesgue space with measure μ.

Definition 2.2 (see [11]). Let $1 \le p < \infty$, $0 < \beta < 1$, and $\omega \in A_\infty$. A locally integral function b is said to be in $\mathrm{Lip}_\beta^p(\omega)$ if

$$\|b\|_{\mathrm{Lip}_\beta^p(\omega)} = \sup_B \frac{1}{\omega(B)^{\beta/n}} \left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x)^{1-p} dx \right)^{1/p} \le C < \infty, \tag{2.9}$$

where $b_B = |B|^{-1} \int_B b(y) dy$ and the supremum is taken over all ball $B \subset R^n$. When $p = 1$, we denote $\mathrm{Lip}_\beta^p(\omega)$ by $\mathrm{Lip}_\beta(\omega)$.

Obviously, for the case $\omega = 1$, then the $\mathrm{Lip}_\beta^p(\omega)$ space is the classical Lip_β^p space.

Remark 2.3. Let $\omega \in A_1$, García-Cuerva [11] proved that the spaces $\|f\|_{\mathrm{Lip}_\beta^p(\omega)}$ coincide, and the norms of $\|\cdot\|_{\mathrm{Lip}_\beta^p(\omega)}$ are equivalent with respect to different values of provided that $1 \le p < \infty$.

Given a locally integrable function f and β, $0 \le \beta < n$, define the fractional maximal function by

$$M_{\beta,r} f(x) = \sup_{x\in B} \left(\frac{1}{|B|^{1-\beta r/n}} \int_B |f(y)|^r dy \right)^{1/r}, \quad r \ge 1, \tag{2.10}$$

when $0 < \beta < n$. If $\beta = 0$ and $r = 1$, then $M_{0,1} f = Mf$ denotes the usual Hardy-Littlewood maximal function.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega f(x) = \sup_{x\in B} \frac{1}{\omega(B)} \int_B |f(y)| dy. \tag{2.11}$$

The fractional weighted maximal operator $M_{\beta,r,\omega}$ is defined by

$$M_{\beta,r,\omega} f(x) = \sup_{x\in B} \left(\frac{1}{\omega(B)^{1-\beta r/n}} \int_B |f(y)|^r \omega(y) dy \right)^{1/r}, \tag{2.12}$$

where $0 \le \beta < n$ and $r \ge 1$. For any $f \in L^p(\mathbb{R}^n), p \ge 1$, the sharp maximal function $M_L^\# f$ associated the generalized approximations to the identity $\{e^{-tL}, t > 0\}$ is given by Martell [7] as follows:

$$M_L^\# f(x) = \sup_{x\in B} \frac{1}{|B|} \int_B \left| f(y) - e^{-t_B L} f(y) \right| dy, \tag{2.13}$$

where $t_B = r_B^2$ and r_B is the radius of the ball B. For $0 < \delta < 1$, we introduce the δ-sharp maximal operator $M_{L,\delta}^{\#}$ as

$$M_{L,\delta}^{\#} f = M_L^{\#}\left(|f|^{\delta}\right)^{1/\delta}, \tag{2.14}$$

which is a modification of the sharp maximal operator $M^{\#}$ of Stein and Murphy [9]. Set $M_{\delta} f = M(|f|^{\delta})^{1/\delta}$. Using the same methods as those of [9, 12], we can get the following.

Lemma 2.4. *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (1.1). Let $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Suppose that $\omega \in A_\infty$, then for every $0 < \eta < 1$, there exists a real number $\gamma > 0$ independent of γ, f such that one has the following weighted version of the local good λ inequality, for $\eta > 0$, $A > 1$,*

$$\omega\left\{x \in \mathbb{R}^n : M_{\delta} f > A\lambda, M_{L,\delta}^{\#} f(x) \le \gamma\lambda\right\} \le \eta\omega\{x \in \mathbb{R}^n : M_{\delta} f(x) > \lambda\}, \tag{2.15}$$

where $A > 1$ is a fixed constant which depends only on n.

If $\mu, \nu \in A_\infty, 1 < p < \infty, 0 \le k < 1$, then

$$\|f\|_{L^{p,k}(\mu,\nu)} \le \|M_{\delta} f\|_{L^{p,k}(\mu,\nu)} \le C\left\|M_{L,\delta}^{\#} f\right\|_{L^{p,k}(\mu,\nu)}. \tag{2.16}$$

In particular, when $\mu = \nu = \omega$ and $\omega \in A_\infty$, we have

$$\|f\|_{L^{p,k}(\omega)} \le \|M_{\delta} f\|_{L^{p,k}(\omega)} \le C\left\|M_{L,\delta}^{\#} f\right\|_{L^{p,k}(\omega)}. \tag{2.17}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

Lemma 3.1 (see [1]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (1.1). Then for $0 < \alpha < 1$, the difference operator $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x,y)$ which satisfies*

$$K_{\alpha,t}(x,y) \le \frac{C}{|x-y|^{n-\alpha}} \frac{t}{|x-y|^2}. \tag{3.1}$$

Lemma 3.2 (see [4]). *Let $0 < \alpha + \beta < n$, $1 < p < n/(\alpha+\beta)$, $1/q = 1/p - (\alpha+\beta)/n$, and $\omega \in A_1$. Then for every $0 < k < p/q$ and $1 < r < p$, one has*

$$\|M_{\alpha+\beta,r} f\|_{L^{q,kq/p}(\omega^q)} \le C\|f\|_{L^{p,q}(\omega^p,\omega^q)}. \tag{3.2}$$

Lemma 3.3 (see [5]). *Let $0 < \beta < n$, $1 < p < n/\beta$, $1/s = 1/p - \beta/n$, and $\omega \in A_{p,s}$. Then for every $0 < k < p/s$, one has*

$$\|M_{\beta,1} f\|_{L^{s,ks/p}(\omega^s)} \le C\|f\|_{L^{p,k}(\omega^p,\omega^s)}. \tag{3.3}$$

Lemma 3.4 (see [4]). *Let $0 < \alpha + \beta < n$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - \alpha/n$, $1/s = 1/q - \beta/n$, and $\omega^q \in A_1$. Then for every $0 < k < p/s$, one has*

$$\left\| M_{\beta,1} f \right\|_{L^{s,ks/p}(\omega^s)} \leq C \| f \|_{L^{q,kq/p}(\omega^q,\omega^s)}. \tag{3.4}$$

Lemma 3.5. *Let $0 < \alpha + \beta < n$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - \alpha/n$, $1/s = 1/q - \beta/n$, and $\omega^q \in A_1$. Then for every $0 < k < p\beta/n$, one has*

$$\left\| L^{-\alpha/2} f \right\|_{L^{q,kq/p}(\omega^q,\omega^s)} \leq C \| f \|_{L^{p,k}(\omega^p,\omega^s)}. \tag{3.5}$$

Proof. Since the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1.1), it is easy to check that $L^{-\alpha/2} f(x) \leq C I_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$. Together with the result (cf. [4]), that is,

$$\| I_\alpha f \|_{L^{q,kq/p}(\omega^q,\omega^s)} \leq C \| f \|_{L^{p,k}(\omega^p,\omega^s)}, \tag{3.6}$$

we can get the desired result. □

Remark 3.6. Using the boundedness property of I_α, we also know $L^{-\alpha/2}$ is bounded from L^1 to weak $L^{n/(n-\alpha)}$. It is easy to check that Lemmas 3.2–3.5 also hold when $k = 0$.

The following lemma plays an important role in the proof of Theorem 1.1.

Lemma 3.7. *Let $0 < \delta < 1$, $0 < \alpha < n$, $0 < \beta < 1$, and $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, one has*

$$\begin{aligned} &M^{\#}_{L,\delta}\left(\left[b, L^{-\alpha/2}\right] f\right)(x) \\ &\quad \leq C \| b \|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left(M_{\beta,1}\left(L^{-\alpha/2} f\right)(x) + M_{\alpha+\beta,r} f(x) + M_{\alpha+\beta,1} f(x) \right). \end{aligned} \tag{3.7}$$

The same method of proof as that of Lemma 4.7 (see below), one omits the details.

Proof of Theorem 1.1. We first prove (a). We only prove Theorem 1.1 in the case $0 < \alpha < 1$. For the general case $0 < \alpha < n$, the method is the same as that of [1]. We omit the details.

For $0 < \alpha + \beta < n$ and $1 < p < n/(\alpha + \beta)$, we can find a number r such that $1 < r < p$. By (2.17) and Lemma 3.7, we obtain the following:

$$\begin{aligned} &\left\| \left[b, L^{-\alpha/2}\right] f \right\|_{L^{q,kq/p}(\omega^q)} \\ &\quad \leq C \left\| M^{\#}_{L,\delta}\left(\left[b, L^{-\alpha/2}\right] f\right) \right\|_{L^{q,kq/p}(\omega^q)} \\ &\quad \leq C \| b \|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \left(\left\| M_{\beta,1}\left(L^{-\alpha/2} f\right) \right\|_{L^{q,kq/p}(\omega^q)} \right. \\ &\qquad\qquad \left. + \left\| M_{\alpha+\beta,r} f \right\|_{L^{q,kq/p}(\omega^q)} + \left\| M_{\alpha+\beta,1} f \right\|_{L^{q,kq/p}(\omega^q)} \right). \end{aligned} \tag{3.8}$$

Let $1/q_1 = 1/p - \alpha/n$ and $1/q = 1/q_1 - \beta/n$. Since $\omega^q \in A_1$, then by (2.5), we have $\omega \in A_{p,q}$. Since $0 < k < \min\{p/q,\ p\beta/n\}$, by Lemmas 3.2–3.5, we yield that

$$\begin{aligned}\left\|\left[b, L^{-\alpha/2}\right] f\right\|_{L^{q,kq/p}(\omega^q)} &\leq C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\left(\left\|L^{-\alpha/2} f\right\|_{L^{q_1,kq_1/p}(\omega^{q_1},\omega^q)}\right.\\ &\qquad\left. + \|f\|_{L^{p,k}(\omega^p,\omega^q)}\right)\\ &\leq C\|b\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)}\|f\|_{L^{p,k}(\omega^p,\omega^q)}.\end{aligned} \tag{3.9}$$

Now we prove (b). Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α. Let $k = 0$ and weight $\omega \equiv 1$, then $L^{p,k}(\omega^p, \omega^q) = L^p$ and $L^{q,kq/p}(\omega^q, \omega) = L^q$. From [2], the (L^p, L^q) boundedness of $[b, I_\alpha]$ implies that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$.

Thus Theorem 1.1 is proved. □

4. Proof of Theorem 1.2

We also need some Lemmas to prove Theorem 1.2.

Lemma 4.1 (see [4]). *Let $0 < \alpha + \beta < n, 1 < p < n/(\alpha+\beta), 1/q = 1/p - \alpha/n, 1/s = 1/q - \beta/n$, and $\omega^{s/p} \in A_1$. Then if $0 < k < p/s$ and $r_\omega > 1/(p/(q-k))$, one has*

$$\|M_{\beta,1} f\|_{L^{s,ks/p}(\omega^{s/p},\omega)} \leq C\|f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}. \tag{4.1}$$

Lemma 4.2 (see [4]). *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$ and $r_\omega > (1-k)/(p/(q-k))$, one has*

$$\|M_{\alpha,1} f\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}. \tag{4.2}$$

Lemma 4.3 (see [4]). *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_\infty$. For any $1 < r < p$, one has*

$$\|M_{\alpha,r,\omega} f\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}. \tag{4.3}$$

Lemma 4.4. *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$ and $r_\omega > (1-k)/(p/(q-k))$, one has*

$$\left\|L^{-\alpha/2} f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}. \tag{4.4}$$

Proof. As before, we know that $L^{-\alpha/2} f(x) \leq C I_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$. Using the boundedness property of I_α on weighted Morrey space (cf. [4]), we have

$$\left\|L^{-\alpha/2} f\right\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq \|I_\alpha f\|_{L^{q,kq/p}(\omega^{q/p},\omega)} \leq C\|f\|_{L^{p,k}(\omega)}, \tag{4.5}$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. □

Remark 4.5. It is easy to check that the above lemmas also hold for $k = 0$.

Lemma 4.6. *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound* (1.1)*, and let $b \in \mathrm{Lip}_\beta(\omega), \omega \in A_1$. Then, for every function $f \in L^p(\mathbb{R}^n), p > 1, x \in \mathbb{R}^n$, and $1 < r < \infty$, one has*

$$\sup_{x\in B} \frac{1}{|B|}\int_B \left| e^{-t_B L}(b(y) - b_{2B}) f(y)\right| dy \le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x) M_{\beta,r,\omega} f(x). \tag{4.6}$$

Proof. Fix $f \in L^p(\mathbb{R}^n), 1 < p < \infty$ and $x \in B$. Then,

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left| e^{-t_B L}((b - b_{2B}) f)(y)\right| dy \\
&\quad\le \frac{1}{|B|}\int_B \int_{\mathbb{R}^n} |p_{t_B}(y,z)||(b(z) - b_{2B}) f(z)| dz dy \\
&\quad\le \frac{1}{|B|}\int_B \int_{2B} |p_{t_B}(y,z)||(b(z) - b_{2B}) f(z)| dz dy \\
&\qquad + \frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^k B} |p_{t_B}(y,z)||(b(z) - b_{2B}) f(z)| dz dy \\
&\quad\doteq \mathcal{M} + \mathcal{N}.
\end{aligned} \tag{4.7}$$

It follows from $y \in B$ and $z \in 2B$ that

$$|p_{t_B}(y,z)| \le C t_B^{-n/2} \le C\frac{1}{|2B|}. \tag{4.8}$$

Thus, Hölder's inequality and Definition 2.2 lead to the following:

$$\begin{aligned}
\mathcal{M} &\le C\frac{1}{|2B|}\int_{2B} |(b(z) - b_{2B}) f(z)| dz \\
&\le C\frac{1}{|2B|}\left(\int_{2B} |b(z) - b_{2B}|^{r'} \omega(z)^{1-r'} dz\right)^{1/r'} \left(\int_{2B} |f(z)|^r \omega(z) dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\frac{1}{|2B|}\omega(2B)^{\beta/n+1/r'}\omega(2B)^{1/r}\left(\frac{1}{\omega(2B)}\int_{2B} |f(z)|^r \omega(z) dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\frac{1}{|2B|}\omega(2B)^{\beta/n+1}\left(\frac{1}{\omega(2B)}\int_{2B} |f(z)|^r \omega(z) dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)\left(\frac{1}{\omega(2B)^{1-\beta r/n}}\int_{2B} |f(z)|^r \omega(z) dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x) M_{\beta,r,\omega} f(x).
\end{aligned} \tag{4.9}$$

Moreover, for any $y \in B$ and $z \in 2^{k+1}B \setminus 2^kB$, we have $|y-z| \geq 2^{k-1}r_B$ and $|p_{t_B}(y,z)| \leq C(e^{-c2^{2(k-1)}}2^{(k+1)n}/|2^{k+1}B|)$

$$\begin{aligned}
\mathcal{N} &= \frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|p_{t_B}(y,z)|\left|(b(z)-b_{2B})f(z)\right|dz\,dy \\
&\leq C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}\left|(b(z)-b_{2B})f(z)\right|dz \\
&\leq C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}\left|(b(z)-b_{2^{k+1}B})f(z)\right|dz \\
&\quad + C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|}\int_{2^{k+1}B}\left|(b_{2^{k+1}B}-b_{2B})f(z)\right|dz \\
&\doteq \mathcal{N}_1+\mathcal{N}_2.
\end{aligned} \tag{4.10}$$

We will estimate the values of terms $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively.

Using Hölder's inequality and Remark 2.3, we have the following:

$$\begin{aligned}
\mathcal{N}_1 &\leq C\sum_{k=1}^{\infty}\frac{e^{-c2^{2(k-1)}}2^{(k+1)n}}{|2^{k+1}B|} \\
&\quad\times\left(\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}\left(\int_{2^{k+1}B}|f(z)|^r\omega(z)dz\right)^{1/r} \\
&\leq C\sum_{k=1}^{\infty}2^{(k+1)n}e^{-c2^{2(k-1)}} \\
&\quad\times\|b\|_{\mathrm{Lip}_\beta(\omega)}\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}\left(\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^r\omega(z)dz\right)^{1/r} \\
&\leq C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).
\end{aligned} \tag{4.11}$$

By a simple calculation, we have

$$|b_{2^{k+1}B}-b_{2B}| \leq Ck\omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega\left(2^{k+1}B\right)^{\beta/n}. \tag{4.12}$$

Since $\omega \in A_1$, by the Hölder inequality, we get

$$\begin{aligned}
\mathcal{N}_2 &\leq C\sum_{k=1}^{\infty}2^{(k+1)n}e^{-c2^{2(k-1)}}\frac{k\omega(2^{k+1}B)^{\beta/n}}{|2^{k+1}B|}\omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\int_{2^{k+1}B}|f(z)|dz \\
&\leq C\sum_{k=1}^{\infty}k2^{(k+1)n}e^{-c2^{2(k-1)}}\omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\frac{\omega(2^{k+1}B)^{\beta/n}}{|2^{k+1}B|^{1/r}}\left(\int_{2^{k+1}B}|f(z)|^r dz\right)^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{k=1}^{\infty} k2^{(k+1)n}e^{-c2^{2(k-1)}} \\
&\quad \times \omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\left(\frac{\omega(2^{k+1}B)}{|2^{k+1}B|}\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^r dz\right)^{1/r} \\
&\le C\sum_{k=1}^{\infty} k2^{(k+1)n}e^{-c2^{2(k-1)}}\omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\left(\frac{1}{\omega(2^{k+1}B)^{1-\beta r/n}}\int_{2^{k+1}B}|f(z)|^r\omega(z)dz\right)^{1/r} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)M_{\beta,r,\omega}f(x).
\end{aligned}
\tag{4.13}
$$

Thus, Lemma 4.6 is proved. □

Lemma 4.7. *Let* $0 < \delta < 1$, $0 < \alpha < 1, \omega \in A_1$, *and* $b \in \mathrm{Lip}_\beta(\omega)$. *Then for all* $r > 1$ *and for all* $x \in \mathbb{R}^n$, *one has*

$$
\begin{aligned}
&M^{\#}_{L,\delta}\left(\left[b, L^{-\alpha/2}\right]f\right)(x) \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(\omega)} \times \left(\omega(x)^{1+\beta/n}M_{\beta,1}\left(L^{-\alpha/2}f\right)(x)\right. \\
&\qquad \left.+ \omega(x)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f(x) + \omega(x)^{1+\beta/n}M_{\alpha+\beta,1}f(x)\right).
\end{aligned}
\tag{4.14}
$$

Proof. For any given $x \in \mathbb{R}^n$, fix a ball $B = B(x_0, r_B)$ which contains x. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Observe that

$$
\begin{aligned}
\left[b, L^{-\alpha/2}\right]f &= (b-b_{2B})L^{-\alpha/2}f - L^{-\alpha/2}((b-b_{2B})f_1) - L^{-\alpha/2}((b-b_{2B})f_2) \\
e^{-t_BL}\left(\left[b, L^{-\alpha/2}\right]f\right) &= e^{-t_BL}\left[(b-b_{2B})L^{-\alpha/2}f - L^{-\alpha/2}((b-b_{2B})f_1) - L^{-\alpha/2}((b-b_{2B})f_2)\right].
\end{aligned}
\tag{4.15}
$$

Then

$$
\begin{aligned}
&\left(\frac{1}{|B|}\int_B\left|\left[b, L^{-\alpha/2}\right]f(y) - e^{-t_BL}\left(\left[b, L^{-\alpha/2}\right]f\right)(y)\right|^\delta dy\right)^{1/\delta} \\
&\quad \le C\left(\frac{1}{|B|}\int_B\left|(b(y)-b_{2B})L^{-\alpha/2}f(y)\right|^\delta dy\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|B|}\int_B\left|L^{-\alpha/2}((b-b_{2B})f_1)(y)\right|^\delta dy\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|B|}\int_B\left|e^{-t_BL}\left((b-b_{2B})L^{-\alpha/2}f\right)(y)\right|^\delta dy\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|B|}\int_B\left|e^{-t_BL}\left(L^{-\alpha/2}((b-b_{2B})f_1)\right)(y)\right|^\delta dy\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{|B|}\int_B\left|\left(L^{-\alpha/2} - e^{-t_BL}L^{-\alpha/2}\right)((b-b_{2B})f_2)(y)\right|^\delta dy\right)^{1/\delta} \\
&\quad \doteq I + II + III + IV + V.
\end{aligned}
\tag{4.16}
$$

□

We are going to estimate each term, respectively. Fix $0 < \delta < 1$ and choose a real number τ such that $1 < \tau < 2$ and $\tau'\delta < 1$. Since $\omega \in A_1$, then it follows from Hölder's inequality that

$$\begin{aligned} I &\le C\left(\frac{1}{|B|}\int_B |(b(y)-b_{2B})|^{\tau\delta}dy\right)^{1/\tau\delta}\left(\int_B \left|L^{-\alpha/2}f(y)\right|^{\tau'\delta}dy\right)^{1/\tau'\delta} \\ &\le C\left(\frac{1}{|B|}\int_{2B}|(b(y)-b_{2B})|dy\right)\left(\int_B\left|L^{-\alpha/2}f(y)\right|dy\right) \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\frac{1}{|2B|}\omega(2B)^{1+\beta/n}\left(\int_B\left|L^{-\alpha/2}f(y)\right|dy\right) \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)^{1+\beta/n}M_{\beta,1}\left(L^{-\alpha/2}f\right)(x). \end{aligned} \tag{4.17}$$

For II, using Hölder's inequality, Kolmogorov's inequality (see page 485 [8],) and Remark 3.6, then we deduce that

$$\begin{aligned} II &\le C\frac{1}{|B|}\int_B\left|L^{-\alpha/2}((b-b_{2B})f_1)(y)\right|dy \\ &\le C\frac{1}{|B|}|B|^{\alpha/n}\left\|L^{-\alpha/2}(b-b_{2B})f_1\right\|_{L^{n/(n-\alpha),\infty}} \\ &\le C\frac{1}{|B|^{1-\alpha/n}}\int_B|(b(y)-b_{2B})f_1(y)|dy \\ &\le C\frac{1}{|B|^{1-\alpha/n}}\left(\int_{2B}|f(y)|^r w(y)dy\right)^{1/r}\left(\int_{2B}|b(y)-b_{2B}|^{r'}w(y)^{-r'/r}dy\right)^{1/r'} \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}M_{\alpha+\beta,r,\omega}f(x)\left(\frac{\omega(2B)}{|2B|}\right)^{1-\alpha/n} \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f(x), \end{aligned} \tag{4.18}$$

where we have used the condition that $\omega \in A_1$.

Using Hölder's inequality and Lemma 4.6, we obtain that

$$III \le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)M_{\beta,r,\omega}\left(L^{-\alpha/2}f\right)(x). \tag{4.19}$$

For IV, using the estimate in II, we get

$$\begin{aligned} IV &\le \frac{C}{|B|}\int_B\int_{2B}|p_{t_B}(y,z)|\left|L^{-\alpha/2}((b-b_{2B})f)(z)\right|dzdy \\ &\le \frac{C}{|2B|}\int_{2B}\left|L^{-\alpha/2}((b-b_{2B})f)(z)\right|dz \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f(x). \end{aligned} \tag{4.20}$$

By virtue of Lemma 3.1, we have the following:

$$
\begin{aligned}
V &\leq \frac{C}{|B|}\int_B\int_{(2B)^c}\left|K_{\alpha,t_B}(y,z)\right|\left|(b(z)-b_{2B})f(z)\right|dzdy\\
&\leq C\sum_{k=1}^{\infty}\int_{2^k r_B\leq|x_0-z|<2^{k+1}r_B}\frac{1}{|x_0-z|^{n-\alpha}}\frac{r_B^2}{|x_0-z|^2}\left|(b(z)-b_{2B})f(z)\right|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{\left|2^{k+1}B\right|^{1-\alpha/n}}\int_{2^{k+1}B}\left|(b(z)-b_{2B})f(z)\right|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-2k}\frac{1}{\left|2^{k+1}B\right|^{1-\alpha/n}}\int_{2^{k+1}B}\left|(b(z)-b_{2^{k+1}B})f(z)\right|dz\\
&\quad+C\sum_{k=1}^{\infty}2^{-2k}|b_{2^{k+1}B}-b_{2B}|\frac{1}{\left|2^{k+1}B\right|^{1-\alpha/n}}\int_{2^{k+1}B}\left|f(z)\right|dz\\
&\doteq VI+VII.
\end{aligned}
\tag{4.21}
$$

Making use of the same argument as that of II, we have

$$
VI \leq C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f(x). \tag{4.22}
$$

Note that $\omega\in A_1$,

$$
|b_{2^{k+1}B}-b_{2B}| \leq Ck\omega(x)\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega\left(2^{k+1}B\right)^{\beta/n}. \tag{4.23}
$$

So, the value of VII can be controlled by

$$
C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)^{1+\beta/n}M_{\alpha+\beta,1}f(x). \tag{4.24}
$$

Combining the above estimates for I–V, we finish the proof of Lemma 4.7.

Proof of Theorem 1.2. We first prove (a). As before, we only prove Theorem 1.2 in the case $0<\alpha<1$. For $0<\alpha+\beta<n$ and $1<p<n/(\alpha+\beta)$, we can find a number r such that $1<r<p$. By Lemma 4.7, we obtain the following:

$$
\begin{aligned}
&\left\|\left[b,L^{-\alpha/2}\right]f\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\quad\leq C\left\|M^{\#}_{L,\delta}\left(\left[b,L^{-\alpha/2}\right]f\right)\right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\Big(\Big\|\omega(\cdot)^{1+\beta/n}M_{\beta,1}\Big(L^{-\alpha/2}f\Big)\Big\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\qquad+\Big\|\omega(\cdot)^{1-\alpha/n}M_{\alpha+\beta,r,\omega}f\Big\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\qquad+\Big\|\omega(\cdot)^{1+\beta/n}M_{\alpha+\beta,1}f\Big\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\Big)\\
&\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\Big(\Big\|M_{\beta,1}\Big(L^{-\alpha/2}f\Big)\Big\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\\
&\qquad+\|M_{\alpha+\beta,r,\omega}f\|_{L^{q,kq/p}(\omega)}+\|M_{\alpha+\beta,1}f\|_{L^{q,kq/p}(\omega^{q/p},\omega)}\Big).
\end{aligned}
\tag{4.25}
$$

Let $1/q_1 = 1/p - \alpha/n$ and $1/q = 1/q_1 - \beta/n$. Lemmas 4.1–4.4 yield that

$$
\begin{aligned}
&\Big\|\Big[b, L^{-\alpha/2}\Big]f\Big\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\Big(\Big\|L^{-\alpha/2}f\Big\|_{L^{q_1,kq_1/p}(\omega^{q_1/p},\omega)}+\|f\|_{L^{p,k}(\omega)}\Big)\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta(\omega)}\|f\|_{L^{p,k}(\omega)}.
\end{aligned}
\tag{4.26}
$$

Now we prove (b). Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α. We use the same argument as Janson [13]. Choose $Z_0 \in \mathbb{R}^n$ so that $|Z_0| = 3$. For $x \in B(Z_0, 2)$, $|x|^{-\alpha+n}$ can be written as the absolutely convergent Fourier series, $|x|^{-\alpha+n} = \sum_{m\in\mathbb{Z}^n} a_m e^{i\langle v_m, x\rangle}$ with $\sum_m |a_m| < \infty$ since $|x|^{-\alpha+n} \in C^\infty(B(Z_0, 2))$. For any $x_0 \in \mathbb{R}^n$ and $\rho > 0$, let $B = B(x_0, \rho)$ and $B_{Z_0} = B(x_0 + Z_0\rho, \rho)$,

$$
\begin{aligned}
\int_B\Big|b(x)-b_{B_{Z_0}}\Big|dx &= \frac{1}{|B_{Z_0}|}\int_B\Big|\int_{B_{Z_0}}(b(x)-b(y))dy\Big|dx\\
&= \frac{1}{\rho^n}\int_B s(x)\Big(\int_{B_{Z_0}}(b(x)-b(y))|x-y|^{-\alpha+n}|x-y|^{n-\alpha}dy\Big)dx,
\end{aligned}
\tag{4.27}
$$

where $s(x) = \mathrm{sgn}(\int_{B_{Z_0}}(b(x)-b(y))dy)$. Fix $x \in B$ and $y \in B_{Z_0}$, then $(y-x)/\rho \in B_{Z_0,2}$, hence,

$$
\begin{aligned}
&\frac{\rho^{-\alpha+n}}{\rho^n}\int_B s(x)\Big(\int_{B_{Z_0}}(b(x)-b(y))|x-y|^{-\alpha+n}\Big(\frac{|x-y|}{\rho}\Big)^{n-\alpha}dy\Big)dx\\
&\quad= \rho^{-\alpha}\sum_{m\in Z^n}a_m\int_B s(x)\Big(\int_{B_{Z_0}}(b(x)-b(y))|x-y|^{n-\alpha}e^{i\langle v_m, y/\rho\rangle}dy\Big)e^{-i\langle v_m, x/\rho\rangle}dx\\
&\quad\le \rho^{-\alpha}\Big|\sum_{m\in Z^n}|a_m|\int_B s(x)\Big[b, L^{-\alpha/2}\Big]\Big(\chi_{B_{Z_0}}e^{i\langle v_m, \cdot/\rho\rangle}\Big)(x)\chi_B(x)e^{-i\langle v_m, x/\rho\rangle}dx\Big|
\end{aligned}
$$

$$\leq \rho^{-\alpha} \sum_{m\in Z^n} |a_m| \left\| \left[b, L^{-\alpha/2}\right] \left(\chi_{B_{Z_0}} e^{i\langle \nu_m, \cdot/\rho\rangle}\right) \right\|_{L^{q,0}(\omega^{1-(1-\alpha/n)q},\omega)} \left(\int_B \omega(x)^{q'(1/q'-\alpha/n)} dx\right)^{1/q'}$$

$$\leq C\rho^{-\alpha} \sum_{m\in Z^n} |a_m| \left\|\chi_{B_{Z_0}}\right\|_{L^{p,0}(\omega)} \left(\int_B \omega(x)^{q'(1/q'-\alpha/n)} dx\right)^{1/q'}$$

$$\leq C\omega(B)^{1/p+1/q'-\alpha/n} = C\omega(B)^{1+\beta/n}. \tag{4.28}$$

This implies that $b \in \mathrm{Lip}_\beta(\omega)$. Thus, (b) is proved. □

Acknowledgment

The authors thank the referee for the useful suggestions. Z. Si was supported by Doctoral Foundation of Henan Polytechnic University. F. Zhao was supported by Shanghai Leading Academic Discipline Project (Grant no. J50101).

References

[1] X. T. Duong and L. X. Yan, "On commutators of fractional integrals," *Proceedings of the American Mathematical Society*, vol. 132, no. 12, pp. 3549–3557, 2004.
[2] M. Paluszyński, "Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss," *Indiana University Mathematics Journal*, vol. 44, no. 1, pp. 1–17, 1995.
[3] S. Shirai, "Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces," *Hokkaido Mathematical Journal*, vol. 35, no. 3, pp. 683–696, 2006.
[4] H. Wang, "On some commutator theorems for fractional integral operators on the weighted morrey spaces ," http://128.84.158.119/abs/1010.2638v1.
[5] Y. Komori and S. Shirai, "Weighted Morrey spaces and a singular integral operator," *Mathematische Nachrichten*, vol. 282, no. 2, pp. 219–231, 2009.
[6] H. Wang, "Some estimates for the commutators of fractional integrals associated to operators with Gaussian kenerl bounds ," http://xxx.tau.ac.il/abs/1102.4380v1.
[7] J. M. Martell, "Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications," *Studia Mathematica*, vol. 161, no. 2, pp. 113–145, 2004.
[8] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, vol. 116 of *North-Holland Mathematics Studies*, North-Holland Publishing, Amsterdam, The Netherlands, 1985.
[9] E. M. Stein and T. S. Murphy, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, vol. 43 of *Monographs in Harmonic Analysis*, Princeton University Press, Princeton, NJ, USA, 1993.
[10] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, vol. 123, Academic Press, Orlando, Fla, USA, 1986.
[11] J. García-Cuerva, "Weighted H^p spaces," *Dissertations Math*, vol. 162, pp. 1–63, 1979.
[12] C. Pérez, "Endpoint estimates for commutators of singular integral operators," *Journal of Functional Analysis*, vol. 128, no. 1, pp. 163–185, 1995.
[13] S. Janson, "Mean oscillation and commutators of singular integral operators," *Arkiv för Matematik*, vol. 16, no. 2, pp. 263–270, 1978.