

Research Article

Strong Convergence Theorems for a Common Fixed Point of Two Countable Families of Relatively Quasi Nonexpansive Mappings and Applications

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The purpose of this paper is to prove strong convergence theorems for common fixed points of two countable families of relatively quasi nonexpansive mappings in a uniformly convex and uniformly smooth real Banach space using the properties of generalized f -projection operator. In order to get the strong convergence theorems, a new iterative scheme by monotone hybrid method is presented and is used to approximate the common fixed points. Then, two examples of countable families of uniformly closed nonlinear mappings are given. The results of this paper modify and improve the results of Li et al. (2010), the results of Takahashi and Zembayashi (2008), and many others.

1. Introduction

Let E be a real Banach space with the dual E^* . We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad (1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The duality mapping J has the following properties:

- (i) if E is smooth, then J is single-valued;
- (ii) if E is strictly convex, then J is one-to-one;
- (iii) if E is reflexive, then J is surjective;

- (iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E ;
- (v) if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is single valued and also one-to-one (see [1–4]).

Let E be a smooth Banach space with the dual E^* . The functional $\phi : E \times E \rightarrow R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad (1.2)$$

for all $x, y \in E$.

Let C be a closed convex subset of E , and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T . A point p in C is said to be an asymptotic fixed point of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [1, 6–8] if $F(T) = \hat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [1, 6–8].

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [9] which is well known as Mann's iteration process and is defined as follows:

$$\begin{aligned} x_0 & \text{ chosen arbitrarily,} \\ x_{n+1} & = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{aligned} \quad (1.3)$$

where the sequence $\{\alpha_n\}$ is chosen in $[0, 1]$. Fourteen years later, Halpern [10] proposed the new innovation iteration process which resembles Mann's iteration (1.3), it is defined by

$$\begin{aligned} x_0 & \text{ chosen arbitrarily,} \\ x_{n+1} & = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{aligned} \quad (1.4)$$

where the element $u \in C$ is fixed. Seven years later, Ishikawa [2] enlarged and improved Mann's iteration (1.3) to the new iteration method, it is often cited as Ishikawa's iteration process which is defined recursively by

$$\begin{aligned} x_0 & \text{ chosen arbitrarily,} \\ y_n & = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} & = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{aligned} \quad (1.5)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$.

In both Hilbert space [10–12] and uniformly smooth Banach space [13–15] the iteration process (1.4) has been proved to be strongly convergent if the sequence $\{\alpha_n\}$ satisfies the following conditions:

- (i) $\alpha_n \rightarrow 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1$.

By the restriction of condition (ii), it is widely believed that Halpern's iteration process (1.4) has slow convergence though the rate of convergence has not been determined. Halpern [10] proved that conditions (i) and (ii) are necessary in the strong convergence of (1.4) for a nonexpansive mapping T on a closed convex subset C of a Hilbert space H . Moreover, Wittmann [12] showed that (1.4) converges strongly to $P_{F(T)}u$ when $\{\alpha_n\}$ satisfies (i), (ii), and (iii), where $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

Both iterations processes (1.3) and (1.5) have only weak convergence, in general Banach space (see [16] for more details). As a matter of fact, process (1.3) may fail to converge while process (1.5) can still converge for a Lipschitz pseudo contractive mapping in a Hilbert space [17]. For example, Reich [18] proved that if E is a uniformly convex Banach space with Frechet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T . However, we note that Manns iteration process (1.3) has only weak convergence even in a Hilbert space [16].

Some attempts to modify the Mann's iteration method so that strong convergence guaranteed has recently been made. Nakajo and Takahashi [19] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H :

$$\begin{aligned}
 x_0 &\in C \quad \text{chosen arbitrarily,} \\
 y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\
 C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
 Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
 x_{n+1} &= P_{C_n \cap Q_n}(x_0),
 \end{aligned} \tag{1.6}$$

where C is a closed convex subset of H , P_K denotes the metric projection from H onto a closed convex subset K of H . They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denotes the fixed points set of T .

The ideas to generalize the process (1.6) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly

smooth Banach space, Matsushita and Takahashi [8] presented their ideas as the following method for a single relatively nonexpansive mapping T in a Banach space E :

$$\begin{aligned}
 x_0 &\in C \quad \text{chosen arbitrarily,} \\
 y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT x_n), \\
 C_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0).
 \end{aligned} \tag{1.7}$$

They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , and let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.7), where J is the duality mapping on E . If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

In 2007, Plubtieng and Ungchittrakool [20] proposed the following hybrid algorithms for two relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

Theorem SK 1. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E , and let T, S be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{aligned}
 x_0 &\in C \quad \text{chosen arbitrarily,} \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
 z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT x_n + \beta_n^{(3)} JS x_n), \\
 H_n &= \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 W_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 x_{n+1} &= \Pi_{H_n \cap W_n}(x_0),
 \end{aligned} \tag{1.8}$$

with the following restrictions:

- (i) $0 \leq \alpha_n < 1$, $\limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (ii) $0 \leq \beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \leq 1$, $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$, $\liminf_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then the $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Theorem SK 2. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , and let T, S be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by

$$\begin{aligned}
x_0 &\in C \quad \text{chosen arbitrarily,} \\
y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n), \\
z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT x_n + \beta_n^{(3)} JS x_n\right), \\
H_n &= \left\{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n \left(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0 \rangle\right)\right\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n}(x_0),
\end{aligned} \tag{1.9}$$

with the following restrictions:

- (i) $0 < \alpha_n < 1$, $\limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (ii) $0 \leq \beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \leq 1$, $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$, $\liminf_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then the $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

In 2010, Su et al. [21] proposed the following hybrid algorithms for two countable families of weak relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

Theorem SXZ 1. Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E , and let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned}
x_0 &\in C \quad \text{chosen arbitrarily,} \\
z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right), \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
C_n &= \left\{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \\
C_0 &= \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
Q_n &= \left\{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\right\}, \\
Q_0 &= C, \\
x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0)
\end{aligned} \tag{1.10}$$

with the conditions:

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (iii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F .

Theorem SXZ 2. Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E , and let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned}
 & x_0 \in C \quad \text{chosen arbitrarily,} \\
 & z_n = J^{-1} \left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right), \\
 & y_n = J^{-1} (\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
 & C_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq (1 - \alpha_n \beta_n^{(1)}) \phi(z, x_n) + \alpha_n \beta_n^{(1)} \phi(z, x_0) \right\}, \\
 & C_0 = \{ z \in C : \phi(z, y_0) \leq \phi(z, x_0) \}, \\
 & Q_n = \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\}, \\
 & Q_0 = C, \\
 & x_{n+1} = \Pi_{C_n \cap Q_n} (x_0),
 \end{aligned} \tag{1.11}$$

with the conditions:

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$;
- (iii) $\limsup_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Recently, Li et al. [22] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f -projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{aligned}
 & y_n = J^{-1} (\alpha_n Jx_n + (1 - \alpha_n) JT_n x_n), \\
 & C_n = \{ w \in C_n : G(w, Jy_n) \leq G(w, Jx_n) \}, \\
 & x_{n+1} = \Pi_{C_{n+1}}^f (x_0), \quad n \geq 1.
 \end{aligned} \tag{1.12}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T . We remark here that the results of Li et al. [22] extended and improved on the results of Matsushita and Takahashi [8].

Motivated by the above-mentioned results and the ongoing research, it is our purpose in this paper to prove a strong convergence theorem for two countable families of relatively quasi nonexpansive mappings in a uniformly convex and uniformly smooth real Banach space using the properties of generalized f -projection operator. Our results extend the results of Li et al. [22], Takahashi and Zembayashi [23], and many other recent known results in the literature.

2. Preliminaries

Let E be a smooth Banach space with the dual E^* . The functional $\phi : E \times E \rightarrow R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \tag{2.1}$$

for all $x, y \in E$. Observe that, in a Hilbert space H , (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

Recall that if C is a nonempty, closed, and convex subset of a Hilbert space H and $P_C : H \rightarrow C$ is the metric projection of H onto C , then P_C is nonexpansive. This is true only when H is a real Hilbert space. In this connection, Alber [24] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \rightarrow C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where \bar{x} is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.2}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J . In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the functional ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(y, x) \leq (\|y\|^2 + \|x\|^2), \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) - 2\langle x - z, Jz - Jy \rangle \tag{2.4}$$

for all $x, y \in E$. See [25] for more details.

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J , we have $Jx = Jy$. Since J is one-to-one, then we have $x = y$; see [12, 15, 26] for more details.

Let C be a closed convex subset of E , and let $\{T_n\}_{n=0}^\infty$ be a countable family of mappings from C into itself. We denote by F the set of common fixed points of $\{T_n\}_{n=0}^\infty$, that is,

$F = \bigcap_{n=0}^{\infty} F(T_n)$, where $F(T_n)$ denote the set of fixed points of T_n , for all $n \geq 0$. A point p in C is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \rightarrow \infty} \|T_n x_n - x_n\| = 0$. The set of asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. A point p in C is said to be a *strong asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n \rightarrow \infty} \|T_n x_n - x_n\| = 0$. The set of strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$ [21].

Definition 2.2. Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of relatively nonexpansive mappings if the following conditions are satisfied:

- (i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;
- (ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$;
- (iii) $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

Definition 2.3. Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of weak relatively nonexpansive mappings if the following conditions are satisfied:

- (i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;
- (ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$;
- (iii) $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

Definition 2.4. Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be countable family of relatively quasi nonexpansive mappings if the following conditions are satisfied:

- (i) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;
- (ii) $\phi(u, T_n x) \leq \phi(u, x)$, for all $u \in F(T_n)$, $x \in C$, $n \geq 0$.

Definition 2.5. A mapping T is said to be relatively nonexpansive mappings if the following conditions are satisfied:

- (i) $F(T)$ is nonempty;
- (ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$;
- (iii) $\widehat{F}(T) = F(T)$.

Definition 2.6. A mapping T is said to be weak relatively nonexpansive mappings if the following conditions are satisfied:

- (i) $F(T)$ is nonempty;
- (ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$;
- (iii) $\widetilde{F}(T) = F(T)$.

Definition 2.7. A mapping T is said to be relatively quasi nonexpansive mappings if the following conditions are satisfied:

- (i) $F(T)$ is nonempty;
- (ii) $\phi(u, Tx) \leq \phi(u, x)$, for all $u \in F(T)$, $x \in C$.

The Definition 2.5 (Definitions 2.6 and 2.7) is a special form of the Definition 2.2 (Definitions 2.3 and 2.4) as $T_n \equiv T$ for all $n \geq 0$. The following conclusions are obvious: (1) relatively nonexpansive mapping must be weak relatively nonexpansive mapping; (2) weak relatively nonexpansive mapping must be relatively quasi nonexpansive mapping.

The hybrid algorithms for fixed point of relatively nonexpansive mappings and applications have been studied by many authors, for example, [1, 6, 7, 17, 27, 28]. In recent years, the definition of relatively quasi nonexpansive mapping has been presented and studied by many authors [7, 17, 26, 28]. Now we give an example which is a countable family of relatively quasi nonexpansive mappings but not a countable family of relatively nonexpansive mappings.

Example 2.8. Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \quad (2.5)$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \dots, \xi_n, \dots), \quad \eta = (\eta_1, \eta_2, \eta_3, \dots, \eta_n, \dots) \in l^2.$$

It is well known that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots), \\ x_1 &= (1, 1, 0, 0, \dots), \\ x_2 &= (1, 0, 1, 0, 0, \dots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \dots), \\ &\vdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \dots, \xi_{n,k}, \dots), \\ &\vdots \end{aligned} \quad (2.6)$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n + 1, \\ 0 & \text{if } k \neq 1, k \neq n + 1, \end{cases} \quad (2.7)$$

for all $n \geq 1$.

Define a countable family of mappings $T_n : E \rightarrow E$ as follows:

$$T_n(x) = \begin{cases} \frac{n}{n+1}x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n, \end{cases} \quad (2.8)$$

for all $n \geq 0$.

Conclusion 2.9. $\{x_n\}$ converges weakly to x_0 .

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_k, \dots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \zeta_{n,k} = \zeta_{n+1} \rightarrow 0, \quad (2.9)$$

as $n \rightarrow \infty$. That is, $\{x_n\}$ converges weakly to x_0 □

Conclusion 2.10. $\{x_n\}$ is not a Cauchy sequence, so that, it does not converge strongly to any element of l^2 .

Proof. In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 2.11. T_n has a unique fixed point 0, that is, $F(T_n) = \{0\}$, for all $n \geq 0$.

Proof. The conclusion is obvious. □

Conclusion 2.12. x_0 is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$.

Proof. Since $\{x_n\}$ converges weakly to x_0 and

$$\|T_n x_n - x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \rightarrow 0 \quad (2.10)$$

as $n \rightarrow \infty$, so that, x_0 is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$. □

Conclusion 2.13. $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi nonexpansive mappings.

Proof. Since $E = l^2$ is a Hilbert space, for any $n \geq 0$ we have

$$\phi(0, T_n x) = \|0 - T_n x\|^2 = \|T_n x\|^2 \leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E. \quad (2.11)$$

then $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi nonexpansive mappings □

Conclusion 2.14. $\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mappings.

Proof. From Conclusions 2.11 and 2.12, we have $\bigcap_{n=0}^{\infty} F(T_n) \neq \widehat{F}(\{T_n\}_{n=0}^{\infty})$, so that, $\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mapping. □

Next, we recall the concept of generalized f -projector operator, together with its properties. Let $G : C \times E^* \rightarrow R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.12}$$

where $\xi \in C$, $\varphi \in E$, ρ is a positive number, and $f : C \rightarrow R \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From the definitions of G and f , it is easy to see the following properties:

- (i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
- (ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.15 (see [29]). Let E be a real Banach space with its dual E^* . Let C be a nonempty, closed and convex subset of E . We say that $\Pi_C^f : E^* \rightarrow 2^C$ is a generalized f -projection operator if

$$\Pi_C^f \varphi = \left\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \right\}, \quad \forall \varphi \in E^*. \tag{2.13}$$

For the generalized f -projection operator, Wu and Huang [42] proved the following theorem basic properties

Lemma 2.16 (see [29]). *Let E be a real reflexive Banach space with its dual E^* . Let C be a nonempty, closed, and convex subset of E . Then the following statements hold:*

- (i) Π_C^f is a nonempty closed convex subset of C for all $\varphi \in E^*$;
- (ii) if E is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f$ if and only if

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.14}$$

- (iii) if E is strictly convex and $f : C \rightarrow R \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$, where $x \in C$), then Π_C^f is a single valued mapping.

Fan et al. [30] showed that the condition f is a positive homogeneous which appeared in Lemma 2.13 can be removed.

Lemma 2.17 (see [30]). *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed, and convex subset of E . Then if E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E$. such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.15) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx \rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.15}$$

Now, we consider the second generalized f -projection operator in a Banach space.

Definition 2.18. Let E be a real Banach space and C a nonempty, closed, and convex subset of E . We say that $\Pi_C^f : E \rightarrow 2^C$ is a generalized f -projection operator if

$$\Pi_C^f(x) = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \right\}, \quad \forall x \in E. \quad (2.16)$$

Obviously, the definition of relatively quasi nonexpansive mapping T is equivalent to

- (R1) $F(T) \neq \emptyset$;
- (R2) $G(p, JT x) \leq G(p, Jx)$, for all $x \in C$, $p \in F(T)$.

Lemma 2.19 (see [31]). *Let E be a Banach space and let $f : E \rightarrow R \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exists $x^* \in E$ and $\alpha \in R$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E. \quad (2.17)$$

We know that the following lemmas hold for operator Π_C^f .

Lemma 2.20 (see [22]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E . Then the following statements hold:*

- (i) $\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$;
- (ii) for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \quad (2.18)$$

- (iii) if E is strictly convex, then $\Pi_C^f x$ is a single valued mapping.

Lemma 2.21 (see [22]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E . let $x \in E$ and $\hat{x} \in \Pi_C^f x$, then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \quad (2.19)$$

The fixed points set $F(T)$ of a relatively quasi nonexpansive mapping is closed convex as given in the following lemma.

Lemma 2.22 (see [32, 33]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E . let T be a closed relatively quasi nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

Lemma 2.23 (see [25]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$.*

Lemma 2.24 (see [24, 25, 27]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.20}$$

Lemma 2.25 (see [28]). *Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E . Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|) \tag{2.21}$$

for all x, y , and $z \in B_r(0)$ and λ, μ , and $\gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

It is easy to prove the following result.

Lemma 2.26. *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E , and let T be a relatively quasi nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.27. *Let E be a p -uniformly convex Banach space with $p \geq 2$. Then, for all $x, y \in E$, $j(x) \in J_p(x)$ and $j(y) \in J_p(y)$,*

$$\langle x - y, j(x) - j(y) \rangle \geq \frac{c^p}{c^{p-2}p} \|x - y\|^p, \tag{2.22}$$

where J_p is the generalized duality mapping from E into E^* and $1/c$ is the p -uniformly convexity constant of E .

Observe that an infinite family of operators $\{T_n\}_{n=1}^\infty$ in a Banach space is said to be uniformly closed, if $x_n \rightarrow x$, $x_n - T_n x_n \rightarrow 0$ then $T_n x = x$ (i.e., $x \in \bigcap_{n=1}^\infty F(T_n)$). Obviously, a countable family of uniformly closed of relatively quasi nonexpansive mappings is a countable family of weak relatively nonexpansive mappings.

3. Main Results

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , and let $\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty$ be two countable families of uniformly closed of relatively quasi nonexpansive mappings of C into itself such that $F := (\bigcap_{n=1}^\infty F(T_n)) \cap (\bigcap_{n=1}^\infty F(S_n)) \neq \emptyset$.*

Let $f : E \rightarrow R$ be a convex and lower semicontinuous mapping with $C \subset \text{int}(D(f))$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned}
z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : G(z, Jy_n) \leq G(z, Jx_n)\}, \\
C_0 &= \{z \in C : G(z, Jy_0) \leq G(z, Jx_0)\}, \\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
Q_0 &= C, \\
x_{n+1} &= \Pi_{C_n \cap Q_n}^f(x_0),
\end{aligned} \tag{3.1}$$

with the conditions

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (iii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to x^* , where $x^* = \Pi_F^f x_0$.

Proof. Step 1. We show that C_n and Q_n are closed and convex for each $n \geq 0$.

From the definitions of C_n and Q_n , it is obvious that Q_n is closed and convex and C_n is closed for each $n \geq 0$. Moreover, since $G(z, Jy_n) \leq G(z, Jx_n)$ is equivalent to

$$\phi(z, y_n) \leq \phi(z, x_n), \tag{3.2}$$

and is equivalent to

$$2\langle z, Jx_n - Jy_n \rangle \leq \|x_n\|^2 + \|y_n\|^2, \tag{3.3}$$

it follows that C_n is convex for each $n \geq 0$. So, $C_n \cap Q_n$ is a closed convex subset of E for all $n \in N \cup \{0\}$.

Step 2. We show that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Observe that

$$z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n). \tag{3.4}$$

Hence from the definition of $G(x, Jy)$ and the convexity of $\| \cdot \|^2$ we have, for all $p \in F$, that

$$\begin{aligned}
 G(p, Jz_n) &= G\left(p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right) \\
 &= \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \rangle \\
 &\quad + \left\| \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\|^2 + 2\rho f(p) \\
 &\leq \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \rangle \\
 &\quad + \beta_n^{(1)} \|Jx_n\|^2 + \beta_n^{(2)} \|JT_n x_n\|^2 + \beta_n^{(3)} \|JS_n x_n\|^2 + 2\rho f(p) \\
 &= \beta_n^{(1)} G(p, Jx_n) + \beta_n^{(2)} G(p, JT_n x_n) + \beta_n^{(3)} G(p, JS_n x_n) \\
 &\leq \beta_n^{(1)} G(p, Jx_n) + \beta_n^{(2)} G(p, Jx_n) + \beta_n^{(3)} G(p, Jx_n) \\
 &= G(p, Jx_n).
 \end{aligned} \tag{3.5}$$

By the similar reason we have, for all $p \in F$, that

$$\begin{aligned}
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
 G(p, Jy_n) &= G(p, \alpha_n Jx_n + (1 - \alpha_n) Jz_n) \\
 &= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1 - \alpha_n) Jz_n \rangle \\
 &\quad + \left\| \alpha_n Jx_n + (1 - \alpha_n) Jz_n \right\|^2 + 2\rho f(p) \\
 &\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1 - \alpha_n) Jz_n \rangle \\
 &\quad + \alpha_n \|Jx_n\|^2 + (1 - \alpha_n) \|Jz_n\|^2 + 2\rho f(p) \\
 &= \alpha_n G(p, Jx_n) + (1 - \alpha_n) G(p, Jz_n) \\
 &\leq \alpha_n G(p, Jx_n) + (1 - \alpha_n) G(p, Jx_n) \\
 &= G(p, Jx_n).
 \end{aligned} \tag{3.6}$$

That is, $p \in C_n$ for all $n \geq 0$.

Next, we show that $F \subset Q_n$ for all $n \geq 0$, we prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0)$, by Definition 2.15 we have

$$G(x_{n+1}, Jx_0) \leq G(z, Jx_0), \quad \forall z \in C_n \cap Q_n. \tag{3.8}$$

It is equivalent to

$$\phi(x_{n+1}, Jx_0) \leq \phi(z, Jx_0), \tag{3.9}$$

and is equivalent to

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n. \quad (3.10)$$

As $F \subset C_n \cap Q_n$, by the induction assumptions, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$.

Step 3. We show that $x_n \rightarrow x^*$ as $n \rightarrow \infty$, and $x^* \in F$.

We now show that $\lim_{n \rightarrow \infty} G(x_n, Jx_0)$ exists. Since $f : E \rightarrow R$ is a convex and lower semi-continuous, applying Lemma 2.19, we see that there exists $u^* \in E^*$ and $\alpha \in R$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E. \quad (3.11)$$

It follows that

$$\begin{aligned} G(x_{n+1}, Jx_0) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho f(x_{n+1}) \\ &\geq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho\langle x_{n+1}, u^* \rangle + 2\rho\alpha \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 - \rho u^* \rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_{n+1}\|^2 - 2\|x_{n+1}\| \|Jx_0 - \rho u^*\| + \|x_0\|^2 + 2\rho\alpha \\ &= (\|x_{n+1}\| - \|Jx_0 - \rho u^*\|)^2 - \|Jx_0 - \rho u^*\|^2 + \|x_0\|^2 + 2\rho\alpha. \end{aligned} \quad (3.12)$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0)$, for each $p \in F$, it follows from (3.12) that

$$\begin{aligned} G(p, Jx_0) &\geq G(x_{n+1}, Jx_0) \\ &\geq (\|x_{n+1}\| - \|Jx_0 - \rho u^*\|)^2 - \|Jx_0 - \rho u^*\|^2 + \|x_0\|^2 + 2\rho\alpha. \end{aligned} \quad (3.13)$$

This implies that $\{x_n\}_{n=1}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_n \cap Q_n \subset C_{n-1} \cap Q_{n-1}$, by Lemma 2.21 we have

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \quad (3.14)$$

It is obvious that

$$G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0), \quad (3.15)$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists.

By the fact that $C_{n+m} \cap Q_{n+m} \subset C_{n-1} \cap Q_{n-1}$ and $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_{n-1} \cap Q_{n-1}$, by Lemma 2.21 we obtain

$$\phi(x_{n+m}, x_n) + G(x_n, Jx_0) \leq G(x_{n+m}, Jx_0). \quad (3.16)$$

Taking the limit as $m, n \rightarrow \infty$ in (3.16), we obtain

$$\lim_{n \rightarrow \infty} \phi(x_{n+m}, x_n) = 0, \quad (3.17)$$

which holds uniformly for all m . By using Lemma 2.21, we get that

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0, \quad (3.18)$$

which holds uniformly for all m . Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $x^* \in C$ such that $x_n \rightarrow x^*$. In particular, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.19)$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_n$, from the definition of C_n , we know that

$$G(x_{n+1}, Jy_n) \leq G(x_{n+1}, Jx_n) \quad (3.20)$$

is equivalent to

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n). \quad (3.21)$$

So, $\phi(x_{n+1}, y_n) \rightarrow 0$.

By using Lemma 2.23, we get that

$$\|x_{n+1} - y_n\| \rightarrow 0, \quad (3.22)$$

as $n \rightarrow \infty$.

Hence $y_n \rightarrow x^*$ as $n \rightarrow \infty$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} (1 - \alpha_n) \|Jz_n - Jx_n\| = \|Jy_n - Jx_n\| = 0. \quad (3.23)$$

Since $0 \leq \alpha_n \leq \alpha < 1$, then

$$\lim_{n \rightarrow \infty} \|Jz_n - Jx_n\| = 0. \quad (3.24)$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0 \quad (3.25)$$

so that $z_n \rightarrow x^*$ as $n \rightarrow \infty$.

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_n x_n\}$, and $\{JS_n x_n\}$. From the definition of $\phi(x, y)$ and

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right), \quad (3.26)$$

we have, for all $p \in F$ that

$$\begin{aligned} \phi(p, z_n) &= \phi\left(p, J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right)\right) \\ &= \|p\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\rangle \\ &\quad + \left\| \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\|^2. \end{aligned} \quad (3.27)$$

Therefore by using Lemma 2.25 (inequality (2.21)), for all $p \in F$, we have

$$\begin{aligned} \phi(p, z_n) &\leq \|p\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\rangle \\ &\quad + \beta_n^{(1)} \|Jx_n\|^2 + \beta_n^{(2)} \|JT_n x_n\|^2 + \beta_n^{(3)} \|JS_n x_n\|^2 \\ &\quad - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \\ &= \beta_n^{(1)} \phi(p, x_n) + \beta_n^{(2)} \phi(p, T_n x_n) + \beta_n^{(3)} \phi(p, S_n x_n) \\ &\quad - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \\ &\leq \beta_n^{(1)} \phi(p, x_n) + \beta_n^{(2)} \phi(p, x_n) + \beta_n^{(3)} \phi(p, x_n) \\ &\quad - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \\ &= \phi(p, x_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|), \end{aligned} \quad (3.28)$$

and hence

$$\beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \leq \phi(p, x_n) - \phi(p, z_n) \longrightarrow 0, \quad (3.29)$$

as $n \rightarrow \infty$. By using the same way, we can prove that

$$\beta_n^{(1)} \beta_n^{(3)} g(\|Jx_n - JS_n x_n\|) \leq \phi(p, x_n) - \phi(p, z_n) \longrightarrow 0, \quad (3.30)$$

as $n \rightarrow \infty$. From the properties of the mapping g , we have

$$\|Jx_n - JT_n x_n\| \longrightarrow 0, \quad (3.31)$$

as $n \rightarrow \infty$, and

$$\|Jx_n - JS_n x_n\| \longrightarrow 0, \quad (3.32)$$

as $n \rightarrow \infty$. Since J^{-1} is also uniformly norm-to-norm continuous on any bounded sets, we have

$$\|x_n - T_n x_n\| \rightarrow 0, \quad (3.33)$$

as $n \rightarrow \infty$, and

$$\|Jx_n - JS_n x_n\| \rightarrow 0, \quad (3.34)$$

as $n \rightarrow \infty$. Since $x_n \rightarrow x^*$ and T_n, S_n are uniformly closed, $x^* = T_n x^*$, and $x^* = S_n x^*$.

Step 4. we show that $x^* = \Pi_F^f x_0$.

Since $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$ is a closed and convex set, from Lemma 2.19, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_{n-1} \cap Q_{n-1}}^f(x_0)$ and $w \in F \subset C_{n-1} \cap Q_{n-1}$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 0. \quad (3.35)$$

We know that $G(\xi, J\varphi)$ is convex and lower semicontinuous with respect to ξ when φ is fixed. This implies that

$$G(x^*, Jx_0) \leq \liminf_{n \rightarrow \infty} G(x_n, Jx_0) \leq \limsup_{n \rightarrow \infty} G(x_n, Jx_0) \leq G(w, Jx_0). \quad (3.36)$$

From the definition of $\Pi_F^f x_0$ and $x^* \in F$, we see that $x^* = w$. This completes the proof. \square

Based on Theorem 3.1, we have the following.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , and let $\{T_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}$ be two countable families of weak relatively nonexpansive mappings of C into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned} z_n &= J^{-1} \left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right), \\ y_n &= J^{-1} (\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ C_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \right\}, \\ C_0 &= \left\{ z \in C : \phi(z, y_0) \leq \phi(z, x_0) \right\}, \\ Q_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\}, \\ Q_0 &= C, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \end{aligned} \quad (3.37)$$

with the conditions:

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

- (iii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to x^* , where $x^* = \Pi_F x_0$.

Proof. Putting $f(x) \equiv 0$, we can conclude from Theorem 3.1 the desired conclusion immediately. \square

Theorem 3.3. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , let $\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty$ be two countable families of uniformly closed of relatively quasi nonexpansive mappings of C into itself such that $F := (\bigcap_{n=1}^\infty F(T_n)) \cap (\bigcap_{n=1}^\infty F(S_n)) \neq \emptyset$. Let $f : E \rightarrow \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \text{int}(D(f))$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned}
 z_n &= J^{-1} \left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right), \\
 y_n &= J^{-1} (\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\
 C_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : G(z, Jy_n) \leq \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)} \right) G(z, Jx_0) \right. \\
 &\quad \left. + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)} \right) G(z, Jx_n) \right\}, \\
 C_0 &= \{z \in C : G(z, Jy_0) \leq G(z, Jx_0)\}, \\
 Q_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\}, \\
 Q_0 &= C, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n}^f(x_0),
 \end{aligned} \tag{3.38}$$

with the conditions:

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$;
- (iii) $\limsup_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to x^* , where $x^* = \Pi_F^f x_0$.

Proof. Step 1. We show that C_n and Q_n are closed and convex for each $n \geq 0$.

From the definitions of C_n and Q_n , it is obvious that Q_n is closed and convex and C_n is closed for each $n \geq 0$. Moreover, since $G(z, Jy_n) \leq (\beta_n^{(1)} - \alpha_n \beta_n^{(1)})G(z, Jx_0) + (1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)})G(z, Jx_n)$ is equivalent to

$$\begin{aligned} & 2 \left\langle z, \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)} \right) Jx_0 + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)} \right) Jx_n - Jy_n \right\rangle \\ & \leq \left(\beta_n^{(1)} - \alpha_n \beta_n^{(1)} \right) \|x_0\|^2 + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)} \right) \|x_n\|^2 - \|y_n\|^2. \end{aligned} \quad (3.39)$$

it follows that C_n is convex for each $n \geq 0$. So, $C_n \cap Q_n$ is a closed convex subset of E for all $n \in N \cup \{0\}$.

Step 2. We show that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Observe that

$$z_n = J^{-1} \left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right). \quad (3.40)$$

Hence from the definition of $G(x, Jy)$ and the convexity of $\|\cdot\|^2$ we have, for all $p \in F$, that

$$\begin{aligned} G(p, Jz_n) &= G \left(p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right) \\ &= \|p\|^2 - 2 \left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\rangle \\ &\quad + \left\| \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\|^2 + 2\rho f(p) \\ &\leq \|p\|^2 - 2 \left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right\rangle \\ &\quad + \beta_n^{(1)} \|Jx_0\|^2 + \beta_n^{(2)} \|JT_n x_n\|^2 + \beta_n^{(3)} \|JS_n x_n\|^2 + 2\rho f(p) \\ &= \beta_n^{(1)} G(p, Jx_0) + \beta_n^{(2)} G(p, JT_n x_n) + \beta_n^{(3)} G(p, JS_n x_n) \\ &\leq \beta_n^{(1)} G(p, Jx_0) + \beta_n^{(2)} G(p, Jx_n) + \beta_n^{(3)} G(p, Jx_n) \\ &= \beta_n^{(1)} G(p, Jx_0) + \left(1 - \beta_n^{(1)} \right) G(p, Jx_n). \end{aligned} \quad (3.41)$$

By the similar reason we have, for all $p \in F$, that

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
G(p, Jy_n) &= G(p, \alpha_n Jx_n + (1 - \alpha_n)Jz_n) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1 - \alpha_n)Jz_n \rangle \\
&\quad + \|\alpha_n Jx_n + (1 - \alpha_n)Jz_n\|^2 + 2\rho f(p) \\
&\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1 - \alpha_n)Jz_n \rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + (1 - \alpha_n)\|Jz_n\|^2 + 2\rho f(p) \\
&= \alpha_n G(p, Jx_n) + (1 - \alpha_n)G(p, Jz_n) \\
&\leq \alpha_n G(p, Jx_n) + (1 - \alpha_n)\left(\beta_n^{(1)} G(p, Jx_0) + \left(1 - \beta_n^{(1)}\right)G(p, Jx_n)\right) \\
&= (1 - \alpha_n)\beta_n^{(1)} G(p, Jx_0) + \left(1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}\right)G(p, Jx_n).
\end{aligned} \tag{3.42}$$

That is, $p \in C_n$ for all $n \geq 0$.

Next, we show that $F \subset Q_n$ for all $n \geq 0$, we prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0)$, by Definition 2.15 we have

$$G(x_{n+1}, Jx_0) \leq G(z, Jx_0), \quad \forall z \in C_n \cap Q_n. \tag{3.43}$$

It is equivalent to

$$\phi(x_{n+1}, Jx_0) \leq \phi(z, Jx_0), \tag{3.44}$$

and is equivalent to

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n. \tag{3.45}$$

As $F \subset C_n \cap Q_n$, by the induction assumptions, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$.

Step 3. We show that $x_n \rightarrow x^*$ as $n \rightarrow \infty$, and $x^* \in F$.

We now show that $\lim_{n \rightarrow \infty} G(x_n, Jx_0)$ exists. Since $f : E \rightarrow R$ is a convex and lower semi-continuous, applying Lemma 2.19, we see that there exists $u^* \in E^*$ and $\alpha \in R$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E. \tag{3.46}$$

It follows that

$$\begin{aligned}
G(x_{n+1}, Jx_0) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho f(x_{n+1}) \\
&\geq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 \rangle + \|x_0\|^2 + 2\rho\langle x_{n+1}, u^* \rangle + 2\rho\alpha \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_0 - \rho u^* \rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_{n+1}\|^2 - 2\|x_{n+1}\| \|Jx_0 - \rho u^*\| + \|x_0\|^2 + 2\rho\alpha \\
&= (\|x_{n+1}\| - \|Jx_0 - \rho u^*\|)^2 - \|Jx_0 - \rho u^*\|^2 + \|x_0\|^2 + 2\rho\alpha
\end{aligned} \tag{3.47}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0)$, for each $p \in F$, it follows from (3.47) that

$$\begin{aligned}
G(p, Jx_0) &\geq G(x_{n+1}, Jx_0) \\
&\geq (\|x_{n+1}\| - \|Jx_0 - \rho u^*\|)^2 - \|Jx_0 - \rho u^*\|^2 + \|x_0\|^2 + 2\rho\alpha.
\end{aligned} \tag{3.48}$$

This implies that $\{x_n\}_{n=1}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_n \cap Q_n \subset C_{n-1} \cap Q_{n-1}$, by Lemma 2.21, we have

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \tag{3.49}$$

It is obvious that

$$G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0), \tag{3.50}$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists.

By the fact that $C_{n+m} \cap Q_{n+m} \subset C_{n-1} \cap Q_{n-1}$ and $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_{n-1} \cap Q_{n-1}$, by Lemma 2.21 we obtain

$$\phi(x_{n+m}, x_n) + G(x_n, Jx_0) \leq G(x_{n+m}, Jx_0). \tag{3.51}$$

Taking the limit as $m, n \rightarrow \infty$ in (3.51), we obtain

$$\lim_{n \rightarrow \infty} \phi(x_{n+m}, x_n) = 0, \tag{3.52}$$

which holds uniformly for all m . By using Lemma 2.23, we get that

$$\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0, \tag{3.53}$$

which holds uniformly for all m . Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $x^* \in C$ such that $x_n \rightarrow x^*$. In particular, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.54}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n}^f(x_0) \in C_n$, from the definition of C_n , we know that

$$\begin{aligned} G(x_{n+1}, Jy_n) &\leq G(z, Jy_n) \\ &\leq (\beta_n^{(1)} - \alpha_n \beta_n^{(1)})G(x_{n+1}, Jx_0) + (1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)})G(x_{n+1}, Jx_n), \end{aligned} \quad (3.55)$$

and is equivalent to

$$\begin{aligned} \phi(x_{n+1}, y_n) + 2\rho f(x_{n+1}) &\leq (\beta_n^{(1)} - \alpha_n \beta_n^{(1)})(\phi(x_{n+1}, x_0) + 2\rho f(x_{n+1})) \\ &\quad + (1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)})(\phi(x_{n+1}, x_n) + 2\rho f(x_{n+1})), \end{aligned} \quad (3.56)$$

and is equivalent to

$$\phi(x_{n+1}, y_n) \leq (\beta_n^{(1)} - \alpha_n \beta_n^{(1)})\phi(x_{n+1}, x_0) + (1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)})\phi(x_{n+1}, x_n), \quad (3.57)$$

So, $\phi(x_{n+1}, y_n) \rightarrow 0$.

By using Lemma 2.23, we get that

$$\|x_{n+1} - y_n\| \rightarrow 0, \quad (3.58)$$

as $n \rightarrow \infty$,

and hence $y_n \rightarrow x^*$ as $n \rightarrow \infty$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} (1 - \alpha_n) \|Jz_n - Jx_n\| = \|Jy_n - Jx_n\| = 0. \quad (3.59)$$

Since $0 \leq \alpha_n \leq \alpha < 1$, then

$$\lim_{n \rightarrow \infty} \|Jz_n - Jx_n\| = 0. \quad (3.60)$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0, \quad (3.61)$$

so that $z_n \rightarrow x^*$ as $n \rightarrow \infty$.

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_n x_n\}$, and $\{JS_n x_n\}$. From the definition of $\phi(x, y)$ and

$$z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \quad (3.62)$$

we have, for all $p \in F$ that

$$\begin{aligned} \phi(p, z_n) &= \phi\left(p, J^{-1}\left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right)\right) \\ &= \|p\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n \right\rangle \\ &\quad + \left\| \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n \right\|^2. \end{aligned} \tag{3.63}$$

Therefore by using Lemma 2.25 (inequality (2.21)), for all $p \in F$, we have

$$\begin{aligned} \phi(p, z_n) &\leq \|p\|^2 - 2\left\langle p, \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n \right\rangle \\ &\quad + \beta_n^{(1)} \|Jx_0\|^2 + \beta_n^{(2)} \|JT_nx_n\|^2 + \beta_n^{(3)} \|JS_nx_n\|^2 \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|JT_nx_n - JS_nx_n\|) \\ &= \beta_n^{(1)} \phi(p, x_0) + \beta_n^{(2)} \phi(p, T_nx_n) + \beta_n^{(3)} \phi(p, S_nx_n) \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|JT_nx_n - JS_nx_n\|) \\ &\leq \beta_n^{(1)} \phi(p, x_0) + \beta_n^{(2)} \phi(p, x_n) + \beta_n^{(3)} \phi(p, x_n) \\ &\quad - \beta_n^{(2)} \beta_n^{(3)} g(\|JT_nx_n - JS_nx_n\|) \\ &= \beta_n^{(1)} \phi(p, x_0) + \left(1 - \beta_n^{(1)}\right) \phi(p, x_n) - \beta_n^{(2)} \beta_n^{(3)} g(\|JT_nx_n - JS_nx_n\|), \end{aligned} \tag{3.64}$$

and hence

$$\beta_n^{(2)} \beta_n^{(3)} g(\|Jx_n - JT_nx_n\|) \leq \beta_n^{(1)} \phi(p, x_0) + \left(1 - \beta_n^{(1)}\right) \phi(p, x_n) - \phi(p, z_n). \tag{3.65}$$

From $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$ and $x_n \rightarrow x^*, z_n \rightarrow x^*$, we have

$$\beta_n^{(2)} \beta_n^{(3)} g(\|JT_nx_n - JS_nx_n\|) \rightarrow 0, \tag{3.66}$$

as $n \rightarrow \infty$. From the properties of the mapping g , we have

$$\|JT_nx_n - JS_nx_n\| \rightarrow 0, \tag{3.67}$$

as $n \rightarrow \infty$.

Since

$$z_n = J^{-1}\left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_nx_n + \beta_n^{(3)} JS_nx_n\right), \tag{3.68}$$

then we have

$$Jz_n = \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n. \quad (3.69)$$

Therefore

$$\begin{aligned} \|Jx_n - Jz_n\| &= \left\| Jx_n - \left(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n \right) \right\| \\ &= \left\| \beta_n^{(1)} (Jx_n - Jx_0) + \beta_n^{(2)} (Jx_n - JT_n x_n) \right. \\ &\quad \left. + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\ &\geq \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\ &\quad - \left\| \beta_n^{(1)} (Jx_n - Jx_0) \right\|, \end{aligned} \quad (3.70)$$

which leads to

$$\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \leq \|Jx_n - Jz_n\| + \left\| \beta_n^{(1)} (Jx_n - Jx_0) \right\|. \quad (3.71)$$

Since $x_n \rightarrow x^*$, $z_n \rightarrow x^*$ and $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$. Then from above inequality we obtain

$$\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \rightarrow 0. \quad (3.72)$$

On the other hand, by using the property of norm $\|\cdot\|$, we have

$$\begin{aligned} &\left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\ &= \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) + \beta_n^{(3)} (Jx_n - JT_n x_n) - \beta_n^{(3)} (Jx_n - JT_n x_n) \right\| \\ &= \left\| \left(\beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) + \beta_n^{(3)} (JT_n x_n - JS_n x_n) \right\| \\ &\geq \left\| \left(\beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) \right\| - \left\| \beta_n^{(3)} (JT_n x_n - JS_n x_n) \right\|, \end{aligned} \quad (3.73)$$

which leads to the following inequality

$$\begin{aligned} \left\| \left(\beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) \right\| &\leq \left\| \beta_n^{(2)} (Jx_n - JT_n x_n) + \beta_n^{(3)} (Jx_n - JS_n x_n) \right\| \\ &\quad + \left\| \beta_n^{(3)} (JT_n x_n - JS_n x_n) \right\|. \end{aligned} \quad (3.74)$$

Therefore, by using (3.67) and (3.72) we have

$$\left\| \left(\beta_n^{(2)} + \beta_n^{(3)} \right) (Jx_n - JT_n x_n) \right\| \longrightarrow 0. \quad (3.75)$$

This together with condition (ii) of Theorem 3.3 implies that

$$\|Jx_n - JT_n x_n\| \longrightarrow 0. \quad (3.76)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, then we have

$$\|x_n - T_n x_n\| \longrightarrow 0, \quad (3.77)$$

as $n \rightarrow \infty$. By using the same way, we can prove that

$$\|x_n - S_n x_n\| \longrightarrow 0 \quad (3.78)$$

as $n \rightarrow \infty$. Since $x_n \rightarrow x^*$ and T_n, S_n are uniformly closed, $x^* = T_n x^*$, and $x^* = S_n x^*$, so that $x^* \in F$.

Step 4. we show that $x^* = \Pi_F^f x_0$.

Since $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n))$ is a closed and convex set, from Lemma 2.19, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_{n-1} \cap Q_{n-1}}^f(x_0)$ and $w \in F \subset C_{n-1} \cap Q_{n-1}$ we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 0. \quad (3.79)$$

We know that $G(\xi, J\varphi)$ is convex and lower semicontinuous with respect to ξ when φ is fixed. This implies that

$$G(x^*, Jx_0) \leq \liminf_{n \rightarrow \infty} G(x_n, Jx_0) \leq \limsup_{n \rightarrow \infty} G(x_n, Jx_0) \leq G(w, Jx_0). \quad (3.80)$$

From the definition of $\Pi_F^f x_0$ and $x^* \in F$, we see that $x^* = w$. This completes the proof. \square

Based on Theorem 3.3, we have the following.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E , and let $\{T_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}$ be two countable families of weak relatively*

nonexpansive mappings of C into itself such that $F := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned} z_n &= J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ C_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq (\beta_n^{(1)} - \alpha_n \beta_n^{(1)}) \phi(z, x_0) \right. \\ &\quad \left. + (1 - \beta_n^{(1)} + \alpha_n \beta_n^{(1)}) \phi(z, x_n) \right\}, \\ C_0 &= \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \right\}, \\ Q_0 &= C, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \end{aligned} \tag{3.81}$$

with the conditions:

- (i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;
- (ii) $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$;
- (iii) $\limsup_{n \rightarrow \infty} \beta_n^{(2)} \beta_n^{(3)} > 0$;
- (iv) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to x^* , where $x^* = \Pi_F x_0$.

Proof. Putting $f(x) \equiv 0$, we can conclude from Theorem 3.3 the desired conclusion immediately. \square

4. Applications

Let E be a real Banach space and let E^* be the dual space of E . Let C be a closed convex subset of E . Let f be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$. The equilibrium problem is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $EP(f)$. Given a mapping $T : C \rightarrow E^*$ let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $p \in EP(f)$ if and only if $\langle Tp, y - p \rangle \geq 0$ for all $y \in C$, that is, p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (4.1). Some methods have been proposed to solve the equilibrium problem in Hilbert spaces.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

- (A1) $f(x, x) = 0$, for all $x \in E$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in E$;
- (A3) for all $x, y, z \in E$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semi-continuous.

Lemma 4.1 (see [34]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{4.2}$$

Lemma 4.2 (see [34]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E , and let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4). For $r > 0$, define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C \right\}, \tag{4.3}$$

for all $x \in E$. Then, the following hold

- (1) T_r is single valued;
- (2) T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \tag{4.4}$$

- (3) $F(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex;
- (5) T_r is also a relatively nonexpansive mapping.

Lemma 4.3 (see [34]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$, $q \in F(T_r)$, then the following holds*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{4.5}$$

Lemma 4.4. *Let E be a p -uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E . Let f be a bifunction from $C \times C$ to $\mathbb{R} = (-\infty, +\infty)$ satisfying (A1)–(A4). Let $\{r_n\}$ be a positive real sequence such that $\lim_{n \rightarrow \infty} r_n = r > 0$. Then the sequence of mappings $\{T_{r_n}\}$ is uniformly closed.*

Proof. (1) Let $\{x_n\}$ be a convergent sequence in C . Let $z_n = T_{r_n}x_n$ for all n , then

$$f(z_n, y) + \frac{1}{r_n} \langle y - z_n, Jz_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \quad (4.6)$$

$$f(z_{n+m}, y) + \frac{1}{r_{n+m}} \langle y - z_{n+m}, Jz_{n+m} - Jx_{n+m} \rangle \geq 0, \quad \forall y \in C. \quad (4.7)$$

Putting $y = z_{n+m}$ in (4.6) and $y = z_n$ in (4.7), we have

$$\begin{aligned} f(z_n, z_{n+m}) + \frac{1}{r_n} \langle z_{n+m} - z_n, Jz_n - Jx_n \rangle &\geq 0, \quad \forall y \in C, \\ f(z_{n+m}, z_n) + \frac{1}{r_{n+m}} \langle z_n - z_{n+m}, Jz_{n+m} - Jx_{n+m} \rangle &\geq 0, \quad \forall y \in C. \end{aligned} \quad (4.8)$$

So, from (A2) we have

$$\left\langle z_{n+m} - z_n, \frac{Jz_n - Jx_n}{r_n} - \frac{Jz_{n+m} - Jx_{n+m}}{r_{n+m}} \right\rangle \geq 0, \quad (4.9)$$

and hence

$$\left\langle z_{n+m} - z_n, Jz_n - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0. \quad (4.10)$$

Thus, we have

$$\left\langle z_{n+m} - z_n, Jz_n - Jz_{n+m} + Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0, \quad (4.11)$$

which implies that

$$\langle z_{n+m} - z_n, Jz_{n+m} - Jz_n \rangle \leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \geq 0. \quad (4.12)$$

By using Lemma 2.25, we obtain

$$\begin{aligned} \frac{c^p}{c^{p-2}p} \|z_{n+m} - z_n\|^p &\leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}} (Jz_{n+m} - Jx_{n+m}) \right\rangle \\ &= \left\langle z_{n+m} - z_n, \left(1 - \frac{r_n}{r_{n+m}}\right) Jz_{n+m} + \frac{r_n}{r_{n+m}} Jx_{n+m} - Jx_n \right\rangle. \end{aligned} \quad (4.13)$$

Therefore, we get

$$\frac{c^p}{c^{p-2}p} \|z_{n+m} - z_n\|^{p-1} \leq \left| 1 - \frac{r_n}{r_{n+m}} \right| \|Jz_{n+m}\| + \left\| \frac{r_n}{r_{n+m}} Jx_{n+m} - Jx_n \right\|. \quad (4.14)$$

On the other hand, for any $p \in EP(f)$, from $z_n = T_{r_n}x_n$, we have

$$\|z_n - p\| = \|T_{r_n}x_n - p\| \leq \|x_n - p\|, \quad (4.15)$$

so that $\{z_n\}$ is bounded. Since $\lim_{n \rightarrow \infty} r_n = r > 0$, this together with (4.14) implies that $\{z_n\}$ is a Cauchy sequence. Hence $T_{r_n}x_n = z_n$ is convergent.

(2) By using the Lemma 4.2, we know that

$$\bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f) \neq \emptyset. \quad (4.16)$$

(3) From (1) we know that, $\lim_{n \rightarrow \infty} T_{r_n}x$ exists for all $x \in C$. So, we can define a mapping T from C into itself by

$$Tx = \lim_{n \rightarrow \infty} T_{r_n}x, \quad \forall x \in C. \quad (4.17)$$

It is obvious that, T is nonexpansive. It is easy to see that

$$EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T). \quad (4.18)$$

On the other hand, let $w \in F(T)$, $w_n = T_{r_n}w$, we have

$$f(w_n, y) + \frac{1}{r_n} \langle y - w_n, Jw_n - Jw \rangle \geq 0, \quad \forall y \in C. \quad (4.19)$$

By (A2) we know

$$\frac{1}{r_n} \langle y - w_n, Jw_n - Jw \rangle \geq f(y, w_n), \quad \forall y \in C. \quad (4.20)$$

Since $w_n \rightarrow Tw = w$ and from (A4), we have $f(y, w) \leq 0$, for all $y \in C$. Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y). \end{aligned} \quad (4.21)$$

Therefore, we have

$$f(ty + (1-t)w, y) \geq 0. \quad (4.22)$$

Letting $t \downarrow 0$ and using (A3), we get

$$f(w, y) \geq 0, \quad \forall y \in C \quad (4.23)$$

and hence $w \in EP(f)$. From the above two respects, we know that $F(T) = \bigcap_{n=0}^{\infty} F(T_{r_n})$.

Next we show $\{T_{r_n}\}$ is uniformly closed. Assume $x_n \rightarrow x$ and $\|x_n - T_{r_n}x_n\| \rightarrow 0$, from the above results we know that $Tx = \lim_{n \rightarrow \infty} T_{r_n}x$. On the other hand, from $\|x_n - T_{r_n}x_n\| \rightarrow 0$, we also get $\lim_{n \rightarrow \infty} T_{r_n}x = x$, so that $x \in F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. That is, the sequence of mappings $\{T_{r_n}\}$ is uniformly closed. This completes the proof. \square

Let A be a multi-valued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \quad (4.24)$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if it is graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well known.

Theorem 4.5. *Let E be a reflexive, strictly convex, and smooth Banach space and let A be a monotone operator from E to E^* . Then A is maximal if and only if $R(J + rA) = E^*$, for all $r > 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space, and let A be a maximal monotone operator from E to E^* . Using Theorem 4.5 and strict convexity of E , one obtains that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r. \quad (4.25)$$

Then one can define a single-valued mapping $J_r : E \rightarrow D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A , one knows that $A^{-1}0 = F(J_r)$ for all $r > 0$.

Theorem 4.6. *Let E be a uniformly convex and a uniformly smooth Banach space, and let A be a maximal monotone operator from E to E^* , let J_r be a resolvent of A for $r > 0$. Then for any sequence $\{r_n\}_{n=1}^{\infty}$ such that $\liminf_{n \rightarrow \infty} r_n > 0$, $\{J_{r_n}\}_{n=1}^{\infty}$ is a uniformly closed sequence of relatively quasi nonexpansive mappings.*

Proof. Firstly, we show that $\{J_{r_n}\}_{n=1}^{\infty}$ is uniformly closed. Let $\{z_n\} \subset E$ be a sequence such that $z_n \rightarrow p$ and $\lim_{n \rightarrow \infty} \|z_n - J_{r_n}z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \rightarrow 0. \quad (4.26)$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \in AJ_{r_n}z_n \quad (4.27)$$

and the monotonicity of A that

$$\left\langle w - J_{r_n} z_n, w^* - \frac{1}{r_n} (Jz_n - JJ_{r_n} z_n) \right\rangle \geq 0 \quad (4.28)$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \rightarrow \infty$, we have $\langle w - p, w^* \rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore from the maximality of A , we obtain $p \in A^{-1}0 = F(J_{r_n})$ for all $n \geq 1$, that is, $p \in \bigcap_{n=1}^{\infty} F(J_{r_n})$.

Next we show J_{r_n} is a relatively quasi nonexpansive mapping for all $n \geq 1$. For any $w \in E$ and $p \in F(J_{r_n}) = A^{-1}0$, from the monotonicity of A , we have

$$\begin{aligned} \phi(p, J_{r_n} w) &= \|p\|^2 - 2\langle p, JJ_{r_n} w \rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w - Jw \rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w \rangle - 2\langle p, Jw \rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle J_{r_n} w - p - J_{r_n} w, Jw - JJ_{r_n} w \rangle - 2\langle p, Jw \rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle J_{r_n} w - p, Jw - JJ_{r_n} w \rangle \\ &\quad + 2\langle J_{r_n} w, Jw - JJ_{r_n} w \rangle - 2\langle p, Jw \rangle + \|J_{r_n} w\|^2 \\ &\leq \|p\|^2 + 2\langle J_{r_n} w, Jw - JJ_{r_n} w \rangle - 2\langle p, Jw \rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 - \|J_{r_n} w\|^2 + 2\langle J_{r_n} w, Jw \rangle - \|w\|^2 \\ &= \phi(p, w) - \phi(J_{r_n} w, w) \\ &\leq \phi(p, w). \end{aligned} \quad (4.29)$$

This implies that J_{r_n} is a relatively quasi nonexpansive mapping for all $n \geq 1$. This completes the proof. \square

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