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Research Article

Toeplitz Operators on the Dirichlet Space of $\mathbb{B}_n$

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We study the algebraic properties of Toeplitz operators on the Dirichlet space of the unit ball $\mathbb{B}_n$. We characterize pluriharmonic symbol for which the corresponding Toeplitz operator is normal or isometric. We also obtain descriptions of conjugate holomorphic symbols of commuting Toeplitz operators. Finally, the commuting problem of Toeplitz operators whose symbols are of the form $z^p\overline{z}^q\phi(|z|^2)$ is studied.

1. Introduction

For any integer $n \geq 1$, let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball of $\mathbb{C}^n$ and dm denote the normalized Lebesgue measure on $\mathbb{B}_n$. The Sobolev space $w^{1,2}$ is defined to be the completion of smooth functions on $\mathbb{B}_n$ which satisfy

$$\|f\|^2 = \left|\int_{\mathbb{B}_n} f\,dm\right|^2 + \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left(\left|\frac{\partial f}{\partial z_i}\right|^2 + \left|\frac{\partial f}{\partial \overline{z}_i}\right|^2\right)dm < \infty. \tag{1.1}$$

The inner product $\langle\cdot,\cdot\rangle$ on $w^{1,2}$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{B}_n} f\,dm \int_{\mathbb{B}_n} \overline{g}\,dm + \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left(\frac{\partial f}{\partial z_i}\overline{\frac{\partial g}{\partial z_i}} + \frac{\partial f}{\partial \overline{z}_i}\overline{\frac{\partial g}{\partial \overline{z}_i}}\right)dm, \quad \forall f, g \in w^{1,2}. \tag{1.2}$$

The Dirichlet space $\mathfrak{D}$ of $\mathbb{B}_n$ is the closed subspace consisting of all holomorphic functions in $w^{1,2}$. It is easily verified that each point evaluation is a bounded linear functional on $\mathfrak{D}$. Hence, for each $z \in \mathbb{B}_n$, there exists a unique reproducing kernel $K_z(w) \in \mathfrak{D}$ such that

$$f(z) = \langle f(w), K_z(w) \rangle, \quad \forall f \in \mathfrak{D}. \tag{1.3}$$

Actually, it can be calculated that $K_z(w) = 1 + \sum_{\alpha \in \mathbb{Z}^{+n}}(((|\alpha| + n - 1)!/|\alpha|n!\alpha!)w^{\alpha}\overline{z}^{\alpha})$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{Z}^+$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. For multi-indexes α and β, the notation $\alpha \succeq \beta$ means that

$$\alpha_i \geq \beta_i, \quad i = 1, \ldots, n \tag{1.4}$$

and $\alpha \succ \beta$ means that $\alpha \succeq \beta$ and $\alpha \neq \beta$.

Let P be the orthogonal projection from $w^{1,2}$ onto $\mathfrak{D}$. By the explicit formula for $K_z(w)$, we have

$$P\psi(z) = \langle P\psi, K_z \rangle = \langle \psi, K_z \rangle = \int_{\mathbb{B}_n} \psi \, dm \int_{\mathbb{B}_n} \overline{K}_z dm + \sum_{i=1}^{n} \int_{\mathbb{B}_n} \frac{\partial \psi}{\partial w_i} \overline{\frac{\partial K_z}{\partial w_i}} dm(w), \quad \forall \psi \in w^{1,2}. \tag{1.5}$$

Let $\Omega = \{\varphi \in w^{1,2} : \varphi, \partial\varphi/\partial z_i, \partial\varphi/\partial\overline{z}_i \in L^{\infty}(\mathbb{B}_n)\}$. Given $\varphi \in \Omega$, the Toeplitz operator T_{φ} with symbol φ is the linear operator on $\mathfrak{D}$ defined by

$$T_{\varphi} f = P(\varphi f), \quad \forall f \in \mathfrak{D}. \tag{1.6}$$

It is easy to verify that the Toeplitz operator $T_{\varphi} : \mathfrak{D} \to \mathfrak{D}$ is always bounded, whenever $\varphi \in \Omega$.

The algebric properties of Toeplitz operators on the classical Hardy spaces and Bergman spaces have been well studied, for example, as in [1–5].

On the Hardy space of the unit circle, a well-known theorem of Brown and Halmos [1] has shown that two Toeplitz operators with bounded symbols commute if and only if one of the followings holds: (i) both symbols are holomorphic; (ii) both symbols are antiholomorphic; (iii) a nontrivial linear combination of the symbols is constant.

On the Bergman space, the commuting problem is more complicated. Axler and Čučković [2] proved that Brown-Halmos Theorem also holds for Toeplitze operators with bounded harmonic symbols. However, the corresponding problem for Toeplitz operator with general symbol remains open.

In recent years, more and more attention has been paid to the Toeplitz operators on Dirichlet spaces. The algebric properties of the Toeplitz operators on the classical Dirichlet spaces of the unit disc have been investigated intensively in [6–13]. Cao considered Fredholm properties of Toeplitz operators with $C^1(\mathbb{D})$ symbols in [6]. Lee showed in [8] that Brown-Halmos's result with harmonic symbols remains vaild on the Dirichlet space of the unit disc. In [12], Duistermaat and Lee gave the following characterizations of the harmonic symbols for which the associated Toeplitz operators are commuting, self-adjoint, or isometric: (1) for a harmonic symbol $u \in \Omega' = \{u \in C^1(\mathbb{D}) : u, \partial u/\partial z, \partial u/\partial\overline{z} \in L^{\infty}(\mathbb{D}, dA)\}$, T_u is self-adjoint if and only if u is a real constant function; (2) for a harmonic symbol $u \in \Omega'$, T_u is an isometry if and only if u is a constant function of modulus 1; (3) for two harmonic symbols $u, v \in \Omega'$, T_u and T_v commute if and only if either u and v are holomorphic or a nontrivial linear combination of u and v is constant on $\mathbb{D}$. In [13], the corresponding problems have been investigated on the polydisc Dirichlet spaces and similar results have been obtained.

Motivated by the work of [12, 13], we study the corresponding problems on the Dirichlet spaces of $\mathbb{B}_n$. In Section 2, we give the characterizations of the pluriharmonic symbol for which the associated Toeplitz operator is self-adjoint or an isometry. In Section 3, we discuss when two Toeplitz operators with conjugate holomorphic symbols commute. At last, we concern with the commuting Toeplitz operators with symbols $z^p\overline{z}^q\phi(|z|^2)$.

2. Characterization of Normality and Isometry

In this section, we will give the condition under which Toeplitz operators with pluriharmonic symbols are self-adjoint or isometric on $\mathfrak{D}$. Before doing this, we first exhibit some properties of Toeplitz operators on $\mathfrak{D}$.

Lemma 2.1. *Let $f = \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^\beta \in \Omega$ be holomorphic. Then the following statements hold:*

(1) $T_{\overline{f}}1 = \overline{f(0)}$;

(2) $T_f^*1 = \overline{f(0)}$;

(3) $T_{\overline{f}}^*1 = \int_{\mathbb{B}_n} f\overline{K_z}dm = f(0) + \sum_{|\beta|>0}((f_\beta/|\beta|(n+|\beta|))z^\beta)$.

Proof. By the definition of the Toeplitz operators and the properties of the reproducing kernel, we obtain that

$$\begin{aligned}
T_{\overline{f}}1 &= \langle T_{\overline{f}}1, K_z\rangle = \left\langle P\left(\overline{f}\right), K_z\right\rangle = \left\langle \overline{f(0)}, K_z\right\rangle = \overline{f(0)},\\
T_f^*1 &= \left\langle T_f^*1, K_z\right\rangle = \langle 1, fK_z\rangle = \int_{\mathbb{B}_n} \overline{fK_z}dm = \overline{\int_{\mathbb{B}_n} fK_z dm} = \overline{f(0)},\\
T_{\overline{f}}^*1 &= \left\langle T_{\overline{f}}^*1, K_z\right\rangle = \left\langle 1, \overline{f}K_z\right\rangle = \int_{\mathbb{B}_n} f\overline{K_z}dm\\
&= \int_{\mathbb{B}_n}\sum_{\beta\in\mathbb{Z}^{n+}} f_\beta w^\beta \sum_{\alpha\in\mathbb{Z}^{+n}} \frac{(|\alpha|+n-1)!}{|\alpha|n!\alpha!} z^\alpha\overline{w}^\alpha dm\\
&= f(0) + \sum_{|\beta|>0} f_\beta \frac{(|\beta|+n-1)!}{|\beta|n!\beta!} z^\beta \int_{\mathbb{B}_n} \left|w^\beta\right|^2 dm\\
&= f(0) + \sum_{|\beta|>0} f_\beta \frac{(|\beta|+n-1)!}{|\beta|n!\beta!} z^\beta \cdot \frac{n!\beta!}{(n+|\beta|)!}\\
&= f(0) + \sum_{|\beta|>0} \frac{f_\beta}{|\beta|(n+|\beta|)} z^\beta.
\end{aligned} \tag{2.1}$$

□

Lemma 2.2. *Let $u = f + \overline{g} \in \Omega$, where f and g are holomorphic. If T_u is normal, then*

$$\sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial f}{\partial z_i} \right|^2 dm = \sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial G}{\partial z_i} \right|^2 dm, \tag{2.2}$$

where $G = T^*_{\overline{g}}(1)$.

Proof. By assumption, we have $T^*_{f+\overline{g}} T_{f+\overline{g}} = T_{f+\overline{g}} T^*_{f+\overline{g}}$. In particular,

$$\left\langle T^*_{f+\overline{g}} T_{f+\overline{g}} 1, 1 \right\rangle = \left\langle T_{f+\overline{g}} T^*_{f+\overline{g}} 1, 1 \right\rangle. \tag{2.3}$$

That is,

$$\left\langle T_{f+\overline{g}} 1, T_{f+\overline{g}} 1 \right\rangle = \left\langle T^*_{f+\overline{g}} 1, T^*_{f+\overline{g}} 1 \right\rangle. \tag{2.4}$$

It follows from Lemma 2.1 that

$$\left\langle f + \overline{g(0)}, f + \overline{g(0)} \right\rangle = \left\langle \overline{f(0)} + G, \overline{f(0)} + G \right\rangle. \tag{2.5}$$

Hence,

$$\left| f(0) + \overline{g(0)} \right|^2 + \sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial f}{\partial z_i} \right|^2 dm = \left| \overline{f(0)} + G(0) \right|^2 + \sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial G}{\partial z_i} \right|^2 dm. \tag{2.6}$$

On the other hand, by the reproducing property and Lemma 2.1, we have

$$G(0) = T^*_{\overline{g}} 1(0) = \left\langle T^*_{\overline{g}} 1, K_0 \right\rangle = \left\langle T^*_{\overline{g}} 1, 1 \right\rangle = g(0). \tag{2.7}$$

Then,

$$\sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial f}{\partial z_i} \right|^2 dm = \sum_{i=1}^{n} \int_{\mathbb{B}_n} \left| \frac{\partial G}{\partial z_i} \right|^2 dm. \tag{2.8}$$

This completes the proof. □

The next lemma shows there is only trivial normal Toeplitz operator with holomorphic (or antiholomorphic) symbols.

Lemma 2.3. *Let $f = \sum_{\beta \in \mathbb{Z}^{+n}} f_\beta z^\beta \in \Omega$ be holomorphic. Then the following statements are equivalent:*

(1) *T_f is normal;*

(2) *$T_{\overline{f}}$ is normal;*

(3) $T_{f+\overline{f}}$ *is normal;*

(4) f *is a constant function on* $\mathbb{B}_n$.

Proof. (1) ⇒ (4) By Lemma 2.2 (with $g = 0$), we have that

$$\sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial f}{\partial z_i}\right|^2 dm = \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial G}{\partial z_i}\right|^2 dm = 0. \tag{2.9}$$

This along with the fact that

$$\sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial f}{\partial z_i}\right|^2 dm = \|f - f(0)\|^2 = \sum_{|\beta|>0}|f_\beta|^2\left\|z^\beta\right\|^2 \tag{2.10}$$

proves that $f_\beta = 0$, for $|\beta| > 0$. This shows that f is a constant.

(2) ⇒ (4) Since $T_{\overline{f}}$ is normal, it follows from Lemma 2.2 that

$$\sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial G}{\partial z_i}\right|^2 dm = \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial \overline{f}}{\partial z_i}\right|^2 dm = 0, \tag{2.11}$$

which implies that G is a constant function for G is holomorphic.

On the other hand, Lemma 2.1 ensures that

$$G = T_{\overline{f}}^{*}1 = f(0) + \sum_{|\beta|>0}\frac{f_\beta}{|\beta|(n+|\beta|)}z^\beta. \tag{2.12}$$

It follows that $f_\beta = 0$, for all $|\beta| > 0$. Hence f is a constant, as desired.

(3) ⇒ (4) Suppose $T_{f+\overline{f}}$ is normal. Using Lemma 2.2,

$$\sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial f}{\partial z_i}\right|^2 dm = \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial G}{\partial z_i}\right|^2 dm, \tag{2.13}$$

where $G = T_{\overline{f}}^{*}(1)$. We conclude that $f_\beta = 0$, for $|\beta| > 0$, since by direct computation and Lemma 2.1

$$\begin{gathered}\sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial f}{\partial z_i}\right|^2 dm = \sum_{|\beta|>0}|f_\beta|^2\left\|z^\beta\right\|^2, \\ \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left|\frac{\partial G}{\partial z_i}\right|^2 dm = \sum_{|\beta|>0}\frac{|f_\beta|^2}{|\beta|^2(n+|\beta|)^2}\left\|z^\beta\right\|^2.\end{gathered} \tag{2.14}$$

Hence f is a constant.

The converse implications are clear. The proof is complete. □

Since $\|T_{\overline{f}}1\| \geq \|T^*_{\overline{f}}1\|$ for hyponormal Toeplitz operator $T_{\overline{f}}$, using Lemma 2.1, $T_{\overline{f}}$ is hyponormal if and only if f is a constant. Consequently, normality of $T_{\overline{f}}$ can be replaced by hyponormality in Lemma 2.3.

On the Hardy space and the Bergman space, we always have $T^*_u = T_{\overline{u}}$. So it is easy to see that T_u is self-adjoint (i.e., $T_u = T^*_u$) if and only if u is a real-valued function. However, on the Dirichlet space of disc and polydisc, the situations are different because T^*_u is not equal to $T_{\overline{u}}$ in both cases. In the following, we will study the adjoint of Toeplitz operators with pluriharmonic symbols on the Dirichlet space of $\mathbb{B}_n$.

Theorem 2.4. *Let $u = f + \overline{g} \in \Omega$, where f and g are holomorphic. Then $T^*_u = T_{\overline{u}}$ if and only if u is a constant function.*

Proof. First, assume that $u = \sum_{\beta \in \mathbb{Z}^{+n}} a_\beta z^\beta$ is holomorphic. Since $T^*_u = T_{\overline{u}}$, for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we have

$$T^*_u z^\alpha = T_{\overline{u}} z^\alpha, \quad \forall |\alpha| > 0. \tag{2.15}$$

Moreover,

$$\langle T^*_u z^\alpha, 1 \rangle = \langle T_{\overline{u}} z^\alpha, 1 \rangle. \tag{2.16}$$

In fact, for $|\alpha| > 0$,

$$\langle T^*_u z^\alpha, 1 \rangle = \langle z^\alpha, T_u 1 \rangle = \langle z^\alpha, u \rangle = \overline{a_\alpha} \|z^\alpha\|^2 = \overline{a_\alpha} \cdot \frac{|\alpha| n! \alpha!}{(n + |\alpha| - 1)!}. \tag{2.17}$$

On the other hand,

$$\langle T_{\overline{u}} z^\alpha, 1 \rangle = \langle \overline{u} z^\alpha, 1 \rangle = \int_{\mathbb{B}_n} \overline{u} z^\alpha dm = \int_{\mathbb{B}_n} \overline{a_\alpha} |z^\alpha|^2 dm = \overline{a_\alpha} \cdot \frac{n! \alpha!}{(n + |\alpha|)!}. \tag{2.18}$$

Note that

$$\frac{|\alpha| n! \alpha!}{(n + |\alpha| - 1)!} - \frac{n! \alpha!}{(n + |\alpha|)!} > 0, \tag{2.19}$$

we conclude that, $a_\alpha = 0$, for $|\alpha| > 0$.

Second, assume that $u = f + \overline{g}$ is the general pluriharmonic symbol and $T^*_u = T_{\overline{u}}$. In particular, we have

$$\left(T^*_f + T^*_{\overline{g}}\right) 1 = T_{\overline{f}} 1 + T_g 1. \tag{2.20}$$

By Lemma 2.1, we get that

$$\overline{f(0)} + \int_{\mathbb{B}_n} g \overline{K_z} dm = \overline{f(0)} + g. \tag{2.21}$$

Since

$$\int_{\mathbb{B}_n} g\overline{K_z}dm = g(0) + \sum_{|\beta|>0} \frac{g_\beta}{|\beta|(n+|\beta|)} z^\beta, \tag{2.22}$$

where $g = \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta z^\beta$, it follows that

$$\sum_{|\beta|>0} \frac{g_\beta}{|\beta|(n+|\beta|)} z^\beta = \sum_{|\beta|>0} g_\beta z^\beta. \tag{2.23}$$

Equivalently,

$$\sum_{|\beta|>0} \left(1 - \frac{1}{|\beta|(n+|\beta|)}\right) g_\beta z^\beta = 0. \tag{2.24}$$

This implies that $g_\beta = 0$, for $|\beta| > 0$. So $u = f + \overline{g(0)}$ is holomorphic. The desired result follows immediately from the previous holomorphic case.

The converse implication is clear. The proof is complete. □

We now characterize pluriharmonic symbols inducing self-adjoint Toeplitz operators.

Theorem 2.5. *Let $u = f + \overline{g} \in \Omega$, where f, g are holomorphic. Then $T_u = T_u^*$ if and only if u is a real constant function.*

Proof. The "if" part is clear. Suppose $f = \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^\beta$ and $g = \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta z^\beta$. It follows from Lemma 2.1 that

$$\begin{aligned} T_u 1 = T_{f+\overline{g}} 1 = f + \overline{g(0)} = f(0) + \overline{g(0)} + \sum_{|\beta|>0} f_\beta z^\beta, \\ T_u^* 1 = T_{f+\overline{g}}^* 1 = T_f^* 1 + T_{\overline{g}}^* 1 = \overline{f(0)} + g(0) + \sum_{|\beta|>0} \frac{g_\beta}{|\beta|(n+|\beta|)} z^\beta. \end{aligned} \tag{2.25}$$

Since $T_u 1 = T_u^* 1$, by comparing the coefficients of the above two equations, we have that

$$\frac{g_\beta}{|\beta|(n+|\beta|)} = f_\beta, \quad |\beta| > 0, \tag{2.26}$$

$$\overline{f(0)} + g(0) = f(0) + \overline{g(0)}. \tag{2.27}$$

Let $w_i = w^{e_i}$, where $e_i = (\underbrace{0,0,\ldots,0,1}_{i},0,\ldots,0)$. Then we have

$$\begin{aligned}
\langle \overline{g}w_i, K_z(w)\rangle &= \int_{\mathbb{B}_n} \overline{g}w_i dm + \sum_{j=1}^{n}\int_{\mathbb{B}_n} \frac{\partial(\overline{g}w_i)}{\partial w_j}\overline{\frac{\partial K_z(w)}{\partial w_j}}dm \\
&= \int_{\mathbb{B}_n} \overline{\sum_{\beta\in\mathbb{Z}^{+n}} g_\beta w^\beta} w_i dm + \int_{\mathbb{B}_n} \overline{g}\,\overline{\frac{\partial K_z(w)}{\partial w_i}}dm \\
&= \overline{g_{e_i}}\cdot\int_{\mathbb{B}_n} |w^{e_i}|^2 dm + \overline{\int_{\mathbb{B}_n} \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta w^\beta \cdot \sum_{\alpha\geq e_i} \frac{(|\alpha|+n-1)!}{|\alpha|n!\alpha!}\alpha_i w^{\alpha-e_i}\overline{z}^\alpha dm} \\
&= \frac{\overline{g_{e_i}}}{n+1} + \overline{g(0)}\cdot z^{e_i}.
\end{aligned} \tag{2.28}$$

This shows that

$$\left(T_{f+\overline{g}}w_i\right)(z) = \left(T_f w_i\right)(z) + \left(T_{\overline{g}}w_i\right)(z) = \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^{\beta+e_i} + \frac{\overline{g_{e_i}}}{n+1} + \overline{g(0)}\cdot z^{e_i}. \tag{2.29}$$

On the other hand, if $h = \sum_{\beta\in\mathbb{Z}^{+n}} h_\beta w^\beta$ is holomorphic, then

$$\int_{\mathbb{B}_n} \frac{\partial h}{\partial w_i} dm = \int_{\mathbb{B}_n} \sum_{\beta\geq e_i} h_\beta \beta_i w^{\beta-e_i} dm = h_{e_i}. \tag{2.30}$$

Therefore,

$$\int_{\mathbb{B}_n} \frac{\partial\left(fK_z(w)\right)}{\partial w_i} dm = f_{e_i} + f(0)\cdot\overline{z}^{e_i}. \tag{2.31}$$

A direct computation shows that

$$\begin{aligned}
\int_{\mathbb{B}_n} \overline{g}\cdot\frac{\partial K_z(w)}{\partial w_i}dm &= \int_{\mathbb{B}_n} \overline{\sum_{\alpha\in\mathbb{Z}^{+n}} g_\alpha w^\alpha}\cdot\sum_{\alpha\geq e_i}\frac{(|\beta|+n-1)!}{|\beta|n!\beta!}\beta_i w^{\beta-e_i}\overline{z}^\beta dm \\
&= \sum_{\alpha\in\mathbb{Z}^{+n}} \overline{g_\alpha}\frac{(|\alpha+e_i|+n-1)!}{|\alpha+e_i|n!(\alpha+e_i)!}(\alpha_i+1)\overline{z}^{\alpha+e_i}\cdot\int_{\mathbb{B}_n}|w^\alpha|^2 dm \\
&= \sum_{\alpha\in\mathbb{Z}^{+n}} \overline{g_\alpha}\frac{\overline{z}^{\alpha+e_i}}{|\alpha|+1}.
\end{aligned} \tag{2.32}$$

Hence,

$$\begin{aligned}\left(T^*_{f+\overline{g}}w_i\right)(z) &= \left\langle T^*_{f+\overline{g}}w_i, K_z(w)\right\rangle = \langle w_i, (f+\overline{g})K_z(w)\rangle \\ &= \int_{\mathbb{B}_n} \overline{\frac{\partial((f+\overline{g})K_z(w))}{\partial w_i}} dm \\ &= \overline{\int_{\mathbb{B}_n}\left[\frac{\partial(fK_z(w))}{\partial w_i} + \overline{g}\cdot\frac{\partial K_z(w)}{\partial w_i}\right]dm} \\ &= \overline{f_{e_i}} + \overline{f(0)}\cdot z^{e_i} + \sum_{\alpha\in\mathbb{Z}^{+n}} g_\alpha \frac{z^{\alpha+e_i}}{|\alpha|+1}.\end{aligned} \tag{2.33}$$

Comparing the expressions of $(T_{f+\overline{g}}w_i)(z)$ and $(T^*_{f+\overline{g}}w_i)(z)$, we obtain

$$\frac{g_\beta}{|\beta|+1} = f_\beta, \quad \forall|\beta| > 0. \tag{2.34}$$

It follows from (2.26) and (2.34) that

$$f_\beta = g_\beta = 0, \quad \text{for } |\beta| > 0,\ n > 1, \tag{2.35}$$

which, according to (2.27), implies that $u = f(0) + \overline{g(0)}$ is a real constant function. This complete the proof of the theorem. □

Note that for Theorem 2.5 the assumption "$u = f + \overline{g} \in \Omega$, where f, g are holomorphic" can not be removed. For example, let $u = 1 - |z|^2$, that is $u = 1 - (z_1^2 + z_2^2 + \cdots + z_n^2)$, then by the below Theorem 4.5 $T_u = T_u^* = 0$. However, u is not a constant function.

Corollary 2.6. *Let $u = f + \overline{g} \in \Omega$, where f and g are holomorphic. Then T_u is a projection operator if and only if $u = 1$ or $u = 0$.*

Proof. The "if" part is clear. If T_u is a projection, then $T_u = T_u^*$. Theorem 2.5 implies that $u = c$ where c is a real. Since $T_u = T_u^2$, we see that $c^2 = c$. This proves $u = 1$ or $u = 0$. □

Next, we will characterize pluriharmonic symbols for which the corresponding Toeplitz operator is an isometry.

Theorem 2.7. *Let $u = f + \overline{g} \in \Omega$, where f and g are holomorphic. Then the following statements are equivalent:*

(1) *T_u is unitary;*

(2) *T_u is isometric;*

(3) *u is a constant function of modulus 1.*

Proof. That (1) implies (2) follows from the fact that unitary operator is isometric.

To prove that (2) implies (3), we denote $f = \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^\beta$ and $g = \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta z^\beta$. Recalling the proof of Theorem 2.5, we have that

$$\begin{aligned} T_{f+\overline{g}}1 &= f(0) + \overline{g(0)} + \sum_{|\beta|>0} f_\beta z^\beta, \\ T_{f+\overline{g}}w_i(z) &= \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^{\beta+e_i} + \frac{\overline{g_{e_i}}}{n+1} + \overline{g(0)}\cdot z^{e_i}, \\ T^*_{f+\overline{g}}1 &= \overline{f(0)} + g(0) + \sum_{|\beta|>0} \frac{g_\beta}{|\beta|(n+|\beta|)} z^\beta, \\ \left(T^*_{f+\overline{g}}w_i\right)(z) &= \overline{f_{e_i}} + \overline{f(0)}\cdot z^{e_i} + \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta \frac{z^{\beta+e_i}}{|\beta|+1}. \end{aligned} \tag{2.36}$$

Calculating the norms of the above items, it follows that

$$\left\|T_{f+\overline{g}}1\right\|^2 = \left|f(0)+\overline{g(0)}\right|^2 + \sum_{|\beta|>0} |f_\beta|^2 \left\|z^\beta\right\|^2, \tag{2.37}$$

$$\left\|T_{f+\overline{g}}w_i(z)\right\|^2 = \sum_{|\beta|>0} |f_\beta|^2 \left\|z^{\beta+e_i}\right\|^2 + \left|\frac{\overline{g_{e_i}}}{n+1}\right|^2 + \left|f(0)+\overline{g(0)}\right|^2, \tag{2.38}$$

$$\left\|T^*_{f+\overline{g}}1\right\|^2 = \left|\overline{f(0)}+g(0)\right|^2 + \sum_{|\beta|>0} \left|\frac{g_\beta}{|\beta|(n+|\beta|)}\right|^2 \left\|z^\beta\right\|^2, \tag{2.39}$$

$$\left\|(T^*_{f+\overline{g}}w_i)(z)\right\|^2 = \left|\overline{f_{e_i}}\right|^2 + \left|\overline{f(0)}+g(0)\right|^2 + \sum_{|\beta|>0} \left|\frac{g_\beta}{|\beta|+1}\right|^2 \left\|z^{\beta+e_i}\right\|^2. \tag{2.40}$$

By the assumption, (2.37), (2.38), (2.39), and (2.40) are all equal to 1 since T_u as well as T_u^* is an isometry.

Note that $\|z^\beta\|^2/[|\beta|(n+|\beta|)]^2 < \|z^{\beta+e_i}\|^2/(1+|\beta|)^2$, for $|\beta|>0$ and $n>1$. Comparing (2.39) and (2.40), we obtain that $|f(0)+\overline{g(0)}| = 1$ and $g_\beta = 0$, for $|\beta|>0$ and $n>1$. Then (2.37) implies that

$$f_\beta = 0, \quad \text{for } |\beta| > 0,\ n > 1. \tag{2.41}$$

Therefore, $u = f(0) + \overline{g(0)}$ and $|u| = |f(0)+\overline{g(0)}| = 1$.

Finally, if $u = c$ is a constant function, by Theorem 2.4, we have $T_u^* = T_{\overline{u}}$. The desired implication (3) $\Rightarrow$ (1) follows from the fact that $T_u^*T_u = T_{\overline{u}}T_u = M_{|u|^2} = 1$ and $T_uT_u^* = T_uT_{\overline{u}} = M_{|u|^2} = 1$. □

3. Commuting Toeplitz Operators with Conjugate Holomorphic Symbols

In this section, we will study the commuting problems of Toeplitz operators with conjugate holomorphic symbols. By the definition of T_ϕ, if $\phi \in \Omega$ is holomorphic, then $T_\phi = M_\phi$. Therefore, for two holomorphic symbols $f, g \in \Omega$, T_f, and T_g commute. It is natural to ask when $T_{\overline{f}}$ and $T_{\overline{g}}$ commute, The following theorem shows that $T_{\overline{f}}$ commutes with $T_{\overline{g}}$ only in the trival case. In this section, we may always assume $f = \sum_{\beta\in\mathbb{Z}^{+n}} f_\beta z^\beta$ and $g = \sum_{\beta\in\mathbb{Z}^{+n}} g_\beta z^\beta$.

Theorem 3.1. *Let $f \in \Omega$ and $g \in \Omega$ be holomorphic. Then $T_{\overline{g}}T_{\overline{f}} = T_{\overline{f}}T_{\overline{g}}$ if and only if for $\alpha,\ \beta,\ \gamma \succ 0$,*

$$\sum_{\gamma+\beta=\alpha} \overline{g_\gamma}\,\overline{f_\beta}\frac{(|\gamma| - |\beta|)}{(n + |\beta|)(n + |\gamma|)} = 0. \tag{3.1}$$

Proof. Suppose reproducing kernel $K_z(w) = 1 + \sum_{|\alpha|>0}((((|\alpha| + n - 1)!/|\alpha|n!\alpha!)w^\alpha\overline{z}^\alpha) = 1 + \sum_{|\alpha|>0} c_\alpha w^\alpha \overline{z}^\alpha$. Without loss of generality, we may assume $f(0) = g(0) = 0$.

Note that for $\alpha \succ \beta \succ 0$,

$$P\left(\overline{z}^\beta z^\alpha\right) = d(\alpha, \alpha - \beta) z^{\alpha-\beta}, \tag{3.2}$$

where $d(\alpha, \alpha - \beta) = (\alpha!/(n + |\alpha| - 1)!)/((\alpha - \beta)!/(n + |\alpha - \beta| - 1)!)$. It follows that

$$T_{\overline{g}} z^\alpha = P(\overline{g} w^\alpha) = \overline{g_\alpha}\|z^\alpha\|_2^2 + \sum_{\gamma<\alpha}\overline{g_\gamma} d(\alpha, \alpha - \gamma) z^{\alpha-\gamma}. \tag{3.3}$$

Therefore, we have

$$\begin{aligned}
T_{\overline{f}}\left[T_{\overline{g}} z^\alpha\right] &= \overline{g_\alpha}\|z^\alpha\|_2^2 T_{\overline{f}}(1) + \sum_{\gamma<\alpha}\overline{g_\gamma} d(\alpha, \alpha - \gamma)\left(T_{\overline{f}} z^{\alpha-\gamma}\right) \\
&= \overline{g_\alpha}\overline{f(0)}\|z^\alpha\|_2^2 + \sum_{\gamma<\alpha}\overline{g_\gamma}\,\overline{f_{\alpha-\gamma}} d(\alpha, \alpha - \gamma)\left\|z^{\alpha-\gamma}\right\|_2^2 \\
&\quad + \sum_{\gamma<\alpha}\sum_{\beta<\alpha-\gamma}\overline{g_\gamma}\,\overline{f_\beta} d(\alpha, \alpha - \gamma) d(\alpha - \gamma, \alpha - \gamma - \beta) z^{\alpha-\gamma-\beta} \\
&= \sum_{\gamma+\beta=\alpha}\overline{g_\gamma}\,\overline{f_\beta}\frac{n!\alpha!}{(n + |\beta|)(n + |\alpha| - 1)!} + \sum_{\gamma+\beta<\alpha}\overline{g_\gamma}\,\overline{f_\beta} d(\alpha, \alpha - \gamma - \beta) z^{\alpha-\gamma-\beta}.
\end{aligned} \tag{3.4}$$

Similarly, we have that

$$\begin{aligned}
T_{\overline{g}}\left[T_{\overline{f}} z^\alpha\right] &= \sum_{\gamma+\beta=\alpha}\overline{f_\gamma}\,\overline{g_\beta}\frac{n!\alpha!}{(n + |\beta|)(n + |\alpha| - 1)!} + \sum_{\gamma+\beta<\alpha}\overline{f_\gamma}\,\overline{g_\beta} d(\alpha, \alpha - \gamma - \beta) z^{\alpha-\gamma-\beta} \\
&= \sum_{\gamma+\beta=\alpha}\overline{g_\gamma}\,\overline{f_\beta}\frac{n!\alpha!}{(n + |\gamma|)(n + |\alpha| - 1)!} + \sum_{\gamma+\beta<\alpha}\overline{g_\gamma}\,\overline{f_\beta} d(\alpha, \alpha - \gamma - \beta) z^{\alpha-\gamma-\beta}.
\end{aligned} \tag{3.5}$$

Observe that (3.4) and (3.5) have the same coefficients of $z^{\alpha-\gamma-\beta}$, it follows that

$$T_{\overline{f}}\left[T_{\overline{g}}z^{\alpha}\right] = T_{\overline{g}}\left[T_{\overline{f}}z^{\alpha}\right] \tag{3.6}$$

if and only if

$$\sum_{\gamma+\beta=\alpha} \overline{g_{\gamma}}\,\overline{f_{\beta}}\frac{(|\gamma| - |\beta|)}{(n+|\beta|)(n+|\gamma|)} \cdot \frac{n!\alpha!}{(n+|\alpha|-1)!} = 0. \tag{3.7}$$

Since $n!\alpha!/(n+|\alpha|-1)! > 0$, the desired result is obtained. □

Corollary 3.2. *If $f_{\beta} = g_{\beta} = 0$ for $|\beta| \neq k_0$, where k_0 is a positive integer, then $T_{\overline{f}}T_{\overline{g}} = T_{\overline{g}}T_{\overline{f}}$.*

Proof. Since each item $\overline{g_{\gamma}}\,\overline{f_{\beta}}(|\gamma| - |\beta|)$ equals to 0, (3.1) is satisfied. Thus the desired result follows by Theorem 3.1. □

For example, $T_{\overline{z}^{(1,1)}}T_{\overline{z}^{(2,0)}} = T_{\overline{z}^{(2,0)}}T_{\overline{z}^{(1,1)}}$ since $|(1,1)| = |(2,0)| = 2$. On the Dirichlet space of the unit disc or polydisc, Dusitermaat, Lee, Geng, and Zhou prove that for holomorphic functions f and g, $T_{\overline{g}}T_{\overline{f}} = T_{\overline{f}}T_{\overline{g}}$ if and only if f, g, and 1 are dependent, see [12, 13]. However, this is not true on the unit ball Dirichlet space by Corollary 3.2. Indeed, the condition that f, g, and 1 are linearly dependent is sufficient but not necessary for the commuting of $T_{\overline{f}}$ and $T_{\overline{g}}$.

Next, we will discuss when Toeplitz operator with holomorphic symbol and Toeplitz operator with conjugate holomorphic symbol will commute.

Theorem 3.3. *Let $f \in \Omega$ and $g \in \Omega$ be holomorphic. Then $T_{\overline{g}}T_f = T_fT_{\overline{g}}$ if and only if f or g is a constant function.*

Proof. The "if" part implication is obvious. Now suppose $T_{\overline{g}}T_f = T_fT_{\overline{g}}$. For each multi-index α, we have

$$\begin{aligned} T_f\left[T_{\overline{g}}z^{\alpha}\right] &= f \cdot T_{\overline{g}}z^{\alpha} \\ &= \sum_{\beta\geq 0} f_{\beta}z^{\beta} \cdot \left[\overline{g_{\alpha}}\|z^{\alpha}\|^2 + \sum_{\gamma<\alpha}\overline{g_{\gamma}}d(\alpha,\alpha-\gamma)z^{\alpha-\gamma}\right] \\ &= \overline{g_{\alpha}}\|z^{\alpha}\|^2 \cdot f(0) + \overline{g_{\alpha}}\|z^{\alpha}\|^2\sum_{\beta>0} f_{\beta}z^{\beta} + \sum_{\gamma<\alpha}\sum_{\beta\geq 0}\overline{g_{\gamma}}f_{\beta}d(\alpha,\alpha-\gamma)z^{\alpha-\gamma+\beta}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} T_{\overline{g}}\left[T_fz^{\alpha}\right] &= T_{\overline{g}}(fz^{\alpha}) \\ &= \sum_{\xi=\beta+\alpha}\overline{g}_{\xi}f_{\beta}\left\|w^{\xi}\right\|^2 + \sum_{\xi<\alpha+\beta}\sum_{\beta\geq 0}\overline{g}_{\xi}f_{\beta}d(\alpha+\beta,\alpha+\beta-\xi)z^{\alpha+\beta-\xi}. \end{aligned} \tag{3.9}$$

Assume that f is not a constant function. Hence there exists $\beta_0 \succ 0$ such that $f_{\beta_0} \neq 0$.

For (3.8), let $\beta = \beta_0$ and $\gamma = \alpha \succ 0$, the coefficient of z^{β_0} is

$$\overline{g}_\alpha f_{\beta_0} \frac{n!\alpha!}{(n+|\alpha|)!}. \tag{3.10}$$

On the other hand, if we let $\beta = \beta_0$ and $\xi = \alpha \succ 0$ in (3.9), then the coefficient of z^{β_0} is

$$\overline{g}_\alpha f_{\beta_0} d(\alpha + \beta_0, \beta_0). \tag{3.11}$$

Since

$$\frac{n!\alpha!}{(n+|\alpha|)!} \neq d(\alpha + \beta_0, \beta_0) \tag{3.12}$$

and $f_{\beta_0} \neq 0$, we deduce that

$$\overline{g}_\alpha = 0, \quad \text{for } |\alpha| > 0, \tag{3.13}$$

which implies that g is a constant function. The proof is complete. □

4. Commuting Toeplitz Operators with Symbols $z^p\overline{z}^q\phi(|z|^2)$

Zhou and Dong [14] discussed the commuting and zero product problems of Toeplitz operators on the Bergman space of the unit ball in $\mathbb{C}^n$ whose symbols are of the form $\xi^k\phi$ where ϕ is a radial function. In [15], they generalized the case of the radial symbols to that of the separately quasi-homogeneous symbols. In [16], Grudsky et al. considered weighted Bergman spaces on the unit ball in $\mathbb{C}^n$. In terms of the Wick symbol of a Toeplitz operator, the complete information about the operator with radial symbols was given. Vasilevski [17] studied the Toeplitz operators with the quasi-radial quasi-homogeneous symbol. For the case of Dirichlet spaces, Chen et al. [18, 19] studied the quasi-radial Toeplitz operaors on the disk. However, little work has been done in the unit ball case. The commuting problem on it is subtle and no general answer is known. Dong and Zhou [15] have shown that any function f in $L^2(\mathbb{B}_n, dm)$ has the decomposition $f(z) = \sum_{k\in\mathbb{Z}} \xi^k f_k(r)$, where $f_k(r)$ is separately radial. In this section, the commuting and zero product problems of Toeplitz operators $T_{z^p\overline{z}^q\phi(|z|^2)}$, p, $q \succeq 0$ will be concerned, which may be helpful to the further study of the commuting Toeplitz operators with general symbols. We denote $\Sigma = \{\phi : \phi, \phi' \in L^1([0,1])\}$ and $\Sigma' = \{\phi$ is absolutely continuous on [0,1): $\phi, \phi' \in L^1([0,1))\}$. In the remaining part of this paper, we will always assume $\phi \in \Sigma$. A direct calculation gives the following lemma.

Lemma 4.1. *Let $p \succeq 0$ and $\phi(|z|^2) \in \Sigma$ be radial functions. Then*

$$T_{z^p\phi(|z|^2)} z^\alpha = \begin{cases} (n + |p+\alpha| - 1)\widehat{\phi}(n + |p+\alpha| - 1) z^{p+\alpha}, & p + \alpha \succ 0, \\ n \displaystyle\int_0^1 \phi(r) r^{n-1} dr, & p = \alpha = 0, \end{cases} \tag{4.1}$$

where $\widehat{\phi}(z) = \int_0^1 r^{z-1}[\phi + \int_r^1 \phi'(t)dt]dr$.

Proof. To simiplify the statement, we denote reproducing kernel $K_z(w)$ by $\sum_{|\gamma|\geq 0} c_\gamma w^\gamma \overline{z}^\gamma$. Notice that $(\partial/\partial w_i)(w^{p+\alpha}\phi(|w|^2)) = (p_i+\alpha_i)\phi w^{p+\alpha-e_i} + w^{p+\alpha}\phi'\overline{w}_i$. For $p+\alpha \succ 0$, with integration in polar coordinates we have

$$
\begin{aligned}
T_{z^p\phi(|z|^2)} z^\alpha &= \left\langle w^{p+\alpha}\phi, K_z(w)\right\rangle \\
&= \sum_{i=1}^{n} \int_{\mathbb{B}_n} \frac{\partial}{\partial w_i}\left(w^{p+\alpha}\phi\right) \overline{\frac{\partial(K_z(w))}{\partial w_i}} dm \\
&= \left[\sum_{i=1}^{n} \int_{\mathbb{B}_n} (p_i+\alpha_i)^2 c_{p+\alpha}\phi \left|w^{p+\alpha-e_i}\right|^2 dm \right. \\
&\qquad \left. + \int_{\mathbb{B}_n} (p_i+\alpha_i) c_{p+\alpha}\phi' |w^{p+\alpha}|^2 dm \right] z^{p+\alpha} \\
&= \left[\left(n + \left|p+\alpha\right| - 1\right) \int_0^1 t^{n+|p+\alpha|-2}\phi(t)dt + \int_0^1 t^{n+|p+\alpha|-1}\phi' dt\right] z^{p+\alpha}.
\end{aligned}
\tag{4.2}
$$

Since $\int_0^1 t^{n+|p+\alpha|-1}\phi' dt = (n+|p+\alpha|-1)\int_0^1 t^{n+|p+\alpha|-2}[\int_t^1 \phi' dr]dt$ with integration by part, the desired result is obvious.

For $p = \alpha = 0$, it is easy to see that

$$
T_{z^p\phi(|z|^2)} z^\alpha = \left\langle \phi, K_z(w)\right\rangle = \int_{\mathbb{B}_n} \phi dm = n\int_0^1 \phi(r) r^{n-1} dr. \tag{4.3}
$$

The proof is complete. □

We now characterize the commuting Toeplitz operators whose symbols are of the form $z^p\phi(|z|^2)$, where $p \succ 0$.

Theorem 4.2. *Let $p, q \succ 0$, $\phi, \varphi \in \Sigma$. $T_{z^p\phi(|z|^2)}T_{z^q\varphi(|z|^2)} = T_{z^q\varphi(|z|^2)}T_{z^p\phi(|z|^2)}$ if and only if $(n+|q+\alpha|-1)\widehat{\varphi}(n+|q+\alpha|-1)\widehat{\phi}(n+|p+q+\alpha|-1) = (n+|p+\alpha|-1)\widehat{\phi}(n+|p+\alpha|-1)\widehat{\varphi}(n+|p+q+\alpha|-1)$ holds for any multi-index $\alpha \succeq 0$.*

Proof. For any multi-index $\alpha \succeq 0$, by Lemma 4.1, it follows that

$$
\begin{aligned}
&T_{z^p\phi(|z|^2)}T_{z^q\varphi(|z|^2)} z^\alpha \\
&\quad = \left(n + \left|q+\alpha\right| - 1\right)\widehat{\varphi}\left(n + \left|q+\alpha\right| - 1\right)\left(n + \left|p+q+\alpha\right| - 1\right)\widehat{\phi}\left(n + \left|p+q+\alpha\right| - 1\right) z^{p+q+\alpha}, \\
&T_{z^q\varphi(|z|^2)}T_{z^p\phi(|z|^2)} z^\alpha \\
&\quad = \left(n + \left|p+\alpha\right| - 1\right)\widehat{\phi}\left(n + \left|p+\alpha\right| - 1\right)\left(n + \left|p+q+\alpha\right| - 1\right)\widehat{\varphi}\left(n + \left|p+q+\alpha\right| - 1\right) z^{p+q+\alpha}.
\end{aligned}
\tag{4.4}
$$

Since $(n+|p+q+\alpha|-1) > 0$, the result is followed. □

A particular case of the above theorem is $\phi = \psi$. In this case

$$T_{z^p\phi(|z|^2)}T_{z^q\phi(|z|^2)} = T_{z^q\phi(|z|^2)}T_{z^p\phi(|z|^2)} \tag{4.5}$$

if and only if

$$\begin{aligned}&\Big[(n+|q+\alpha|-1)\widehat{\phi}(n+|q+\alpha|-1)-(n+|p+\alpha|-1)\widehat{\phi}(n+|p+\alpha|-1)\Big]\\ &\quad\cdot\widehat{\phi}(n+|p+q+\alpha|-1)=0.\end{aligned} \tag{4.6}$$

Thus we immediately have the following result.

Corollary 4.3. *Let $p,\ q \succ 0$, $\phi \in \Sigma$. If $|p| = |q|$, then $T_{z^p\phi(|z|^2)}T_{z^q\phi(|z|^2)} = T_{z^q\phi(|z|^2)}T_{z^p\phi(|z|^2)}$.*

If ϕ is absolutely continuous on [0,1), integrating by parts, one has $m\widehat{\phi}(m) = \lim_{r\to 1^-}\phi(r) = \phi(1^-)$, for any positive integer m. Thus, using Lemma 4.1 one can get the following lemma which will be often used in the sequel.

Lemma 4.4. *Let $p \geq 0$, $\phi \in \Sigma'$, one has*

$$T_{z^p\phi(|z|^2)}z^\alpha = \begin{cases}\phi(1^-)z^{p+\alpha}, & p+\alpha \succ 0;\\ n\displaystyle\int_0^1 \phi(r)r^{n-1}dr, & p=\alpha=0.\end{cases} \tag{4.7}$$

By Lemma 4.4, a regular argument shows the results below.

Theorem 4.5. *Let $p_i \succ 0$ and $\phi_i \in \Sigma'$. Then the followings hold.*

(1) $T_{z^{p_1}\phi_1}T_{z^{p_2}\phi_2} = T_{z^{p_2}\phi_2}T_{z^{p_1}\phi_1} = T_{z^{p_1+p_2}\phi_1\phi_2}$.

(2) $T_{z^{p_1}\phi_1}\times\cdots\times T_{z^{p_k}\phi_k} = 0$ *if and only if* $\phi_1(1^-)\times\cdots\times\phi_k(1^-) = 0$.

(3) *Let $p_i \neq p_j$ for $i \neq j$, $T_{z^{p_1}\phi_1}+\cdots+T_{z^{p_k}\phi_k} = 0$ if and only if each $\phi_i(1^-) = 0, 1 \leq i \leq k$.*

Before discussing the commutivity of Toeplitz operator with symbols $\overline{z}^q\phi(|z|^2)$, one needs the following lemma which can be obtained by direct computation.

Lemma 4.6. *Let multi-index $q \geq 0$ and $\phi \in \Sigma$. Then*

$$T_{\overline{z}^q\phi(|z|^2)}z^\alpha = \begin{cases}d(\alpha,\alpha-q)(n+|\alpha|-1)\widehat{\phi}(n+|\alpha|-1)z^{\alpha-q}, & \alpha \succ q;\\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\phi(r)dr, & \alpha = q;\\ 0, & \alpha \not\geq q,\end{cases} \tag{4.8}$$

where $d(\alpha,\alpha-q) = (\alpha!/(n+|\alpha|-1)!)/((\alpha-q)!/(n+|\alpha-q|-1)!)$ and $\alpha \not\geq q$ means that there exists i_0 such that $\alpha_{i_0} < q_{i_0}$.

Proof. For $\alpha \succ q$, We get that

$$\begin{aligned}
T_{\overline{z}^q\phi(|z|^2)}z^\alpha &= \langle w^\alpha \overline{w}^q \phi, K_z(w)\rangle \\
&= \sum_{i=1}^{n}\int_{\mathbb{B}_n}\frac{\partial}{\partial w_i}\left(w^\alpha\overline{w}^q\phi\right)\overline{\frac{\partial(K_z(w))}{\partial w_i}}dm \\
&= \sum_{i=1}^{n}\int_{\mathbb{B}_n}\left(\alpha_i w^{\alpha-e_i}\phi + w^\alpha\overline{w}_i\phi'\right)\overline{w}^q\cdot\overline{\sum_r c_r r_i w^{r-e_i}\overline{z}^r}dm \\
&= \left[\sum_{i=1}^{n}\int_{\mathbb{B}_n}\alpha_i\phi\left|w^{\alpha-e_i}\right|^2 c_{\alpha-q}(\alpha_i-q_i) + |w^\alpha|^2\phi' c_{\alpha-q}(\alpha_i-q_i)dm\right]z^{\alpha-q} \\
&= c_{\alpha-q}z^{\alpha-q}\frac{|\alpha-q|n!\alpha!}{(n+|\alpha|-1)!}\left[(n+|\alpha|-1)\int_0^1 t^{n+|q+\alpha|-2}\phi dt + \int_0^1 t^{n+|q+\alpha|-1}\phi' dt\right] \\
&= d(\alpha,\alpha-q)(n+|\alpha|-1)\widehat{\phi}(n+|\alpha|-1)z^{\alpha-q}.
\end{aligned} \tag{4.9}$$

For $\alpha = q$, we have

$$\begin{aligned}
T_{\overline{z}^q\phi(|z|^2)}z^\alpha = \langle w^\alpha\overline{w}^q\phi, K_z(w)\rangle &= \int_{\mathbb{B}_n}|z^q|^2\phi\left(|w|^2\right)dm \\
&= \frac{n!q!}{(n+|q|-1)!}\int_0^1 t^{n+|q|-1}\phi(t)dt.
\end{aligned} \tag{4.10}$$

If there exists $1 \le i \le n$ such that $\alpha_i < q_i$, then

$$T_{\overline{z}^q\phi(|z|^2)}z^\alpha = \langle w^\alpha\overline{w}^q\phi, K_z(w)\rangle = 0. \tag{4.11}$$

Thus the proof is complete. □

Note that if $\phi \in \Sigma'$, then $(n+|\alpha|-1)\widehat{\phi}(n+|\alpha|-1) = \phi(1^-)$. It follows that

$$T_{\overline{z}^q\phi(|z|^2)}z^\alpha = d(\alpha,\alpha-q)\phi(1^-)z^{\alpha-q}, \quad \text{for} \quad \alpha \succ q. \tag{4.12}$$

The following theorem gives some properties of the Toeplitz operator with symbols $\overline{z}^p\phi(|z|^2)$.

Theorem 4.7. *Let $p, q, p_i \succ 0$, $p_i \ne p_j$ for $i \ne j$ and $\phi, \psi, \phi_i \in \Sigma'$. Then the following assertions hold.*

(1) $T_{\overline{z}^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)} = T_{\overline{z}^q\psi(|z|^2)}T_{\overline{z}^p\phi(|z|^2)}$ *if and only if* $\psi(1^-)\int_0^1 r^{n+|p|-1}\phi(r)dr = \phi(1^-)\int_0^1 r^{n+|q|-1}\psi(r)dr$.

(2) $T_{\overline{z}^{p_1}\phi_1} \times \cdots \times T_{\overline{z}^{p_k}\phi_k} = 0$ *if and only if one of the following holds:*

  (i) $\phi_1(1^-) = 0$ *and* $\int_0^1 r^{n+|p_1|-1}\phi_1(r)dr = 0$;

  (ii) *There exists i_0 where $2 \le i_0 \le k$ such that $\phi_{i_0}(1^-) = 0$.*

(3) $T_{\overline{z}^{p_1}\phi_1} + \cdots + T_{\overline{z}^{p_k}\phi_k} = 0$ *if and only if* $\phi_i(1^-) = 0$ *and* $\int_0^1 r^{n+|p_i|-1}\phi_i(r)dr = 0$ *for each* i, $1 \le i \le k$.

Proof. Assertions (2) and (3) are the direct consequence of Lemma 4.6. We only need to prove assertion (1). By Lemma 4.6, for $h \succ p+q$, since $\phi, \psi \in \Sigma'$ we have

$$\begin{aligned} T_{\overline{z}^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)}\left(z^h\right) &= d(h,h-q)d(h-q,h-q-p)\phi(1^-)\psi(1^-)z^{h-q-p} \\ &= d(h,h-q-p)\phi(1^-)\psi(1^-)z^{h-q-p}, \\ T_{\overline{z}^q\psi(|z|^2)}T_{\overline{z}^p\phi(|z|^2)}\left(z^h\right) &= d(h,h-p)d(h-p,h-q-p)\phi(1^-)\psi(1^-)z^{h-q-p} \\ &= d(h,h-q-p)\phi(1^-)\psi(1^-)z^{h-q-p}. \end{aligned} \tag{4.13}$$

It is obvious that

$$T_{\overline{z}^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)}\left(z^h\right) = T_{\overline{z}^q\psi(|z|^2)}T_{\overline{z}^p\phi(|z|^2)}\left(z^h\right) \tag{4.14}$$

holds for $h \succ p+q$.

For $h = p+q$, we obtain

$$\begin{aligned} T_{\overline{z}^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)}\left(z^h\right) &= d(p+q,p)\psi(1^-)\frac{n!p!}{(n+|p|-1)!}\int_0^1 r^{n+|p|-1}\phi(r)dr \\ &= \frac{n!(p+q)!}{(n+|p+q|-1)!}\psi(1^-)\int_0^1 r^{n+|p|-1}\phi(r)dr, \\ T_{\overline{z}^q\psi(|z|^2)}T_{\overline{z}^p\phi(|z|^2)}\left(z^h\right) &= d(p+q,q)\phi(1^-)\frac{n!q!}{(n+|q|-1)!}\int_0^1 r^{n+|q|-1}\psi(r)dr \\ &= \frac{n!(p+q)!}{(n+|p+q|-1)!}\phi(1^-)\int_0^1 r^{n+|q|-1}\psi(r)dr. \end{aligned} \tag{4.15}$$

Since $n!(p+q)!/(n+|p+q|-1)! > 0$, then the desired result is obvious. □

In the assertion (1) of Theorem 4.7, if $\phi = \psi = 1$, then we get

$$\psi(1^-)\int_0^1 r^{n+|p|-1}\phi(r)dr = \frac{1}{n+|p|}, \qquad \phi(1^-)\int_0^1 r^{n+|q|-1}\psi(r)dr = \frac{1}{n+|q|}. \tag{4.16}$$

Therefore, it is easy to get the following corollary.

Corollary 4.8. *Let* $p, q \succ 0$. *Then* $T_{\overline{z}^p}T_{\overline{z}^q} = T_{\overline{z}^q}T_{\overline{z}^p}$ *if and only if* $|p| = |q|$.

It is well known that $T_\phi T_\psi = T_h$ on the Hardy space if and only if either $\overline{\phi}$ or ψ is holomorphic. However, Lemma 4.6 and Theorem 4.7 implies that a similar result does not hold on the Dirichlet space of the unit ball. Indeed, by the computation in Lemma 4.6 and Theorem 4.7, it is easy to verify that for any $p, q \succ 0$, $T_{\overline{z}^p}T_{\overline{z}^q}z^{p+q} \neq T_{\overline{z}^{p+q}}z^{p+q}$.

Theorem 4.9. *Let p, $q \succ 0$ and ϕ, $\psi \in \Sigma'$. Then $T_{z^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)} = T_{\overline{z}^q\psi(|z|^2)}T_{z^p\phi(|z|^2)}$ if and only if $\phi(1^-) = 0$ or $\psi(1^-) = 0$ and $\int_0^1 r^{n+|q|-1}\psi(r)dr = 0$.*

Proof. For each multi-index $h \geq 0$, by Lemmas 4.4 and 4.6, we have that

$$T_{z^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)}z^h = \begin{cases} d(h, h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ q \\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr\phi(1^-)z^p, & h = q \\ 0, & \text{others}, \end{cases}$$
$$T_{\overline{z}^q\psi(|z|^2)}T_{z^p\phi(|z|^2)}z^h = \begin{cases} d(h+p, h+p-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & p+h \succ q \\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\phi(r)dr\phi(1^-), & p+h = q \\ 0, & \text{others}. \end{cases} \tag{4.17}$$

Consequently, for $h \succ q$, we conclude

$$d(h, h-q)\psi(1^-)\phi(1^-)z^{h+p-q} = d(h+p, h+p-q)\psi(1^-)\phi(1^-)z^{h+p-q}. \tag{4.18}$$

Since $p, q \succ 0$, it is clear that $d(h, h-q)$ can not always equal to $d(h+p, h+p-q)$ for all $h \succ q$. Thus, we get

$$\psi(1^-)\phi(1^-) = 0. \tag{4.19}$$

On the other hand, for $h = q$, we have

$$\frac{n!q!}{(n+|q|-1)!}\int_0^1 r^{n+|q|-1}\psi(r)dr\phi(1^-) = d(h+p, h+p-q)\psi(1^-)\phi(1^-). \tag{4.20}$$

Combining (4.19) and (4.20), we obtain the desired result. □

Notice the assertion (3) of Theorem 4.5 and the assertion (3) of Theorem 4.7, Theorem 4.9 above shows that $T_{z^p\phi(|z|^2)}T_{\overline{z}^q\psi(|z|^2)} = T_{\overline{z}^q\psi(|z|^2)}T_{z^p\phi(|z|^2)}$ if and only if $T_{z^p\phi(|z|^2)} = 0$ or $T_{\overline{z}^q\psi(|z|^2)} = 0$ holds. That is, $T_{z^p\phi(|z|^2)}$ commutes with $T_{\overline{z}^q\psi(|z|^2)}$ only in the trival case.

Finally, we will discuss when Toepitze operaor $T_{\phi(|z|^2)}$ commute with $T_{z^p\overline{z}^q\psi(|z|^2)}$.

Theorem 4.10. *Let p, $q \succ 0$ and ϕ, $\psi \in \Sigma'$. Then the following assertions hold.*

(1) *If $p \succ q$, $T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)} = T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}$ if and only if $\psi(1^-) = 0$ or $\phi(1^-) = n\int_0^1 r^{n-1}\phi(r)dr$.*

(2) *If $q \succ p$, $T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)} = T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}$ if and only if $\int_0^1 r^{n+|q|-1}\psi(r)dr = 0$ or $n\int_0^1 r^{n-1}\phi(r)dr = \phi(1^-)$.*

(3) *If $p \not\succeq q$ and $q \not\succeq p$ or $p = q$, $T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)} = T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}$.*

Proof. For each multi-index $h \succeq 0$, by Lemmas 4.4 and 4.6 we have

$$T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)}z^h = \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h+p \succ q \\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr \cdot n\int_0^1 r^{n-1}\phi(r)dr, & h+p = q, \end{cases}$$

$$T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}z^h = \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ 0,\ h+p \succ q \\ \phi(1^-)\dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr, & h \succ 0,\ h+p = q \\ 0, & h \succ 0,\ h+p \not\succeq q \\ n\displaystyle\int_0^1 r^{n-1}\phi(r)dr d(p,p-q)\psi(1^-)z^{h+p-q}, & h = 0,\ p \succ q \\ n\displaystyle\int_0^1 r^{n-1}\phi(r)dr\dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr, & h = 0,\ p = q \\ 0, & h = 0,\ p \not\succeq q. \end{cases} \tag{4.21}$$

Case 1. Suppose $p \succ q$. We have

$$T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)}z^h = d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q},$$
$$T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}z^h = \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ 0 \\ n\displaystyle\int_0^1 r^{n-1}\phi(r)dr d(p,p-q)\psi(1^-)z^{h+p-q}, & h = 0. \end{cases} \tag{4.22}$$

$T_{z^p\overline{z}^q\psi(|z|^2)}$ commutes with $T_{\phi(|z|^2)}$ if and only if

$$d(p,p-q)\psi(1^-)\phi(1^-) = d(p,p-q)\psi(1^-)n\int_0^1 r^{n-1}\phi(r)dr, \tag{4.23}$$

which is equivalent to $\psi(1^-) = 0$ or $\phi(1^-) = n\int_0^1 r^{n-1}\phi(r)dr$.

Case 2. Suppose $q \succ p$. Note that $h+p \succ q$ if and only if $h \succ q-p$. We have

$$T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)}z^h = \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ q-p \\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr \cdot n\int_0^1 r^{n-1}\phi(r)dr, & h = q-p \\ 0, & h \not\succeq q-p, \end{cases}$$
$$T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}z^h = \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ q-p \\ \phi(1^-)\dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr, & h = q-p \\ 0, & h \not\succeq q-p. \end{cases} \tag{4.24}$$

It follows that $T_{z^p\overline{z}^q\psi(|z|^2)}$ commutes with $T_{\phi(|z|^2)}$ if and only if

$$\int_0^1 r^{n+|q|-1}\psi(r)dr \cdot n\int_0^1 r^{n-1}\phi(r)dr = \int_0^1 r^{n+|q|-1}\psi(r)dr\phi(1^-), \tag{4.25}$$

which is equivalent to $\int_0^1 r^{n+|q|-1}\psi(r)dr = 0$ or $n\int_0^1 r^{n-1}\phi(r)dr = \phi(1^-)$.

Case 3. Suppose $p\not\succeq q$ and $q\not\succeq p$. Let $h' = \{h_i\}$, where $h_i = \max\{q_i - p_i, 0\}$ for $1 \le i \le n$. Then for $h \ge 0$, $h + p \succ q$ if and only if $h \succ h'$. Thus,

$$\begin{aligned} T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)}z^h &= \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succeq h' \\ 0, & h \not\succeq h', \end{cases} \\ T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}z^h &= \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succeq h' \\ 0, & h \not\succeq h'. \end{cases} \end{aligned} \tag{4.26}$$

It is obvious that $T_{z^p\overline{z}^q\psi(|z|^2)}$ commutes with $T_{\phi(|z|^2)}$.

Case 4. Suppose $p = q$. We have

$$\begin{aligned} T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)}z^h &= \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ 0 \\ \dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr \cdot n\int_0^1 r^{n-1}\phi(r)dr, & h = 0, \end{cases} \\ T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}z^h &= \begin{cases} d(p+h,p+h-q)\psi(1^-)\phi(1^-)z^{h+p-q}, & h \succ 0 \\ n\displaystyle\int_0^1 r^{n-1}\phi(r)dr\dfrac{n!q!}{(n+|q|-1)!}\displaystyle\int_0^1 r^{n+|q|-1}\psi(r)dr, & h = 0. \end{cases} \end{aligned} \tag{4.27}$$

It is easy to see that $T_{z^p\overline{z}^q\psi(|z|^2)}$ commutes with $T_{\phi(|z|^2)}$. This completes the proof. □

Corollary 4.11. *Let $p \perp q$ and ϕ, $\psi \in \Sigma'$. Then $T_{\phi(|z|^2)}T_{z^p\overline{z}^q\psi(|z|^2)} = T_{z^p\overline{z}^q\psi(|z|^2)}T_{\phi(|z|^2)}$.*

Proof. Note that $p \perp q$ implies $p\not\succeq q$ and $q\not\succeq p$. The desired result is immediately followed by Theorem 4.10. □

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