Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 273040, 6 pages
http://dx.doi.org/10.1155/2013/273040

Research Article

Results on Difference Analogues of Valiron-Mohon'ko Theorem

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Received 6 April 2013; Accepted 4 May 2013

Academic Editor: Allan Peterson

The classical Valiron-Mohon'ko theorem has many applications in the study of complex equations. In this paper, we investigate rational functions in $f(z)$ and the shifts of $f(z)$. We get some results on their characteristic functions. These results may be viewed as difference analogues of Valiron-Mohon'ko theorem.

1. Introduction and Results

We use the basic notions of Nevanlinna's theory in this work (see [1, 2]). Let $f(z)$ be a meromorphic function. We say that a meromorphic function $\alpha(z)$ is a small function of $f(z)$ if $T(r,\alpha) = S(r,f)$, where $S(r,f) = o(T(r,f))$ outside a possible exceptional set of finite logarithmic measure.

The Valiron-Mohon'ko theorem has been proved to be an extremely useful tool in the study of meromorphic solutions of differential, difference, and functional equations. It is stated as follows.

Theorem A (see [3, page 29]). *Let f be a meromorphic function. Then for all irreducible rational functions in f*

$$R(z,f) = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j} \tag{1}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that

$$\begin{aligned} T(r,a_i) &= S(r,f), \quad i = 0,\ldots,p, \\ T(r,b_j) &= S(r,f), \quad j = 0,\ldots,q, \end{aligned} \tag{2}$$

the characteristic function of $R(r,f(z))$ satisfies

$$T(r,R(z,f)) = \max\{p,q\}\, T(r,f) + S(r,f). \tag{3}$$

Recently, a number of papers have focused on difference analogues of Nevanlinna's theory; see, for instance, [4–12]. Among these papers, difference polynomials are investigated extensively (see [5, 9–11]). But the difference analogues of Valiron-Mohon'ko theorem have not been established. In this paper, we are devoted to this work.

A difference polynomial of $f(z)$ is an expression of the form

$$H(z,f) = \sum_{\lambda\in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f\left(z+\delta_{\lambda,j}\right)^{\mu_{\lambda,j}}, \tag{4}$$

where J is an index set, $\delta_{\lambda,j}$ are complex constants, and $\mu_{\lambda,j}$ are nonnegative integers. In what follows, we assume that the coefficients of difference polynomials are, unless otherwise stated, small functions. The maximal total degree of $H(z,f)$ in $f(z)$ and the shifts of $f(z)$ is defined by

$$\deg_f H = \max_{\lambda\in J} \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j}. \tag{5}$$

First, we investigate the rational function

$$R_1(z,f) = \frac{P(z,f)}{d_1(z) f(z+c) + d_0(z)}, \tag{6}$$

where c is an arbitrary complex number, and $d_0(z)$ and $d_1(z)$ are small functions of $f(z)$ with $d_0(z) \not\equiv 0$ or $d_1(z) \not\equiv 0$. Our result is stated as follows.

Theorem 1. *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) = S(r,f)$. Suppose that $P(z,f) \not\equiv 0$ is a* *difference polynomial in $f(z)$ and that $R_1(z,f)$ is of the form* (6). *Then*

$$T(r,R_1) \le (\deg_f P)\, T(r,f) + S(r,f). \quad (7)$$

In many papers (see, for instance, [7, 13, 14]), linear difference expressions often appear. Concerning their characteristic functions, we have the following corollary, which is obtained easily from Theorem 1.

Corollary 2. *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) = S(r,f)$. Suppose that $L(z,f) \not\equiv 0$ is a linear combination in $f(z)$ and the shifts of $f(z)$. Then*

$$T(r,L) \le T(r,f) + S(r,f). \quad (8)$$

Next we consider the rational function

$$R_2(z,f) = \frac{P(z,f)}{f(z+c_1)\cdots f(z+c_n)}, \quad (9)$$

where $c_1,\dots,c_n$ are different complex constants. We get the following result.

Theorem 3. *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) = S(r,f)$. Suppose that $P(z,f) \not\equiv 0$ is a difference polynomial in $f(z)$ and that $R_2(z,f)$ is of the form* (9). *Then*

$$T(r,R_2) \le \max\{\deg_f P, n\}\, T(r,f) + S(r,f). \quad (10)$$

As for the general rational function in $f(z)$ and the shifts of $f(z)$,

$$R_3(z,f) = \frac{P(z,f)}{Q(z,f)}, \quad (11)$$

we get the following two results.

Theorem 4. *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) = S(r,f)$. Suppose that $P(z,f) \not\equiv 0$ and $Q(z,f) \not\equiv 0$ are difference polynomials in $f(z)$ and that $R_3(z,f)$ is of the form* (11).

(i) *If $\deg_f P \ge \deg_f Q$ and $P(z,f)$ contains just one term of maximal total degree, then*

$$T(r,R_3) \ge (\deg_f P - \deg_f Q)\, T(r,f) + S(r,f). \quad (12)$$

(ii) *If $\deg_f P \le \deg_f Q$ and $Q(z,f)$ contains just one term of maximal total degree, then*

$$T(r,R_3) \ge (\deg_f Q - \deg_f P)\, T(r,f) + S(r,f). \quad (13)$$

Theorem 5. *Let $f(z)$ be a meromorphic function of finite order such that $N(r,f) + N(r,1/f) = S(r,f)$. Suppose that $P(z,f) \not\equiv 0$ and $Q(z,f) \not\equiv 0$ are difference polynomials in $f(z)$ and that $R_3(z,f)$ is of the form* (11). *Then*

$$T(r,R_3) \le \max\{\deg_f P, \deg_f Q\}\, T(r,f) + S(r,f). \quad (14)$$

The following two examples show that the results in Theorems 1–5 are sharp; that is, "≤" and "≥" cannot be replaced by "<", ">" or "=".

Example 6. Let $f(z) = e^z$ and

$$P(z,f) = f(z)^2 f(z+\pi i) + f(z)^2 + 2f(z+\pi i)\, f(z) + 2f(z) + f(z+\pi i) + 1. \quad (15)$$

Let

$$R_{11}(z,f) = \frac{P(z,f)}{f(z+\pi i)+2}, \qquad R_{12}(z,f) = \frac{P(z,f)}{f(z+\pi i)+1}. \quad (16)$$

Then $R_{11}(z,f) = (1+e^z)^2(1-e^z)/(-e^z+2)$ and $R_{12}(z,f) = (1+e^z)^2$. Clearly,

$$T(r,R_{11}) = 3T(r,f) + S(r,f), \qquad T(r,R_{12}) = 2T(r,f) + S(r,f). \quad (17)$$

Therefore,

$$\begin{aligned}(\deg_f P - 1)\, T(r,f) + S(r,f) &< T(r,R_{11}) = (\deg_f P)\, T(r,f) + S(r,f),\\ (\deg_f P - 1)\, T(r,f) + S(r,f) &= T(r,R_{12}) < (\deg_f P)\, T(r,f) + S(r,f).\end{aligned} \quad (18)$$

Example 7. Let $f(z) = \sin z$ and

$$P(z,f) = f\left(z+\frac{\pi}{2}\right)^2 f(z) + f(z)^2 + f(z+\pi)\, f(z) - f(z). \quad (19)$$

Let

$$R_{21}(z,f) = \frac{P(z,f)}{f(z+\pi/2)^3}, \qquad R_{22}(z,f) = \frac{P(z,f)}{f(z+\pi)^2}. \quad (20)$$

Then $R_{21}(z,f) = -\tan^3 z$ and $R_{22}(z,f) = -\sin z$. Clearly,

$$T(r,R_{21}) = 3T(r,f) + S(r,f), \qquad T(r,R_{22}) = T(r,f) + S(r,f). \quad (21)$$

Therefore,

$$\begin{aligned}(\deg_f P - 3)\, T(r,f) + S(r,f) &< T(r,R_{21}) = (\deg_f P)\, T(r,f) + S(r,f),\\ (\deg_f P - 2)\, T(r,f) + S(r,f) &= T(r,R_{22}) < (\deg_f P)\, T(r,f) + S(r,f).\end{aligned} \quad (22)$$

2. Proof of Theorem 1

We need the following lemmas for the proof of Theorem 1.

The difference analogue of the logarithmic derivative lemma was given by Halburd-Korhonen [8, Corollary 2.2] and Chiang-Feng [7, Corollary 2.6], independently. The following Lemma 8 is a variant of [8, Corollary 2.2].

Lemma 8. *Let $f(z)$ be a nonconstant meromorphic function of finite order, and let η_1, η_2 be two arbitrary complex numbers. Then,*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = S(r,f). \tag{23}$$

In the remark of [15, page 15], it is pointed out that the following lemma holds.

Lemma 9. *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then,*

$$\begin{aligned} T(r+|c|, f) &= T(r,f) + S(r,f), \\ N(r+|c|, f) &= N(r,f) + S(r,f). \end{aligned} \tag{24}$$

Let $f(z)$ be a meromorphic function. It is shown in [16, page 66] that for an arbitrary $c \neq 0$, the following inequalities:

$$\begin{aligned} (1+o(1))\, T(r-|c|, f(z)) &\leq T(r, f(z+c)) \\ &\leq (1+o(1))\, T(r+|c|, f(z)) \end{aligned} \tag{25}$$

hold as $r \to \infty$. From its proof we see that the above relations are also true for counting functions. So by these relations and Lemma 9, we get the following lemma.

Lemma 10. *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then,*

$$\begin{aligned} T(r, f(z+c)) &= T(r,f) + S(r,f), \\ N(r, f(z+c)) &= N(r,f) + S(r,f), \\ N\left(r, \frac{1}{f(z+c)}\right) &= N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned} \tag{26}$$

Remark 11. In [7], Chiang and Feng proved a similar result. Let $f(z)$ be a meromorphic function with $\sigma(f) < \infty$, and let $\eta \neq 0$ be fixed; then for each $\varepsilon > 0$, we have

$$T(r, f(z+\eta)) = T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + O(\log r). \tag{27}$$

Proof of Theorem 1. Let

$$P(z,f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f\left(z+\alpha_{\lambda,j}\right)^{l_{\lambda,j}}, \tag{28}$$

and $\deg_f P = p$.

Rearranging the expression of $P(z,f)$ by collecting together all terms having the same total degree, we get

$$P(z,f) = \sum_{i=0}^{p} h_i(z) f(z)^i, \tag{29}$$

where, for $i = 0, \ldots, p$,

$$\begin{aligned} h_i(z) &= \sum_{\lambda \in I_i} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} \left(\frac{f\left(z+\alpha_{\lambda,j}\right)}{f(z)}\right)^{l_{\lambda,j}}, \\ I_i &= \left\{\lambda \in I \mid \sum_{j=1}^{\sigma_\lambda} l_{\lambda,j} = i\right\}. \end{aligned} \tag{30}$$

Since the coefficients $a_\lambda(z)$ of $P(z,f)$ are small functions of $f(z)$, we have

$$m(r, a_\lambda) \leq T(r, a_\lambda) = S(r,f). \tag{31}$$

So by Lemma 8, we have, for all $i = 0, 1, \ldots, p$ the estimates

$$m(r, h_i) = S(r,f). \tag{32}$$

Without loss of generality, we may assume $c = 0$ in (6). Otherwise, substituting $z - c$ for z, we get

$$R_1(z-c, f) = \frac{P(z-c, f)}{d_1(z-c) f(z) + d_0(z-c)}. \tag{33}$$

By Lemma 10, we see that

$$T(r, R_1(z-c,f)) = T(r, R_1(z,f)) + S(r,f). \tag{34}$$

So, in the following discussion, we only discuss the form

$$R_1(z,f) = \frac{P(z,f)}{d_1(z) f(z) + d_0(z)}. \tag{35}$$

Assume first that $d_1(z) = 0$. Clearly, we may assume that $d_0(z) = 1$. By (29), we get

$$\begin{aligned} R_1(z,f) &= P(z,f) \\ &= h_p(z) f(z)^p + h_{p-1}(z) f(z)^{p-1} \\ &\quad + \cdots + h_1(z) f(z) + h_0(z). \end{aligned} \tag{36}$$

If $p = 1$, then $R_1(z,f) = h_1(z) f(z) + h_0(z)$. So by (32), we get

$$m(r, R_1) \leq m(r,f) + S(r,f). \tag{37}$$

If $p > 1$, then rewrite $R_1(z,f)$ in the form

$$R_1(z,f) = f(z)\left(h_p(z) f(z)^{p-1} + \cdots + h_1(z)\right) + h_0(z). \tag{38}$$

So we have

$$\begin{aligned} m(r, R_1) &\leq m(r,f) \\ &\quad + m\left(r, h_p(z) f(z)^{p-1} + \cdots + h_1(z)\right) + S(r,f). \end{aligned} \tag{39}$$

By (39) and the inductive argument, we have

$$m(r, R_1) \leq pm(r, f) + S(r, f). \tag{40}$$

To estimate $N(r, R_1)$, we use the form

$$R_1(z, f) = P(z, f) = \sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}. \tag{41}$$

Clearly,

$$N(r, R_1) \leq \sum_{\lambda \in I} \left(N(r, a_\lambda) + \sum_{j=1}^{\sigma_\lambda} l_{\lambda,j} N\left(r, f\left(z + \alpha_{\lambda,j}\right)\right) \right) + O(1). \tag{42}$$

So by (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$N(r, R_1) = S(r, f). \tag{43}$$

Combining this equality with (40), we get

$$T(r, R_1) \leq pT(r, f) + S(r, f), \tag{44}$$

and we have completed the case $d_1(z) = 0$.

We now proceed to the case $d_1(z) \neq 0$. Clearly, in this case we may assume that $d_1(z) = 1$. By (29), we see that (6) becomes

$$\begin{aligned} R_1(z, f) &= \left(h_p(z) f(z)^p + h_{p-1}(z) f(z)^{p-1} \right. \\ &\quad \left. + \cdots + h_1(z) f(z) + h_0(z)\right) \\ &\quad \times \left(f(z) + d_0(z)\right)^{-1}. \end{aligned} \tag{45}$$

By (45), we get

$$\begin{aligned} R_1(z, f) &= h_p(z) f(z)^{p-1} \\ &\quad + \left(h_{p-1}^*(z) f(z)^{p-1} + h_{p-2}(z) f(z)^{p-2} \right. \\ &\quad\quad \left. + \cdots + h_1(z) f(z) + h_0(z)\right) \\ &\quad \times \left(f(z) + d_0(z)\right)^{-1} \\ &= h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} \\ &\quad + \frac{h_{p-2}^*(z) f(z)^{p-2} + \cdots + h_1(z) f(z) + h_0(z)}{f(z) + d_0(z)} \\ &= \cdots \\ &= h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} \\ &\quad + \cdots + h_2^*(z) f(z) + h_1^*(z) + \frac{h_0^*(z)}{f(z) + d_0(z)}, \end{aligned} \tag{46}$$

where

$$\begin{aligned} h_{p-1}^*(z) &= h_{p-1}(z) - h_p(z) d_0(z), \\ h_{p-2}^*(z) &= h_{p-2}(z) - h_{p-1}^*(z) d_0(z), \\ &\vdots \\ h_1^*(z) &= h_1(z) - h_2^*(z) d_0(z), \\ h_0^*(z) &= h_0(z) - h_1^*(z) d_0(z). \end{aligned} \tag{47}$$

By (32), we get, for $j = 0, 1, \ldots, p-1$, the estimates

$$m(r, h_j^*) = S(r, f). \tag{48}$$

By (46), using the same method as in (36)–(40), we get

$$\begin{aligned} m(r, R_1) &\leq m\left(r, h_p(z) f(z)^{p-1} + h_{p-1}^*(z) f(z)^{p-2} + \cdots + h_1^*(z)\right) \\ &\quad + m\left(r, \frac{h_0^*(z)}{f(z) + d_0(z)}\right) \\ &\leq (p-1) m(r, f) \\ &\quad + m\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f). \end{aligned} \tag{49}$$

To estimate $N(r, R_1)$, we use the form

$$\begin{aligned} R_1(z, f) &= \frac{P(z, f)}{f(z) + d_0(z)} \\ &= \frac{\sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}}{f(z) + d_0(z)}. \end{aligned} \tag{50}$$

By (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$N(r, R_1) = N\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f). \tag{51}$$

Combining this equality with (49), we get

$$\begin{aligned} T(r, R_1) &\leq (p-1) m(r, f) \\ &\quad + T\left(r, \frac{1}{f(z) + d_0(z)}\right) + S(r, f) \\ &\leq pT(r, f) + S(r, f). \end{aligned} \tag{52}$$

Theorem 1 is proved. □

3. Proof of Theorem 3

Proof. Let $P(z, f)$ be of the form (28) and $\deg_f P = p$. Rearranging the expression of $P(z, f)$, we get (29) and (30). We only discuss the case $p \geq n$ since the case $p < n$ is easier.

Rewrite $R_2(z, f)$ in the form

$$R_2(z, f) = \frac{P(z, f)}{s(z) f(z)^n}, \quad (53)$$

where

$$s(z) = \frac{f(z + c_1) \cdots f(z + c_n)}{f(z)^n}. \quad (54)$$

By Lemma 8, we get

$$m\left(r, \frac{1}{s}\right) = S(r, f). \quad (55)$$

By (29) and (53), we get

$$\begin{aligned} R_2(z, f) &= \frac{\sum_{i=0}^{p} h_i(z) f(z)^i}{s(z) f(z)^n} \\ &= \sum_{i=n}^{p} \frac{h_i(z)}{s(z)} f(z)^{i-n} \\ &\quad + \frac{h_{n-1}(z) f(z)^{n-1} + \cdots + h_0(z)}{s(z) f(z)^n} \\ &= \sum_{i=n}^{p} \frac{h_i(z)}{s(z)} f(z)^{i-n} \\ &\quad + \sum_{j=1}^{n} \frac{h_{n-j}(z)}{s(z)} \left(\frac{1}{f(z)}\right)^j. \end{aligned} \quad (56)$$

By (32) and (55), we have, for all $i = 0, \ldots, p$, the estimates

$$m\left(r, \frac{h_i(z)}{s(z)}\right) = S(r, f). \quad (57)$$

By (57), using the same method as in (36)–(40), we get

$$\begin{aligned} m\left(r, \sum_{i=n}^{p} \frac{h_i(z)}{s(z)} f(z)^{i-n}\right) &\leq (p - n) m(r, f) + S(r, f), \\ m\left(r, \sum_{j=1}^{n} \frac{h_{n-j}(z)}{s(z)} \left(\frac{1}{f(z)}\right)^j\right) &\leq nm\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (58)$$

Combining the above two inequalities with (56), we get

$$m(r, R_2) \leq (p - n) m(r, f) + nm\left(r, \frac{1}{f}\right) + S(r, f). \quad (59)$$

To estimate $N(r, R_2)$, we use the form

$$R_2(z, f) = \frac{\sum_{\lambda \in I} a_\lambda(z) \prod_{j=1}^{\sigma_\lambda} f(z + \alpha_{\lambda,j})^{l_{\lambda,j}}}{f(z + c_1) \cdots f(z + c_n)}. \quad (60)$$

By (31), $N(r, f) = S(r, f)$, and Lemma 10, we get

$$\begin{aligned} N(r, R_2) &= N\left(r, \frac{1}{f(z + c_1) \cdots f(z + c_n)}\right) \\ &\quad + S(r, f) \leq nN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \quad (61)$$

Combining this inequality with (59), we get

$$\begin{aligned} T(r, R_2) \leq\ & (p - n) m(r, f) + nT\left(r, \frac{1}{f}\right) \\ & + S(r, f) \leq pT(r, f) + S(r, f). \end{aligned} \quad (62)$$

Theorem 3 is proved. □

4. Proof of Theorem 4

We need the following lemma for the proof of Theorem 4.

Lemma 12 (see [11]). *Let $f(z)$ be a meromorphic function of finite order such that $N(r, f) = S(r, f)$. Suppose that $H(z, f)$ is a difference polynomial in $f(z)$ and $H(z, f)$ contains just one term of maximal total degree. Then,*

$$T(r, H) = \left(\deg_f H\right) T(r, f) + S(r, f). \quad (63)$$

Proof of Theorem 4. We have the following.

Case 1. Suppose that $\deg_f P \geq \deg_f Q$ and $P(z, f)$ contains just one term of maximal total degree.

Let $\deg_f P = p$ and $\deg_f Q = q$. By Lemma 12, we get

$$T(r, P) = pT(r, f) + S(r, f). \quad (64)$$

By Theorem 1, we get

$$T(r, Q) \leq qT(r, f) + S(r, f). \quad (65)$$

By (11), we get

$$P(z, f) = R_3(z, f) Q(z, f). \quad (66)$$

By (64)–(66), we get

$$\begin{aligned} pT(r, f) &+ S(r, f) \\ &= T(r, P(z, f)) \\ &= T(r, R_3(z, f) Q(z, f)) \\ &\leq T(r, R_3(z, f)) + T(r, Q(z, f)) \\ &\leq T(r, R_3(z, f)) + qT(r, f) + S(r, f). \end{aligned} \quad (67)$$

So we have,

$$T(r, R_3) \geq (p - q) T(r, f) + S(r, f). \quad (68)$$

Case 2. Suppose that $\deg_f P \leq \deg_f Q$ and $Q(z, f)$ contains just one term of maximal total degree.

In this case, we consider $1/R_3(z, f)$. Using the same method as in Case 1, we can easily get

$$T(r, R_3) = T\left(r, \frac{1}{R_3}\right) \geq (q - p) T(r, f) + S(r, f). \quad (69)$$

Theorem 4 is proved. □

5. Proof of Theorem 5

Proof. Let $P(z, f)$ be of the form (28) and $\deg_f P = p$. Let

$$Q(z, f) = \sum_{\mu \in J} b_\mu(z) \prod_{j=1}^{\tau_\mu} f(z + \beta_{\mu,j})^{m_{\mu,j}}, \tag{70}$$

and $\deg_f Q = q$.

Rearranging the expression of $P(z, f)$, we get (29) and (30).

Similarly, rearranging the expression of $Q(z, f)$, we get

$$Q(z, f) = \sum_{k=0}^{q} t_k(z) f(z)^k, \tag{71}$$

where, for $k = 0, \ldots, q$,

$$\begin{aligned} t_k(z) &= \sum_{\mu \in J_k} b_\mu(z) \prod_{j=1}^{\tau_\mu} \left(\frac{f(z + \beta_{\mu,j})}{f(z)} \right)^{m_{\mu,j}}, \\ J_k &= \left\{ \lambda \in J \mid \sum_{j=1}^{\tau_\mu} m_{\mu,j} = k \right\}. \end{aligned} \tag{72}$$

By (29) and (71), we get

$$R_3(z, f) = \frac{\sum_{i=0}^{p} h_i(z) f(z)^i}{\sum_{k=0}^{q} t_k(z) f(z)^k}. \tag{73}$$

Since $N(r, f) + N(r, 1/f) = S(r, f)$, by Lemma 10, we have, for an arbitrary η,

$$\begin{aligned} N\left(r, \frac{f(z+\eta)}{f(z)}\right) &\le N\left(r, \frac{1}{f}\right) + N(r, f(z+\eta)) \\ &= N\left(r, \frac{1}{f}\right) + N(r, f) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{74}$$

By (74) and Lemma 8, we have, for an arbitrary η,

$$T\left(r, \frac{f(z+\eta)}{f(z)}\right) = S(r, f). \tag{75}$$

Since the coefficients $a_\lambda(z)$ and $b_\mu(z)$ of $P(z, f)$ and $Q(z, f)$ are small functions of $f(z)$, by (30), (72), and (75), we get

$$\begin{aligned} T(r, h_i) &= S(r, f), \quad i = 0, \ldots, p \\ T(r, t_k) &= S(r, f), \quad k = 0, \ldots, q. \end{aligned} \tag{76}$$

By (73), we are not clear whether $R_3(z, f)$ is an irreducible rational function in $f(z)$. So by Theorem A, we get

$$T(r, R_3) \le \max\{p, q\} T(r, f) + S(r, f). \tag{77}$$

Theorem 5 is proved. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11226090 and 11171119) and Guangdong Natural Science Foundation (S2012040006865).

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