Research Article

Self-Adjointness, Symmetries, and Conservation Laws for a Class of Wave Equations Incorporating Dissipation

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In this work, we study the nonlinear self-adjointness and conservation laws for a class of wave equations with a dissipative source. We show that the equations are nonlinear self-adjoint. As a result, from the general theorem on conservation laws proved by Ibragimov and the symmetry generators, we find some conservation laws for this kind of equations.

1. Introduction

There have been abundant literatures that have contributed to the studies of Lie symmetry classification of various classes of $(1 + 1)$-dimensional nonlinear wave equations and their individual members. Probably, Barone et al. [1] were the first to study the nonlinear wave equation $u_{tt} = u_{xx} + F(u)$ by means of symmetry method. Motivated by a number of physical problems, Ames et al. [2] investigated group properties of quasilinear $u_{tt} = [f(u)u_x]_x$. Later, Torrisi and Valenti [3, 4], generalizing above equations, have investigated the symmetries of the following equation:

$$u_{tt} = [f(u)u_x + g(x,u)]_x. \tag{1}$$

Furthermore, classification results for the equation $u_{tt} + K(u)u_t = [F(u)u_x]_x$ can be found in [5, 6]. An expanded form of the latter equation

$$u_{tt} + K(u)u_t = [F(u)u_x]_x + H(u)u_x \tag{2}$$

was studied by Kingston and Sophocleous [7]. In the papers mentioned above, many interesting results including Lie point and nonlocal symmetries classification were systematically investigated.

In this paper, we consider a subclass of (2) with $K(u) = \alpha$:

$$u_{tt} + \alpha u_t - [f(u)u_x + g(u)]_x = 0, \tag{3}$$

which is also viewed as a special case of (1) with an additional dissipation, where $f(u)$ and $g(u)$ are arbitrary differentiable

functions, t is the time coordinate, x is the one-space coordinate, and α is a nonzero constant. Here, we will focus on the nonlinear self-adjointness and conservation laws for (3).

For (3), we cannot find easily the variational structure so it is inconvenient to apply the Noether theorem to construct conservation laws straightforward for this equation. However, it is fortunate that Ibragimov recently proved a result on conservation laws [8], which does not require the existence of a Lagrangian. The Ibragimov theorem on conservation laws provides an elegant way to establish local conservation laws for the equations under consideration.

Since the seminal work of Ibragimov [8], more and more works are dedicated to studying the self-adjointness and conservation laws for some equations in mathematical physics and there are many new developments, including strict self-adjointness [9–11], quasi-self-adjointness [12–15], weak self-adjointness [16, 17], and nonlinear self-adjointness [18–21] and some results which have been communicated in the recent literature; for more references, see [21] and references therein.

For the sake of completeness, we briefly present the notations, definition of nonlinear self-adjointness, and Ibragimov's theorem on conservation laws. Let $x = (x_1, x_2, \ldots, x_n)$ be n independent variables, $u = u(x)$ a dependent variable,

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \tag{4}$$

a symmetry of an equation

$$F\left(x, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right) = 0, \qquad (5)$$

$$F^*\left(x, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\right) := \frac{\delta \mathscr{L}}{\delta u} = 0 \qquad (6)$$

the *adjoint equation* to (5), where $\mathscr{L} = vF$ is called *formal Lagrangian*, $v = v(x)$ is a new dependent variable, and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{s} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}} \qquad (7)$$

denotes the Euler-Lagrange operator.

Now, let us state the definition of nonlinear self-adjointness for a equation.

Definition 1. Equation (5) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint equation (6) by the substitution $v = \phi(x, u)$ with a certain function $\phi(x, u) \neq 0$ is identical with the original equation (5); in other words, the following equations hold:

$$F^*|_{v=\phi} = \lambda\left(x, u, u_{(1)}, \ldots\right) F \qquad (8)$$

for some differential function $\lambda = \lambda(x, u, u_{(1)}, \ldots)$.

Particularly, if (8) holds for a certain function ϕ such that $\phi_u \neq 0$ and $\phi_{x_i} \neq 0$ for some x_i, (5) is called *weak self-adjoint*; if (8) holds for a certain function ϕ such that $\phi = \phi(u) \neq u$ and $\phi'(u) \neq 0$, then (5) is called *quasi-self-adjoint*; if (8) holds for $\phi = u$, then (5) is called *(strictly) self-adjoint*.

In the following, we recall the "new conservation theorem" given by Ibragimov in [8]. We will find conservation laws for (3) by this theorem.

Theorem 2. *Any Lie point, Lie-Bäcklund, and nonlocal symmetry*

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \qquad (9)$$

of (5) provides a conservation law $D_i(C^i) = 0$ for the system comprising (5) and its adjoint equation (6). The conserved vector is given by

$$C^i = \xi^i \mathscr{L} + W \left[\frac{\partial \mathscr{L}}{\partial u_i} - D_j \left(\frac{\partial \mathscr{L}}{\partial u_{ij}} \right) + D_k D_j \left(\frac{\partial \mathscr{L}}{\partial u_{ijk}} \right) \right.$$
$$\left. - D_l D_k D_j \left(\frac{\partial \mathscr{L}}{\partial u_{ijkl}} \right) + \cdots \right]$$
$$+ D_j(W) \left[\frac{\partial \mathscr{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathscr{L}}{\partial u_{ijk}} \right) + \cdots \right]$$
$$+ D_k D_j(W) \left[\frac{\partial \mathscr{L}}{\partial u_{ijk}} - D_l \left(\frac{\partial \mathscr{L}}{\partial u_{ijkl}} \right) + \cdots \right] + \cdots, \qquad (10)$$

where $W = \eta - \xi^j u_j$ is the Lie characteristic function and $\mathscr{L} = vF$ is the formal Lagrangian.

The paper is organized as the follows. In Section 2, we discuss the nonlinear self-adjointness of (3). In Section 3, we establish conservation laws for some particular cases of (3) using Theorem 2.

2. Nonlinear Self-Adjointness of (3)

In this section, we determine the nonlinearly self-adjoint subclasses of (3). Let

$$F = u_{tt} + \alpha u_t - f_u(u) u_x^2 - f(u) u_{xx} - g_u(u) u_x; \qquad (11)$$

then we have the following formal Lagrangian for (3):

$$\mathscr{L} = v\left(u_{tt} + \alpha u_t - f_u u_x^2 - f u_{xx} - g_u u_x\right). \qquad (12)$$

Computing the variational derivative of this formal Lagrangian

$$\frac{\delta \mathscr{L}}{\delta u} = \frac{\partial \mathscr{L}}{\partial u} - D_t \frac{\partial \mathscr{L}}{\partial u_t} - D_x \frac{\partial \mathscr{L}}{\partial u_x} + D_t^2 \frac{\partial \mathscr{L}}{\partial u_{tt}} + D_x^2 \frac{\partial \mathscr{L}}{\partial u_{xx}}, \qquad (13)$$

we obtain the adjoint equation of (3):

$$F^* = -\alpha v_t + v_{tt} - f v_{xx} + g_u v_x = 0. \qquad (14)$$

Assume that $F^*|_{v=h(t,x,u)} = \lambda F$, for a certain function λ, where F is given by (11); then we have

$$h_u u_{tt} + h_{uu} u_x^2 + (2h_{tu} - \alpha h_u) u_t - f h_u u_{xx} - f h_{uu} u_x^2$$
$$+ (g_u h_u - 2f h_{xu}) u_x + h_{tt} - \alpha u_t - f h_{xx} + g_u h_x \qquad (15)$$
$$= \lambda \left(u_{tt} + \alpha u_t - f_u u_x^2 - f u_{xx} - g_u u_x \right).$$

The comparison of the coefficients of u_{tt}, u_t^2, u_x^2, u_t, and u_x in both sides of (15) yields

$$\lambda = h_u, \qquad (16)$$

$$h_{uu} = 0, \qquad (17)$$

$$\lambda f_u = f h_{uu}, \qquad (18)$$

$$\lambda \alpha = 2h_{tu} - \alpha h_u, \qquad (19)$$

$$\lambda g_u = 2f h_{xu} + g_u h_u; \qquad (20)$$

thus, (15) is reduced as

$$h_{tt} - \alpha h_t - f h_{xx} + g_u h_x = 0. \qquad (21)$$

It follows from (17) and (18) that

$$\lambda f_u = 0. \qquad (22)$$

Equation (22) splits into the following two cases.

Case 1 ($\lambda = 0$). In this case, from (16) we get that $h_u = 0$; thus, (16)–(20) are all satisfied. Noting that h does not depend on the function u, f and g are two functions of u, so from (21) we obtain that

$$h_{tt} - \alpha h_t = 0, \qquad -f h_{xx} + g_u h_x = 0. \qquad (23)$$

TABLE 1: Nonlinear self-adjointness of (3).

f	g	v	Self-adjointness
$\forall (f_u \neq 0)$	$\forall$	$C_1 + C_2 e^{\alpha t}$	Nonlinear
$\forall (f_u \neq 0)$	C_5	$(C_1 + C_2 e^{\alpha t})(C_3 + C_4 x)$	Nonlinear
$\forall (f_u \neq 0)$	$C_5 \int f(u) du$	$\begin{array}{c} C_1 + C_2 e^{\alpha t} \\ + (C_3 + C_4 e^{\alpha t}) e^{x/C_5} \end{array}$	Nonlinear
$C_0 (\neq 0)$	$C_1 + C_2 u$	$C_1 e^{(C_2/C_0)x + \alpha t} u + b(t,x)$	Weak

α is a nonzero constant; $b(t,x)$ satisfies (27).

TABLE 2: Symmetries of (3) for some special choices of f and g.

f	g	Symmetries
u^{-2}	$\ln u$	$X_3 = x \dfrac{\partial}{\partial x} - u \dfrac{\partial}{\partial u}$
u^n $(n \neq 0)$	1	$X_3 = x \dfrac{\partial}{\partial x} + \dfrac{2u}{n} \dfrac{\partial}{\partial u}$
1	u	$X_3 = u \dfrac{\partial}{\partial u},$ $X_4 = t \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial t} - \dfrac{1}{2}(t + \alpha x) u \dfrac{\partial}{\partial u}$
1	1	$X_3 = u \dfrac{\partial}{\partial u}, X_4 = t \dfrac{\partial}{\partial x} + x \dfrac{\partial}{\partial t} - \dfrac{1}{2}\alpha x u \dfrac{\partial}{\partial u}$

Equation (23) can be satisfied by taking some appropriate function $h = h(t, x)$. Therefore, we can take $v = h(x, t) \neq 0$ such that $F^*|_{v=h(t,x,u)} = \lambda F$ holds. Thus, (3) is nonlinear self-adjoint in this case.

Case 2 ($f_u = 0$). That is, f is a constant with respect to u; without loss of generality, we set $f = C_0 \neq 0$. From (17), we assume that

$$h(t, x, u) = a(t, x) u + b(t, x). \tag{24}$$

Taking into account (16) and (19), we have $a_t(t, x) = \alpha a(t, x)$; thus, $a(t, x) = k(x)e^{\alpha t}$. From (20), we deduce that

$$g_u(u) = \frac{C_0 k'(x)}{k(x)}, \tag{25}$$

which implies that $g(u) = C_2 u + C_1$ is a linear function of u and $k(x) = C_1 e^{(C_2/C_0)x}$, where C_2 and C_1 are constants. Hence, we have

$$h(t, x, u) = C_1 e^{(C_2/C_0)x + \alpha t} u + b(t, x). \tag{26}$$

Substituting (26) into (22) and using the original equation (3), we derive that $b(t, x)$ satisfies

$$b_{tt} - \alpha b_t - C_0 b_{xx} + C_2 b_x = 0, \tag{27}$$

which is easy to be solved.

In this case, $f(u) = C_0$ is a constant, $g(u) = C_2 u + C_1$, and $\lambda = h_u = C_1 e^{(C_2/C_0)x + \alpha t}$; therefore, (3) is a weak self-adjoint.

Here, we omit the tedious calculations to obtain the solutions of (23) and (27). In Table 1, we summarize the classification of nonlinear self-adjointness of (3) with the conditions that $f(u)$ and $g(u)$ should satisfy. In what follows, the symbol for all means that the corresponding function has no restrictions and c_i $(1 \leq i \leq 5)$ are arbitrary constants.

Thus, we have demonstrated the following statement.

Theorem 3. *Let α be a nonzero constant, $f(u)$ an arbitrary function of u, and the adjoint equation of (3) given by (14); then (3) is nonlinear self-adjointness with the substitution v and function $g(u)$ given in Table 1.*

3. Symmetries and Conservation Laws

In this section, we will apply Theorem 2 to construct some conservation laws for (3). First, we show the Lie classical symmetries for (3) for some special choice of f and g. Then applying formula in Theorem 2 to the formal Lagrangian (12), and to the symmetries X_i and eliminating v by the substitution v given in Table 1, we obtain the conservation law

$$D_t \left(C^t \right) + D_x \left(C^x \right) = 0. \tag{28}$$

3.1. Lie Symmetries. Now, let us show the Lie classical symmetries for (3). Based on Lie group theory [22], we assume that a Lie point symmetry of (3) is a vector field

$$X = \xi(t, x, u) \frac{\partial}{\partial x} + \phi(t, x, u) \frac{\partial}{\partial t} + \eta(t, x, u) \frac{\partial}{\partial u} \tag{29}$$

on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ such that $X^{(2)}F = 0$ when $F = 0$, and taking into account (3), the operator $X^{(2)}$ is given as follows:

$$\begin{aligned} X^{(2)} = {} & \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \eta_x^{(1)} \frac{\partial}{\partial u_x} \\ & + \eta_t^{(1)} \frac{\partial}{\partial u_t} + \eta_{xx}^{(2)} \frac{\partial}{\partial u_{xx}} + \eta_{tt}^{(2)} \frac{\partial}{\partial u_{tt}}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \eta_x^{(1)} &= D_x \eta - (D_x \xi) u_x - (D_x \phi) u_t, \\ \eta_t^{(1)} &= D_t \eta - (D_t \xi) u_x - (D_t \phi) u_t, \\ \eta_{xx}^{(2)} &= D_x \left(\eta_x^{(1)} \right) - (D_x \xi) u_{xx} - (D_x \phi) u_{xt}, \\ \eta_{tt}^{(2)} &= D_t \left(\eta_t^{(1)} \right) - (D_t \xi) u_{xt} - (D_t \phi) u_{tt}. \end{aligned} \tag{31}$$

The condition $X^{(2)}F = 0$, when $F = 0$, will yield determining equations. Solving these determining equations, we can obtain the symmetries of (3). For $f(u)$ and $g(u)$ arbitrary, the symmetries that are admitted by (3) are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}. \tag{32}$$

For some special choices of the functions $f(u)$ and $g(u)$, here we omit the details of routine calculations and present the symmetries (besides X_1 and X_2) that are admitted by (3) in Table 2.

TABLE 3: Conservation laws of (3) for some special choices of f and g ($n \neq 0$).

f	g	Symmetries	Conserved vector with substitution v
$\dfrac{1}{u^2}$	$\ln u$	$x\dfrac{\partial}{\partial x} - u\dfrac{\partial}{\partial u}$	$C^t = (v_t - \alpha v)(u + xu_x) - v(u_t + xu_{xt})$ $C^x = xvu_{tt} + \alpha xvu_t + v$ $v = C_1 + C_2 e^{\alpha t}$
u^n	1	$x\dfrac{\partial}{\partial x} + \dfrac{2u}{n}\dfrac{\partial}{\partial u}$	$C^t = \left(\dfrac{2}{n}u - xu_x\right)(\alpha v - v_t) + v\left(\dfrac{2}{n}u_t - xu_{xt}\right)$ $C^x = x\left(vu_{tt} + \alpha vu_t - u^n u_x v_x\right) + \dfrac{2}{n}u^{n+1}v_x - \left(1 + \dfrac{2}{n}\right)vu^n u_x$ $v = \left(C_1 + C_2 e^{\alpha t}\right)\left(C_3 + C_4 x\right)$
1	u	$t\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial t} - \dfrac{(t + \alpha x)u}{2}\dfrac{\partial}{\partial u}$	$C^t = -v\left[(1 + x + \alpha t)u_x + \dfrac{1}{2}(t + \alpha x)u_t + \dfrac{1}{2}(1 + \alpha t + \alpha^2 x)u + tu_{xt} + xu_{xx}\right] + v_t\left[\dfrac{1}{2}(t + \alpha x)u + xu_t + tu_x\right]$ $C^x = v\left[(1 + x + \alpha t)u_t + \dfrac{1}{2}(t + \alpha x)u_x + \dfrac{1}{2}(t + \alpha + \alpha x)u + tu_{tt} + xu_{xt}\right] - v_x\left[\dfrac{1}{2}(t + \alpha x)u + xu_t + tu_x\right]$ $v = C_1 + C_2 e^{\alpha t} + (C_3 + C_4 e^{\alpha t})e^x$ or $b(t, x)$
1	1	$t\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial t} - \dfrac{\alpha x u}{2}\dfrac{\partial}{\partial u}$	$C^t = -v\left(\dfrac{\alpha}{2}xu_t + xu_{xx} + \dfrac{\alpha^2}{2}xu + \alpha tu_x + u_x + tu_{xt}\right)$ $+ v_t\left(xu_t + tu_x + \dfrac{\alpha}{2}xu\right)$ $C^x = v\left(\dfrac{\alpha}{2}xu_x + tu_{tt} + \dfrac{\alpha}{2}u + \alpha tu_t + u_t + xu_{xt}\right)$ $- v_x\left(xu_t + tu_x + \dfrac{\alpha}{2}xu\right)$ $v = (C_1 + C_2 e^{\alpha t})(C_3 + C_4 x)$ or $C_1 e^{\alpha t}u + b(t, x)$

3.2. Conservation Laws.

For the symmetries X, from the formula (10) in Theorem 2, we can obtain readily some conservation laws for (3). For example, we take $f(u) = u^{-2}$ and $g(u) = \ln u$; let us construct the conserved vector corresponding to the time translation group with the generator

$$X_2 = \frac{\partial}{\partial t}. \tag{33}$$

For this operator, we have $\xi^t = 1, \xi^x = 0, \eta = 0$, and $W = -u_t$. In this case, (3) is nonlinear self-adjoint and becomes

$$F = u_{tt} + \alpha u_t + 2u^{-3}u_x^2 - u^{-2}u_{xx} - u^{-1}u_x. \tag{34}$$

The formal Lagrangian is

$$\mathscr{L} = v\left(u_{tt} + \alpha u_t + 2u^{-3}u_x^2 - u^{-2}u_{xx} - u^{-1}u_x\right), \tag{35}$$

and the adjoint equation of (34) is

$$\mathscr{L}^* = v_{tt} - \alpha v_t - u^{-2}v_{xx} + u^{-1}v_x. \tag{36}$$

Therefore, we obtain the following conserved vector:

$$
\begin{aligned}
C^t &= \mathscr{L} + W\left(\frac{\partial\mathscr{L}}{\partial u_t} - D_t\frac{\partial\mathscr{L}}{\partial u_{tt}}\right) + (D_t W)\frac{\partial\mathscr{L}}{\partial u_{tt}} \\
&= v\left(u_{tt} + \alpha u_t + 2u^{-3}u_x^2 - u^{-2}u_{xx} - u^{-1}u_x\right) \\
&\quad - u_t(\alpha v - v_t) - u_{tt}v \\
&= 2vu_x^2 u^{-3} - vu_{xx}u^{-2} - vu_x u^{-1} + u_t v_t, \\
C^x &= W\left(\frac{\partial\mathscr{L}}{\partial u_x} - D_x\frac{\partial\mathscr{L}}{\partial u_{xx}}\right) + (D_x W)\frac{\partial\mathscr{L}}{\partial u_{xx}} \\
&= u^{-1}u_t\left(v - 2u^{-2}vu_x - u^{-1}v_x\right) + u^{-2}vu_{xt},
\end{aligned}
\tag{37}
$$

where v satisfies (36). The reckoning shows that the vector (37) satisfies the conservation equation (28).

In this case, since (3) is nonlinear self-adjoint, from Table 1 we take $v = C_1 + C_2 e^{\alpha t}$. So, the conserved vector is simplified as follows:

$$
\begin{aligned}
C^t &= 2vu_x^2 u^{-3} - vu_{xx}u^{-2} - vu_x u^{-1} + u_t v_t, \\
C^x &= u^{-1}u_t v\left(1 - 2u^{-2}u_x\right) + u^{-2}vu_{xt}.
\end{aligned}
\tag{38}
$$

Particularly, if setting $v = 1$, we have

$$C^t = -u_x u^{-1} - D_x \left(u^{-2} u_x \right), \qquad C^x = u^{-1} u_t + D_t \left(u^{-2} u_x \right). \tag{39}$$

Thus, the conserved vector can be reduced to the form

$$C^t = -u_x u^{-1}, \qquad C^x = u^{-1} u_t. \tag{40}$$

If setting $v = e^{\alpha t}$, then the conserved vector (37) is simplified as

$$C^t = \alpha e^{\alpha t} u_t - D_x \left[e^{\alpha t} \left(u^{-2} u_x + \ln u \right) \right],$$
$$C^x = -\alpha e^{\alpha t} \left(u^{-2} u_x + \ln u \right) + D_t \left[e^{\alpha t} \left(u^{-2} u_x + \ln u \right) \right]; \tag{41}$$

therefore, it can be reduced as

$$C^t = \alpha e^{\alpha t} u_t, \qquad C^x = -\alpha e^{\alpha t} \left(u^{-2} u_x + \ln u \right). \tag{42}$$

In what follows, we omit the tedious calculations and list only the conservation laws of (3) for some special choices of functions $f(u)$ and $g(u)$ in Table 3.

In Table 3, the function $b(t, x)$ satisfies (27), v and the symmetry are taken from Tables 1 and 2, respectively. The reckoning shows that the vector listed in Table 3 satisfies the conservation equation (28) with corresponding substitution v. In the same way above, we can simplify the conserved vector using corresponding substitution v.

4. Conclusions

Recently, the new outstanding concepts of nonlinear self-adjoint equations, containing quasi-self-adjoint and weak self-adjoint equations, which extend the self-adjointness to a more generalized meaning, have been introduced in order to find formal Lagrangians of differential equations without variational structure. Using these concepts and the general theorem on conservation laws that is, developed recently [8], nonlinear self-adjointness and conservation laws for (3) for different classes of $f(u)$ and $g(u)$ have been discussed. These conservation laws may be useful in mathematical analysis as they provide basic conserved quantity for obtaining various estimates for smooth solutions and defining suitable norms for weak solutions. Furthermore, it could make the construction of the bi-Hamiltonian form easier.

Acknowledgments

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