Research Article

Stability of Impulsive Cohen-Grossberg Neural Networks with Time-Varying Delays and Reaction-Diffusion Terms

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This work concerns the stability of impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms as well as Dirichlet boundary condition. By means of Poincaré inequality and Gronwall-Bellman-type impulsive integral inequality, we summarize some new and concise sufficient conditions ensuring the global exponential stability of equilibrium point. The proposed criteria are relevant to the diffusion coefficients and the smallest positive eigenvalue of corresponding Dirichlet Laplacian. In conclusion, two examples are illustrated to demonstrate the effectiveness of our obtained results.

1. Introduction

Cohen-Grossberg neural networks (CGNNs) were introduced by Cohen and Grossberg in 1983 [1] and have been a hot topic due to their important applications in various fields such as parallel computation, associative memory, image processing, and optimization problems.

By reason that time delays are unavoidably encountered for the finite switching speed of neurons and amplifiers in the implementation of neural networks, a more powerful model of delayed Cohen-Grossberg neural networks (DCGNNs) is afterwards proposed. This kind of mathematical models is widely applied in dynamic image processing and pattern recognition problems. It is worth noting that all these applications depend heavily on the dynamical behaviors such as stability, convergence, and oscillatory [2–6]. Meanwhile, stability is an important consideration in the designs and applications of neural networks. The stability of delayed neural networks is a subject of current interest, and therefore considerable theoretical efforts have been put into this topic followed by a large number of stability criteria reported; for example, see [7–12] and the references therein.

In real world, however, many evolutionary processes are characterized by abrupt changes at certain instants which may be caused by switching phenomena, frequency changes, or other sudden noises. As such, in the past few years, scientists have become gradually interested in the influence that impulses may have on the CGNNs and DCGNNs, thus obtaining some related results; for example, see [13–18] and the references therein.

Actually, besides impulsive effects, we have to recognize that diffusion effects are also nonignorable in reality as diffusion is unavoidable when electrons are moving in asymmetric electromagnetic fields. On this account, the model of neural networks with both impulses and diffusion should be more effective for describing the evolutionary process of practical systems. Based on this consideration, we wonder what the influence of diffusion on the stability of CGNNs and DCGNNs is.

So far there have appeared a few theoretical achievements [19–29] on the stability of impulsive reaction-diffusion neural networks with or without delays. Particularly, in [21–26], the main research technique is the impulsive differential inequality whereby the authors discussed the stability of equilibrium point and provided a series of sufficient conditions independent of diffusion. From these results, we fail to see the influence of diffusion on the stability of CGNNs and DCGNNs.

Encouragingly, recently there were reported some new results on the stability of CGNNs and DCGNNs in [19, 20, 27]; thereinto, the presented stability criteria derived from

the impulsive differential inequality are related to the diffusion terms, and thereby we know the diffusion do contribute to the stability of impulsive neural networks.

In this paper, different from [20, 27], we shall consider the case where the boundary condition is Dirichlet boundary condition rather than Neumann boundary condition. Moreover, unlike [19], we shall utilize the new method of Poincarè inequality to deal with the reaction-diffusion terms, and Gronwall-Bellman-type impulsive integral inequality is also introduced for stability analysis. The obtained results show that not only the reaction-diffusion coefficients but also the first eigenvalue of corresponding Dirichlet Laplacian can affect the stability.

The rest of this paper is structured as follows. In Section 2, the model of impulsive delayed Cohen-Grossberg neural networks with reaction-diffusion terms as well as Dirichlet boundary condition is outlined and some facts and lemmas are introduced for later reference. By the new agencies of Gronwall-Bellman-type impulsive integral inequality and Poincaré inequality, we discuss the global exponential stability of equilibrium point and develop some new and concise algebraic criteria in Section 3. To conclude, two illustrative examples are given in Section 4 to verify the effectiveness of our results.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space, and let $\Omega \subset R^m$ be an open bounded domain with smooth boundary $\partial\Omega$ and mes$\Omega > 0$. Let $R_+ = [0, \infty)$ and $t_0 \in R_+$.

Consider the following impulsive CGNN with time-varying delays and reaction-diffusion terms:

$$\frac{\partial u_i(t,x)}{\partial t} = \sum_{s=1}^m \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,x)}{\partial x_s}\right)$$

$$- a_i(u_i(t,x))\left[\omega_i(u_i(t,x)) - \sum_{j=1}^n b_{ij}f_j(u_j(t,x))\right.$$

$$\left. - \sum_{j=1}^n c_{ij}f_j(u_j(t-\tau_j(t),x))\right],$$

$$t \geq t_0, t \neq t_k, x \in \Omega, i = 1, 2, \ldots, n, k = 1, 2, \ldots,$$
(1)

$$u_i(t_k + 0, x) = u_i(t_k, x) + P_{ik}(u_i(t_k, x)),$$

$$x \in \Omega, k = 1, 2, \ldots, i = 1, 2, \ldots, n,$$
(2)

where n corresponds to the numbers of units in a neural network, $x = (x_1, \ldots, x_m)^T \in \Omega$, $u_i(t,x)$ denotes the state of the ith neuron at time t and in space x, $D_{is} = $ const > 0 represents transmission diffusion of the ith unit, $a_i(u_i(t,x))$ represents the amplification function, $\omega_i(u_i(t,x))$ is the appropriate behavior function, activation function $f_j(u_j(t,x))$ stands for the output of the jth unit at time i and in space x and b_{ij} and c_{ij} are constants: b_{ij} indicates the connection strength of the jth unit on the ith unit at time

t and in space x, while c_{ij} denotes the connection weight of the jth unit on the ith unit at time $t - \tau_j(t)$ and in space x, where $\tau_j(t)$ corresponds to the transmission delay along the axon of the jth unit satisfying $0 \leq \tau_j(t) \leq \tau$ ($\tau = $ const) and $\dot{\tau}_j(t) < 1 - (1/h)(h > 0)$. $\{t_k\}$ ($k = 1, 2, \ldots$) is the sequence of impulsive moments meeting $0 \leq t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $u_i(t_k + 0, x)$ and $u_i(t_k - 0, x)$ represent the right-hand and left-hand limit of $u_i(t,x)$ at time t_k and in space x, respectively. $P_{ik}(u_i(t_k, x))$ stands for the abrupt change of $u_i(t,x)$ at impulsive moment t_k and in space x.

Denote by $u(t,x) = u(t,x; t_0, \varphi)$, $u \in R^n$, the solution of systems (1)-(2), satisfying the initial condition

$$u(s, x; t_0, \varphi) = \varphi(s, x), \quad t_0 - \tau \leq s \leq t_0, x \in \Omega, \quad (3)$$

and Dirichlet boundary condition

$$u(t, x; t_0, \varphi) = 0, \quad t \geq t_0, x \in \partial\Omega, \quad (4)$$

where the vector-valued function $\varphi(s, x) = (\varphi_1(s, x), \ldots, \varphi_n(s, x))^T$ is such that $\int_\Omega \sum_{i=1}^n \varphi_i^2(s, x)dx$ is bounded on $[t_0 - \tau, t_0]$.

The solution $u(t, x) = u(t, x; t_0, \varphi) = (u_1(t, x; t_0, \varphi), \ldots, u_n(t, x; t_0, \varphi))^T$ of problems (1)–(4) is, for the time variable t, a piecewise continuous function with the first kind discontinuity at the points t_k ($k = 1, 2, \ldots$), where it is left-continuous; that is, the following relations are valid:

$$u_i(t_k - 0, x) = u_i(t_k, x),$$

$$u_i(t_k + 0, x) = u_i(t_k, x) + P_{ik}(u_i(t_k, x)).$$
(5)

Throughout this paper, we define the norm of $u(t, x; t_0, \varphi)$ as

$$\|u(t, x; t_0, \varphi)\|_\Omega = \sqrt{\sum_{i=1}^n \int_\Omega u_i^2(t, x; t_0, \varphi)\, dx} \quad (6)$$

and make the following assumptions for convenience.

(H1) $a_i(\cdot) : R \to R^+$ is continuous and bounded; that is, there exist constants $\underline{a}_i$ and $\bar{a}_i$ such that

$$0 < \underline{a}_i \leq a_i(\zeta) \leq \bar{a}_i < \infty, \quad \text{for } i = 1, \ldots, n. \quad (7)$$

(H2) $\omega_i(\cdot) : R \to R$ is continuous and $\omega_i(0) = 0$; moreover, there exists constant $p_i > 0$ such that

$$\frac{\omega_i(\zeta_1) - \omega_i(\zeta_2)}{\zeta_1 - \zeta_2} \geq p_i > 0, \quad \text{for } \zeta_1 \neq \zeta_2, i = 1, \ldots, n. \quad (8)$$

(H3) $f_i(\cdot) : R \to R$ is continuous and $f_i(0) = 0$; furthermore, there exists constant $l_i > 0$ such that

$$l_i = \sup_{\zeta_1 \neq \zeta_2} \frac{f_i(\zeta_1) - f_i(\zeta_2)}{\zeta_1 - \zeta_2} \quad \text{for } \zeta_1 \neq \zeta_2, i = 1, 2, \ldots, n. \quad (9)$$

(H4) $P_{ik}(\cdot) : R \to R$ is continuous and $P_{ik}(0) = 0$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$.

In the light of (H1)–(H4), it is easy to see that problems (1)-(2) admit an equilibrium point $u = 0$.

Definition 1. The equilibrium point $u = 0$ of problems (1)-(2) is said to be globally exponentially stable if there exist constants $\kappa > 0$ and $M \geq 1$ such that

$$\|u(t, x; t_0, \varphi)\|_\Omega \leq M\overline{\|\varphi\|}_\Omega e^{-\kappa(t-t_0)}, \quad t \geq t_0, \qquad (10)$$

where $\overline{\|\varphi\|}_\Omega^2 = \sup_{t_0-\tau \leq s \leq t_0} \sum_{i=1}^n \int_\Omega \varphi_i^2(s, x)dx$.

Lemma 2 (see [30] (Gronwall-Bellman-type impulsive integral inequality)). *Assume the following.*

(A1) *The sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots$, with $\lim_{k \to \infty} t_k = \infty$.*

(A2) *$q \in PC^1[R_+, R]$ and $q(t)$ is left-continuous at t_k, $k = 1, 2, \ldots$.*

(A3) *$p \in C[R_+, R_+]$ and for $k = 1, 2, \ldots$,*

$$q(t) \leq c + \int_{t_0}^t p(s) q(s) ds + \sum_{t_0 < t_k < t} \eta_k q(t_k), \quad t \geq t_0, \quad (11)$$

where $\eta_k \geq 0$ and $c = $ const. Then,

$$q(t) \leq c \prod_{t_0 < t_k < t} (1 + \eta_k) \exp\left(\int_{t_0}^t p(s) ds\right), \quad t \geq t_0. \quad (12)$$

Lemma 3 (see [31] (Poincaré inequality)). *Let $\mathcal{S}$ be a bounded region in R^n, $v(x) \in C^1(\mathcal{S})$, and $v = 0$ on the boundary of $\mathcal{S}$; then*

$$\lambda_1 \int_{\mathcal{S}} v^2(x) dx \leq \int_{\mathcal{S}} |\nabla v(x)|^2 dx, \qquad (13)$$

where λ_1 is the smallest positive eigenvalue of the following problem:

$$\Delta\Psi(x) + \lambda\Psi(x) = 0, \quad x \in \mathcal{S}, \qquad \Psi(x) = 0, \quad x \in \partial\mathcal{S}. \qquad (14)$$

Lemma 4. *If $a > 0$ and $b > 0$, then $ab \leq (1/\varepsilon)a^2 + \varepsilon b^2$ holds for any $\varepsilon > 0$.*

3. Main Results

Theorem 5. *Assume the following.*

(1) *$\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ and denote $2\underline{D}\lambda_1 = \chi$.*

(2) *$P_{ik}(u_i(t_k, x)) = -\theta_{ik}u_i(t_k, x)$, $0 \leq \theta_{ik} \leq 2$.*

(3) *There exists a constant $\gamma > 0$ satisfying $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$, where*

$$\lambda = \max_{i=1,\ldots,n}\left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^n b_{ij}^2 + \overline{a}_i \sum_{j=1}^n c_{ij}^2\right) + \rho, \qquad (15)$$

$$\rho = \max_{i=1,\ldots,n}\left(l_i^2\right)\sum_{i=1}^n \overline{a}_i.$$

Then, the equilibrium point $u = 0$ of systems (1)-(2) is globally exponentially stable with convergence rate $-(\lambda + h\rho e^{\gamma\tau})/2$.

Proof. Multiplying both sides of (1) by $u_i(t, x)$, we get

$$\frac{\partial u_i^2(t, x)}{\partial t} = 2\sum_{s=1}^m u_i(t, x)\frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t, x)}{\partial x_s}\right)$$

$$- 2u_i(t, x) a_i(u_i(t, x))$$

$$\times \left[\omega_i(u_i(t, x)) - \sum_{j=1}^n b_{ij}f_j(u_j(t, x))\right.$$

$$\left. - \sum_{j=1}^n c_{ij}f_j(u_j(t - \tau_j(t), x))\right],$$

$$t \geq t_0, \ t \neq t_k, \ x \in \Omega, \quad k = 1, 2, \ldots, \qquad (16)$$

which yields, after integrating with respect to spatial variable x on Ω,

$$\frac{d\left(\int_\Omega u_i^2(t, x) dx\right)}{dt} = J_1 + J_2, \qquad (17)$$

$$t \geq t_0, \ t \neq t_k, \ k = 1, 2, \ldots,$$

where $J_1 = 2\int_\Omega \sum_{s=1}^m (u_i(t, x)(\partial/\partial x_s)(D_{is}(\partial u_i(t, x))/\partial x_s))dx$,

$$J_2 = -2\int_\Omega u_i(t, x) a_i(u_i(t, x))$$

$$\times \left[\omega_i(u_i(t, x)) - \sum_{j=1}^n b_{ij}f_j(u_j(t, x))\right.$$

$$\left. - \sum_{j=1}^n c_{ij}f_j(u_j(t - \tau_j(t), x))\right] dx. \qquad (18)$$

By combining Green formula, Dirichlet boundary condition, Lemma 3, and condition (1) of Theorem 5, we obtain

$$J_1 = -2\sum_{s=1}^m \int_\Omega D_{is}\left(\frac{\partial u_i(t, x)}{\partial x_s}\right)^2 dx$$

$$\leq -2\underline{D}\lambda_1 \int_\Omega u_i^2(t, x) dx \triangleq -\chi \int_\Omega u_i^2(t, x) dx. \qquad (19)$$

Moreover, it follows from assumptions (H1), (H2), and (H3) that

$$2 \int_{\Omega} u_i(t,x) a_i(u_i(t,x)) \omega_i(u_i(t,x)) dx \qquad (20)$$
$$\geq 2\underline{a}_i p_i \int_{\Omega} |u_i(t,x)|^2 dx,$$

$$2 \int_{\Omega} u_i(t,x) a_i(u_i(t,x)) \sum_{j=1}^{n} b_{ij} f_j(u_j(t,x)) dx$$
$$\leq \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} \left(b_{ij}^2 u_i^2(t,x) + f_j^2(u_j(t,x)) \right) dx$$
$$\leq \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} 2 |b_{ij}| |u_i(t,x)| |f_j(u_j(t,x))| dx \qquad (21)$$
$$\leq \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} \left(b_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t,x) \right) dx,$$

$$2 \int_{\Omega} u_i(t,x) a_i(u_i(t,x)) \sum_{j=1}^{n} c_{ij} f_j(u_j(t-\tau_j(t),x)) dx$$
$$\leq \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} \left(c_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t-\tau_j(t),x) \right) dx. \qquad (22)$$

Consequently, substituting (19)–(22) into (17) produces

$$\frac{d \left(\int_{\Omega} u_i^2(t,x) dx \right)}{dt}$$
$$\leq -\chi \int_{\Omega} u_i^2(t,x) dx - 2\underline{a}_i p_i \int_{\Omega} u_i^2(t,x) dx$$
$$+ \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} \left(b_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t,x) \right) dx \qquad (23)$$
$$+ \overline{a}_i \sum_{j=1}^{n} \int_{\Omega} \left(c_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t-\tau_j(t),x) \right) dx$$

for $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \ldots$.

Now define Lyapunov function $V_i(t)$ as $V_i(t) = \int_{\Omega} u_j^2(t,x) dx$. It is not difficult to see that $V_i(t)$ is a piecewise continuous function with points of discontinuity of the first kind t_k $(k = 1, 2, \ldots)$, where it is continuous from the left; that is, $V_i(t_k - 0) = V_i(t_k)$ $(k = 1, 2, \ldots)$. In addition, for $t = t_k$ $(k = 0, 1, 2, \ldots)$, we know

$$V_i(t_k + 0) \leq V_i(t_k), \qquad k = 0, 1, 2, \ldots, \qquad (24)$$

as $V_i(t_0 + 0) \leq V_i(t_0)$ and $u_i^2(t_k + 0, x) = (1 - \theta_{ik})^2 u_i^2(t_k, x) \leq u_i^2(t_k, x)$ $(k = 1, 2, \ldots)$, supported by condition 2 of Theorem 5.

Put $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$. It is derived from (23) that

$$\frac{dV_i(t)}{dt} \leq \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2 \right) V_i(t)$$
$$+ \overline{a}_i \max_{i=1,\ldots,n} (l_i^2) \sum_{j=1}^{n} V_j(t) \qquad (25)$$
$$+ \overline{a}_i \max_{i=1,\ldots,n} (l_i^2) \sum_{j=1}^{n} V_j(t-\tau_j(t)),$$
$$t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots.$$

Define function $V(t)$ of the form $V(t) = \sum_{i=1}^{n} V_i(t)$ again. From (25), one then reads

$$\frac{dV(t)}{dt} \leq \lambda V(t)$$
$$+ \rho \sum_{j=1}^{n} V_j(t-\tau_j(t)), \quad t \in (t_k, t_{k+1}), \ k = 0, 1, 2, \ldots, \qquad (26)$$

where $\rho = \max_{i=1,\ldots,n}(l_i^2) \sum_{i=1}^{n} \overline{a}_i$ and $\lambda = \max_{i=1,\ldots,n}(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2) + \rho$.

Construct $V^*(t) = e^{\gamma(t-t_0)} V(t)$, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$. Evidently, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k $(k = 1, 2, \ldots)$, in which it is continuous from the left; that is, $V^*(t_k - 0) = V^*(t_k)$ $(k = 1, 2, \ldots)$. Moreover, at $t = t_k$ $(k = 0, 1, 2, \ldots)$, we find by the use of (24)

$$V^*(t_k + 0) \leq V^*(t_k), \quad k = 0, 1, 2, \ldots. \qquad (27)$$

Set $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$. By virtue of (26), one has

$$\frac{dV^*(t)}{dt} = \gamma e^{\gamma(t-t_0)} V(t) + e^{\gamma(t-t_0)} \frac{dV(t)}{dt}$$
$$\leq \gamma e^{\gamma(t-t_0)} V(t)$$
$$+ \left(\lambda V(t) + \rho \sum_{j=1}^{n} V_j(t-\tau_j(t)) \right) e^{\gamma(t-t_0)} \qquad (28)$$
$$= (\gamma + \lambda) V^*(t) + \rho e^{\gamma(t-t_0)} \sum_{j=1}^{n} V_j(t-\tau_j(t)),$$
$$t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots.$$

Choose small enough $\varepsilon > 0$. Integrating (28) from $t_k + \varepsilon$ to t gives

$$V^*(t) \leq V^*(t_k + \varepsilon) + (\gamma + \lambda) \int_{t_k+\varepsilon}^{t} V^*(s) ds$$
$$+ \int_{t_k+\varepsilon}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s)) ds, \qquad (29)$$
$$t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots$$

which yields, after letting $\varepsilon \to 0$ in (29),

$$V^* (t) \le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t} V^* (s)\, ds$$

$$+ \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds, \qquad (30)$$

$$t \in (t_k, t_{k+1}), \; k = 0, 1, 2, \ldots.$$

Next, we estimate the value of $V^*(t)$ at $t = t_{k+1}$, $k = 0, 1, 2, \ldots$. For small enough $\varepsilon > 0$, we put $t = t_{k+1} - \varepsilon$. Now an application of (30) leads to, for $k = 0, 1, 2, \ldots$,

$$V^* (t_{k+1} - \varepsilon) \le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1}-\varepsilon} V^* (s)\, ds$$

$$+ \int_{t_k}^{t_{k+1}-\varepsilon} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds. \qquad (31)$$

If we let $\varepsilon \to 0$ in (31), there results

$$V^* (t_{k+1} - 0)$$

$$\le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1}} V^* (s)\, ds$$

$$+ \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds, \quad k = 0, 1, 2, \ldots.$$

$$(32)$$

Note that $V^*(t_{k+1}-0) = V^*(t_{k+1})$ is applicable for $k = 0, 1, 2, \ldots$. Thus,

$$V^* (t_{k+1}) \le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t_{k+1}} V^* (s)\, ds$$

$$+ \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds \qquad (33)$$

holds for $k = 0, 1, 2, \ldots$. By synthesizing (30) and (33), we then arrive at

$$V^* (t) \le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^{t} V^* (s)\, ds$$

$$+ \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds, \qquad (34)$$

$$t \in (t_k, t_{k+1}], \; k = 0, 1, 2, \ldots.$$

This, together with (27), results in

$$V^* (t) \le V^* (t_k) + (\gamma + \lambda) \int_{t_k}^{t} V^* (s)\, ds$$

$$+ \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds \qquad (35)$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$.

Recalling the assumptions that $0 \le \tau_j(t) \le \tau$ and $\dot{\tau}_j(t) < 1 - (1/h)(h > 0)$, we therefore obtain

$$\int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j \left(s - \tau_j (s)\right) ds$$

$$= \sum_{j=1}^{n} \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} \rho e^{\gamma(\theta + \tau_j(s) - t_0)} V_j (\theta) \frac{1}{1 - \dot{\tau}_j (s)}\, d\theta \qquad (36)$$

$$\le h \rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j (\theta)\, d\theta.$$

Hence,

$$V^* (t) \le V^* (t_k) + (\gamma + \lambda) \int_{t_k}^{t} V^* (s)\, ds$$

$$+ h \rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(s-t_0)} V_j (s)\, ds, \qquad (37)$$

$$t \in (t_k, t_{k+1}], \; k = 0, 1, 2, \ldots.$$

By induction argument, we reach

$$V^* (t_k) \le V^* (t_{k-1}) + (\gamma + \lambda) \int_{t_{k-1}}^{t_k} V^* (s)\, ds$$

$$+ h \rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_{k-1} - \tau_j(t_{k-1})}^{t_k - \tau_j(t_k)} e^{\gamma(s-t_0)} V_j (s)\, ds,$$

$$\vdots$$

$$V^* (t_2) \le V^* (t_1) + (\gamma + \lambda) \int_{t_1}^{t_2} V^* (s)\, ds \qquad (38)$$

$$+ h \rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_1 - \tau_j(t_1)}^{t_2 - \tau_j(t_2)} e^{\gamma(s-t_0)} V_j (s)\, ds,$$

$$V^* (t_1) \le V^* (t_0) + (\gamma + \lambda) \int_{t_0}^{t_1} V^* (s)\, ds$$

$$+ h \rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_0 - \tau_j(t_0)}^{t_1 - \tau_j(t_1)} e^{\gamma(s-t_0)} V_j (s)\, ds.$$

Thus,

$$V^* (t) \le V^* (t_0) + (\gamma + \lambda) \int_{t_0}^t V^* (s)\, ds$$

$$+ h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t - \tau_j(t)} e^{\gamma(s - t_0)} V_j (s)\, ds$$

$$\le V^* (t_0) + (\gamma + \lambda) \int_{t_0}^t V^* (s)\, ds$$

$$+ h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^t e^{\gamma(s - t_0)} V_j (s)\, ds \qquad (39)$$

$$= V^* (t_0) + (\gamma + \lambda + h\rho e^{\gamma \tau}) \int_{t_0}^t V^* (s)\, ds$$

$$+ h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t_0} e^{\gamma(s - t_0)} V_j (s)\, ds,$$

$$t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots.$$

Since

$$h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_0 - \tau_j(t_0)}^{t_0} e^{\gamma(s - t_0)} V_j (s)\, ds$$

$$\le h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_0 - \tau}^{t_0} V_j (s)\, ds$$

$$= h\rho e^{\gamma \tau} \int_{t_0 - \tau}^{t_0} \left(\sum_{j=1}^n \int_\Omega \varphi_j^{\,2} (s, x)\, dx \right) ds \qquad (40)$$

$$\le \tau h\rho e^{\gamma \tau} \overline{\|\varphi\|}_\Omega^2,$$

we claim

$$V^* (t) \le V^* (t_0) + \tau h\rho e^{\gamma \tau} \overline{\|\varphi\|}_\Omega^2$$

$$+ (\gamma + \lambda + h\rho e^{\gamma \tau}) \int_{t_0}^t V^* (s)\, ds, \qquad (41)$$

$$t \in (t_k, t_{k+1}],\ k = 0, 1, 2 \ldots.$$

According to Lemma 2, we assert that

$$V^* (t) \le \left(V^* (t_0) + \tau h\rho e^{\gamma \tau} \overline{\|\varphi\|}_\Omega^2 \right)$$

$$\times \exp \left\{ (\gamma + \lambda + h\rho e^{\gamma \tau})(t - t_0) \right\}, \quad t \ge t_0, \qquad (42)$$

which reduces to

$$\|u (t, x; t_0, \varphi)\|_\Omega$$

$$\le \sqrt{1 + \tau h\rho e^{\gamma \tau}} \, \overline{\|\varphi\|}_\Omega \qquad (43)$$

$$\times \exp \left\{ \left(\frac{\lambda + h\rho e^{\gamma \tau}}{2} \right) (t - t_0) \right\}, \quad t \ge t_0.$$

This completes the proof. □

Remark 6. According to the conditions of Theorem 5, we see that the reaction-diffusion terms can influence the stability of equilibrium point $u = 0$. Specifically, the acting factors include the reaction-diffusion coefficients and the first eigenvalue of corresponding Dirichlet Laplacian.

Remark 7. It is not difficult to see that there must exist constant $\gamma > 0$ satisfying condition 3 of Theorem 5 if $\lambda < -h\rho$.

Theorem 8. *Assume the following.*

(1) $\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ *and denote* $2\underline{D}\lambda_1 = \chi$.

(2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik} u_i(t_k, x),\ 1 - \sqrt{1 + \alpha} \le \theta_{ik} \le 1 + \sqrt{1 + \alpha},\ \alpha \ge 0$.

(3) $\inf_{k=1,2\ldots}(t_k - t_{k-1}) \ge \mu$.

(4) *There exists a constant* $\gamma > 0$ *satisfying* $\gamma + \lambda + h\rho e^{\gamma \tau} > 0$ *and* $\lambda + h\rho e^{\gamma \tau} + \ln(1 + \alpha)/\mu < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^n b_{ij}^2 + \overline{a}_i \sum_{j=1}^n c_{ij}^2) + \rho$ *and* $\rho = \max_{i=1,\ldots,n}(l_i^2) \sum_{i=1}^n \overline{a}_i$.

Then, the equilibrium point $u = 0$ of systems (1)-(2) is globally exponentially stable with convergence rate $-(1/2)(\lambda + h\rho e^{\gamma \tau} + \ln(1 + \alpha)/\mu)$.

Proof. Define Lyapunov function V of the form $V(t) = \sum_{i=1}^n V_i(t)$, where $V_i(t) = \int_\Omega u_i^2(t, x) dx$. Obviously, $V(t)$ is a piecewise continuous function with the first kind discontinuous points t_k, $k = 1, 2, \ldots$, where it is continuous from the left; that is, $V(t_k - 0) = V(t_k)$ $(k = 1, 2, \ldots)$. Furthermore, for $t = t_k$ $(k = 0, 1, 2, \ldots)$, we derive from condition 2 of Theorem 8 that

$$u_i^2 (t_k + 0, x) - u_i^2 (t_k, x)$$

$$= (1 - \theta_{ik})^2 u_i^2 (t_k, x) - u_i^2 (t_k, x) \le \alpha u_i^2 (t_k, x). \qquad (44)$$

Thereby,

$$V (t_k + 0) \le \alpha V (t_k) + V (t_k), \quad k = 0, 1, 2, \ldots. \qquad (45)$$

Construct function $V^*(t) = e^{\gamma(t - t_0)} V(t)$ again, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho e^{\gamma \tau} > 0$ and $\lambda + h\rho e^{\gamma \tau} + \ln(1 + \alpha)/\mu < 0$. Then, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k, $k = 1, 2, \ldots$, where it is continuous from the left; that is, $V^*(t_k - 0) = V^*(t_k)$ $(k = 1, 2, \ldots)$. And for $t = t_k$ $(k = 0, 1, 2, \ldots)$, it follows from (45) that

$$V^* (t_k + 0) \le \alpha V^* (t_k) + V^* (t_k), \quad k = 0, 1, 2, \ldots \qquad (46)$$

Set $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$. Following the same procedure as shown in the proof of Theorem 5, we get

$$V^* (t) \le V^* (t_k + 0) + (\gamma + \lambda) \int_{t_k}^t V^* (s)\, ds$$

$$+ h\rho e^{\gamma \tau} \sum_{j=1}^n \int_{t_k - \tau_j(t_k)}^{t - \tau_j(t)} e^{\gamma(\theta - t_0)} V_j (\theta)\, d\theta, \qquad (47)$$

$$t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots.$$

The relations (46) and (47) yield

$$V^*(t) - V^*(t_k)$$

$$\leq \alpha V^*(t_k) + (\gamma + \lambda) \int_{t_k}^t V^*(s)\,ds$$

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta,$$

$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots.$$

(48)

By induction argument, we obtain

$$V^*(t_k) - V^*(t_{k-1})$$

$$\leq \alpha V^*(t_{k-1}) + (\gamma + \lambda) \int_{t_{k-1}}^{t_k} V^*(s)\,ds$$

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta,$$

$$\vdots$$

$$V^*(t_2) - V^*(t_1)$$

$$\leq \alpha V^*(t_1) + (\gamma + \lambda) \int_{t_1}^{t_2} V^*(s)\,ds$$

(49)

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta,$$

$$V^*(t_1) - V^*(t_0)$$

$$\leq \alpha V^*(t_0) + (\gamma + \lambda) \int_{t_0}^{t_1} V^*(s)\,ds$$

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta.$$

Hence,

$$V^*(t) - V^*(t_0)$$

$$\leq \alpha V^*(t_0) + (\gamma + \lambda) \int_{t_0}^t V^*(s)\,ds$$

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta$$

$$+ \alpha \sum_{t_0 < t_k < t} V(t_k)$$

$$\leq \alpha V^*(t_0) + (\gamma + \lambda + h\rho e^{\gamma\tau}) \int_{t_0}^t V^*(s)\,ds$$

$$+ h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta + \alpha \sum_{t_0 < t_k < t} V(t_k),$$

$$t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots.$$

(50)

Introducing $h\rho e^{\gamma\tau} \sum_{j=1}^n \int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta \leq \tau h\rho e^{\gamma\tau} \overline{\|\varphi\|}_\Omega^2$ as shown in the proof of Theorem 5 into (50), (50) becomes, for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$,

$$V^*(t) - V^*(t_0)$$

$$\leq \alpha V^*(t_0) + \tau h\rho e^{\gamma\tau} \overline{\|\varphi\|}_\Omega^2$$

$$+ (\gamma + \lambda + h\rho e^{\gamma\tau}) \int_{t_0}^t V^*(s)\,ds + \alpha \sum_{t_0 < t_k < t} V(t_k).$$

(51)

It then results from Lemma 2 that, for $t \geq t_0$,

$$V^*(t) \leq \left((\alpha + 1) V^*(t_0) + \tau h\rho e^{\gamma\tau} \overline{\|\varphi\|}_\Omega^2 \right)$$

$$\times \prod_{t_0 < t_k < t} (1 + \alpha) \exp\left((\gamma + \lambda + h\rho e^{\gamma\tau})(t - t_0) \right)$$

(52)

$$= \left((\alpha + 1) V^*(t_0) + \tau h\rho e^{\gamma\tau} \overline{\|\varphi\|}_\Omega^2 \right)$$

$$\times (1 + \alpha)^k \exp\left((\gamma + \lambda + h\rho e^{\gamma\tau})(t - t_0) \right).$$

On the other hand, since $\inf_{k=1,2,\ldots}(t_k - t_{k-1}) \geq \mu$, one has $k \leq (t_k - t_0)/\mu$. Thereby,

$$(1 + \alpha)^k \leq \exp\left\{ \frac{\ln(1 + \alpha)}{\mu} (t_k - t_0) \right\}$$

$$\leq \exp\left\{ \frac{\ln(1 + \alpha)}{\mu} (t - t_0) \right\}$$

(53)

and (52) can be rewritten as

$$V^*(t) \leq \left((\alpha + 1) V^*(t_0) + \tau h\rho e^{\gamma\tau} \overline{\|\varphi\|}_\Omega^2 \right)$$

$$\times \exp\left(\left(\gamma + \lambda + h\rho e^{\gamma\tau} + \frac{\ln(1 + \alpha)}{\mu} \right)(t - t_0) \right)$$

(54)

which implies

$$\|u(t, x; t_0, \varphi)\|_\Omega \leq \sqrt{(\alpha + 1 + \tau h\rho e^{\gamma\tau})} \overline{\|\varphi\|}_\Omega$$

$$\times \exp\left(\frac{1}{2} \left(\lambda + h\rho e^{\gamma\tau} + \frac{\ln(1 + \alpha)}{\mu} \right)(t - t_0) \right), \quad t \geq t_0.$$

(55)

The proof is completed. $\qquad\square$

Due to Lemma 4, we know that the following inequalities:

$$2\sum_{j=1}^{n} b_{ij} \int_{\Omega} u_i(t,x) f\left(u_j(t,x)\right) dx$$

$$\leq \sum_{j=1}^{n} \int_{\Omega} \left(\varepsilon_1 b_{ij}^2 u_i^2(t,x) + \frac{l_j^2}{\varepsilon_1} u_j^2(t,x)\right) dx,$$

$$2\sum_{j=1}^{n} c_{ij} \int_{\Omega} u_i(t,x) f\left(u_j\left(t-\tau_j,x\right)\right) dx \qquad (56)$$

$$\leq \sum_{j=1}^{n} \int_{\Omega} \left(\varepsilon_2 c_{ij}^2 u_i^2(t,x) + \frac{l_j^2}{\varepsilon_2} u_i^2\left(t-\tau_j,x\right)\right) dx$$

hold for any $\varepsilon_1, \varepsilon_2 > 0$. Thus, in a similar way to the proofs of Theorems 5–8, we can prove the following theorems.

Theorem 9. *Assume the following.*

(1) $\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ *and denote* $2\underline{D}\lambda_1 = \chi$.

(2) $P_{ik}(u_i(t_k,x)) = -\theta_{ik}u_i(t_k,x), 0 \leq \theta_{ik} \leq 2$.

(3) *There exist constants* $\gamma > 0$ *and* $\varepsilon_1, \varepsilon_2 > 0$ *such that* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2\underline{a}_i p_i + \bar{a}_i \sum_{j=1}^{n}(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2)) + (\max_{i=1,\ldots,n}(l_i^2)/\varepsilon_1)\sum_{i=1}^{n}\bar{a}_i$, *and* $\rho = (\max_{i=1,\ldots,n}(l_j^2)/\varepsilon_2)\sum_{i=1}^{n}\bar{a}_i$.

Then, the equilibrium point $u = 0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(\lambda + h\rho e^{\gamma\tau})/2$.

Remark 10. There must exist constant $\gamma > 0$ satisfying condition 3 of Theorem 9 if there are constants $\varepsilon_1, \varepsilon_2 > 0$ such that $\lambda < -h\rho$.

Theorem 11. *Assume the following.*

(1) $\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ *and denote* $2\underline{D}\lambda_1 = \chi$.

(2) $P_{ik}(u_i(t_k,x)) = -\theta_{ik}u_i(t_k,x), 1 - \sqrt{1+\alpha} \leq \theta_{ik} \leq 1 + \sqrt{1+\alpha}, \alpha \geq 0$.

(3) $\inf_{k=1,2,\ldots}(t_k - t_{k-1}) \geq \mu$.

(4) *There exist constants* $\gamma > 0$ *and* $\varepsilon_1, \varepsilon_2 > 0$ *satisfying* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu < 0$, *where*

$$\lambda = \max_{i=1,\ldots,n}\left(-\chi - 2\underline{a}_i p_i + \bar{a}_i \sum_{j=1}^{n}\left(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2\right)\right)$$

$$+ \frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_1}\sum_{i=1}^{n}\bar{a}_i, \qquad (57)$$

$$\rho = \frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_2}\sum_{i=1}^{n}\bar{a}_i.$$

Then, the equilibrium point $u = 0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu)$.

Further, on the condition that $|P_{ik}(u_i(t_k,x))| \leq \theta_{ik}|u_i(t_k,x)|$, where $\theta_{ik}^2 < (\alpha - 1)/2$ and $\alpha \geq 1$, we obtain

$$u_i^2(t_k + 0, x) - u_i^2(t_k, x)$$

$$\leq 2(u_i(t_k,x))^2 + 2(P_{ik}(u_i(t_k,x)))^2 - u_i^2(t_k,x)$$

$$\leq \left(2 + 2\theta_{ik}^2\right)(u_i(t_k,x))^2 - u_i^2(t_k,x) \leq \alpha u_i^2(t_k,x) \qquad (58)$$

for $t = t_k$ $(k = 1, 2, \ldots)$. In an identical way with the proof of Theorem 8, we can present the following.

Theorem 12. *Assume the following.*

(1) *Let* $\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ *and denote* $2\underline{D}\lambda_1 = \chi$.

(2) $|P_{ik}(u_i(t_k,x))| \leq \theta_{ik}|u_i(t_k,x)|$, *where* $\theta_{ik}^2 \leq (\alpha - 1)/2$ *and* $\alpha \geq 1$.

(3) $\inf_{k=1,2,\ldots}(t_k - t_{k-1}) \geq \mu$.

(4) *There exist constants* $\gamma > 0$ *and* $\varepsilon_1, \varepsilon_2 > 0$ *such that* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu < 0$, *where*

$$\lambda = \max_{i=1,\ldots,n}\left(-\chi - 2\underline{a}_i p_i + \bar{a}_i \sum_{j=1}^{n}\left(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2\right)\right)$$

$$+ \frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_1}\sum_{i=1}^{n}\bar{a}_i, \qquad (59)$$

$$\rho = \frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_2}\sum_{i=1}^{n}\bar{a}_i.$$

Then, the equilibrium point $u = 0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu)$.

Remark 13. Different from Theorems 5–11, the impulsive part in Theorem 12 could be nonlinear, and this will be of more applicability. Actually, Theorems 5–11 can be regarded as the special cases of Theorem 12.

4. Examples

Example 14. Consider problems (1)–(4) with $P_{ik}(u_i(t_k,x)) = 1.343u_i(t_k,x)$; moreover, $n = 2$, $m = 2$, $\Omega = \{(x_1,x_2)^T \mid x_1^2 + x_2^2 < 1\}$, $a_i(u_1(t,x)) = 1$, $\omega_1(u_1(t,x)) = 6.5u_1(t,x)$, $\omega_2(u_2(t,x)) = 8.5u_2(t,x)$, $(D_{is}) = \left(\begin{smallmatrix} 1.2 & 2.3 \\ 2.2 & 1.5 \end{smallmatrix}\right)$, $(b_{ij}) = \left(\begin{smallmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{smallmatrix}\right)$, $(c_{ij}) = \left(\begin{smallmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{smallmatrix}\right)$, $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$, and $\tau_j(t) = (3/4)\arctan(t)$.

As $\lambda_1 = 5.783$ and $\underline{D} = 1.2$, we know $\chi = 13.8792$. Further, for $l_i = \sqrt{2}/2$, $\underline{a}_i = \overline{a}_i = 1$, $p_1 = 6.5$, and $p_2 = 8.5$, we compute

$$\rho = \max_{i=1,\ldots,n} \left(l_i^2 \right) \sum_{i=1}^{n} \overline{a}_i = 1,$$

$$\lambda = \max_{i=1,\ldots,n} \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2 \right) \quad (60)$$

$$+ \rho = -15.5402.$$

Let $h = 4$. Since $\lambda = -15.5402 < -4 = -h\rho$, we therefore conclude from Theorem 5 that the zero solution of this system is globally exponential stable.

Example 15. Consider problems (1)–(4) with $P_{ik}(u_i(t_k, x)) = \arctan(0.5u_i(t_k, x))$; moreover, $n = 2$, $m = 2$, $\Omega = \{(x_1, x_2)^T \mid x_1^2 + x_1^2 < 1\}$, $a_i(u_1(t, x)) = 1$, $\omega_1(u_1(t, x)) = 6.5u_1(t, x)$, $\omega_2(u_2(t, x)) = 8.5u_2(t, x)$, $(D_{is}) = \left(\begin{smallmatrix} 1.2 & 2.3 \\ 2.2 & 1.5 \end{smallmatrix} \right)$, $(b_{ij}) = \left(\begin{smallmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{smallmatrix} \right)$, $(c_{ij}) = \left(\begin{smallmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{smallmatrix} \right)$, $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$, $\tau_j(t) = (1/\pi) \arctan(t)$, and $t_k = t_{k-1} + k$.

As $\lambda_1 = 5.783$ and $\underline{D} = 1.2$, we know $\chi = 13.8792$. Further, for $l_i = \sqrt{2}/2$, $\underline{a}_i = \overline{a}_i = 1$, $p_1 = 6.5$, $p_2 = 8.5$, and $\varepsilon_i = 1$, we compute

$$\rho = \frac{\max_{i=1,\ldots,n}}{\varepsilon_2} \sum_{i=1}^{n} \overline{a}_i = 1,$$

$$\lambda = \max_{i=1,\ldots,n} \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} \left(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2 \right) \right) \quad (61)$$

$$+ \frac{\max_{i=1,\ldots,n} \left(l_i^2 \right)}{\varepsilon_1} \sum_{i=1}^{n} \overline{a}_i = -15.5402.$$

Let $\tau = 0.5$, $h = 4$, $\mu = 1$, $\theta_{ik} = 0.5$, and $\alpha = 1.5$; we can find $\gamma = 2.4$ such that

$$\gamma + \lambda + h\rho e^{\gamma\tau} = 0.1403 > 0,$$

$$\lambda + h\rho e^{\gamma\tau} + \frac{\ln(1+\alpha)}{\mu} = -1.3434 < 0. \quad (62)$$

Therefore it is concluded from Theorem 12 that the zero solution of this system is globally exponential stable.

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