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Research Article

Connectedness of Solution Sets for Weak Vector Variational Inequalities on Unbounded Closed Convex Sets

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We study the connectedness of solution set for set-valued weak vector variational inequality in unbounded closed convex subsets of finite dimensional spaces, when the mapping involved is scalar C-pseudomonotone. Moreover, the path connectedness of solution set for set-valued weak vector variational inequality is established, when the mapping involved is strictly scalar C-pseudomonotone. The results presented in this paper generalize some known results by Cheng (2001), Lee et al. (1998), and Lee and Bu (2005).

1. Introduction

The concept of vector variational inequality (VVI), which was first introduced by Giannessi [1] in 1980, has wide applications in many problems such as finance and economics, transportation and optimization, operations research, and engineering sciences. Many authors have devoted to the study of VVI and its various extensions. The main topic of these papers is to establish existence theorems of solution for (VVI), see, for example, [1–5] and the references therein. Another important and interesting problem for (VVI) is to study the topological properties of solutions set. Among them, the connectedness property of the solution set is quite of interest as it provides the possibility of continuously moving from one solution to any other solution.

Some authors have discussed the connectedness of solution set for single-valued weak vector variational inequalities ($WVVIs$) under the assumption that the mapping involved is monotone and the constrained set involved is compact. In [6], Chen discussed the connectedness of solution set for a single-valued $WVVI$ in compact subsets of R^n, when the mapping involved is strictly pseudomonotone. Gong [7] and Gong and Yao [8] studied the connectedness of solution sets for vector equilibrium problems and generalized system in compact subsets of infinite dimensional spaces, when the mapping involved is monotone. For more related work, we refer the readers to [9] and references therein.

It is noted that in the works mentioned above, a compactness assumption of the constrained set is necessary. As for the noncompactness case, we observe that only few papers in the literature have dealt with this. In [10], Lee et al. established the connectedness of the solution set for a single-valued strongly monotone $WVVI$ on unbounded closed convex sets. Lee and Bu [11] discussed the connectedness of solution set for affine vector variational inequalities with noncompact polyhedral constraint sets and positive semidefinite (or monotone) matrices.

Inspired and motivated by the work in [6, 10, 11], we further study the connectedness properties of solution set for set-valued $WVVI$ in noncompact subsets of finite dimensional spaces. We establish the connectedness and path-connectedness results of solution set for set-valued $WVVI$ when the mapping involved is scalar C-pseudomonotone and strictly scalar C-pseudomonotone, which is weaker than monotone mapping and strictly monotone (strongly monotone) mapping, respectively. Compared with the previous connectedness results, we establish our results without putting the compactness assumption on the constrained set, and the mapping involved is set-valued. Moreover, we would like to point out that the image space associated with the problem discussed is infinite dimensional. The results presented in this paper generalize the corresponding results in [6, 10, 11].

The paper is organized as follows. In Section 2, we introduce some basic notations and preliminary results. In Section 3, we establish the connectedness and path-connectedness result of the solution set for set-valued $WVVI$.

2. Preliminaries

Let X be a finite-dimensional norm space and Y a normed space with its dual space of Y^*. Let K be a nonempty closed convex subset of X and $T : K \to 2^{L(X,Y)}$ a set-valued mapping with nonempty values, where $L(X,Y)$ denotes the space of all continuous linear mappings from X to Y. Let C be a closed convex pointed cone in Y with $\operatorname{int} C \neq \emptyset$. The cone C induces a partial ordering in Y, which was defined by

$$z_1 \leq_C z_2 \quad \text{iff } z_2 - z_1 \in C. \tag{1}$$

Let $C^* := \{x^* \in Y^* : \langle x^*, x\rangle \geq 0, \forall x \in C\}$ be the dual cone of C. It is clear that

$$x \in C \Longleftrightarrow \langle x^*, x\rangle \geq 0, \quad \forall x^* \in C^*, \tag{2}$$

$$x \in \operatorname{int} C \Longleftrightarrow \langle x^*, x\rangle > 0, \quad \forall x^* \in C^* \setminus \{0\}. \tag{3}$$

Let $e \in \operatorname{int} C$ be fixed and

$$C^{*0} := \{x^* \in Y^* : \langle x^*, e\rangle = 1\}. \tag{4}$$

The dual cone C^* is said to admit a w^*-compact base if and only if there exists a w^*-compact set $S_1 \subset C^*$ such that $0 \notin S_1$ and $C^* \subset \cup_{t\geq 0} tS_1$. From [12], we know that C^{*0} is a w^*-compact base of C^*.

The recession cone of K, denoted by K_∞, is defined by

$$K_\infty = \left\{d \in X : \exists t_k \longrightarrow +\infty,\ x_k \in K \right. \\ \left. \text{such that } d = \lim_{k\to+\infty} \frac{x_k}{t_k}\right\}. \tag{5}$$

The negative polar cone of K, denoted by K^-, is defined by

$$K^- = \{x^* \in X^* : \langle x^*, x\rangle \leq 0, \forall x \in K\}. \tag{6}$$

Let $\mathscr{L} \subset L(X,Y)$ be a nonempty set. The weak and strong C-polar cones of $\mathscr{L}$, which were introduced in [13], are defined, respectively, by

$$L_C^{w\circ} := \{x \in X : \langle l, x\rangle \not\geq_{\operatorname{int} C} 0, \forall l \in \mathscr{L}\}, \\ L_C^{s\circ} := \{x \in X : \langle l, x\rangle \leq_C 0, \forall l \in \mathscr{L}\}. \tag{7}$$

In this paper, we consider the following set-valued weak vector variational inequality, denoted by $(WVVI)$, which is to find $x^* \in K$ and $u^* \in T(x^*)$ such that

$$\langle u^*, x - x^*\rangle \notin -\operatorname{int} C, \quad \forall x \in K. \tag{8}$$

Let $\xi \in C^{*0}$ be any given point. It is known that $(WVVI)$ is closely related to the following scalar variational inequality problem, denoted by $(VI)_\xi$, which is to find $x^* \in K$ and $u^* \in T(x^*)$ such that

$$\langle \xi(u^*), x - x^*\rangle \geq 0, \quad \forall x \in K. \tag{9}$$

Here, $\xi(u^*)$ means the composition of $\xi \in Y^*$ and $u^* \in L(X,Y)$. Hence, $\langle \xi(u^*), x\rangle = \langle \xi \circ u^*, x\rangle = \langle \xi, \langle u^*, x\rangle\rangle$, for all $x \in K$.

The solution sets of $(WVVI)$ and $(VI)_\xi$ are denoted by $S^w(T,K)$ and $S_\xi(T,K)$, respectively. From [10], the following relationship between $S^w(T,K)$ and $S_\xi(T,K)$ holds:

$$S^w(T,K) = \bigcup_{\xi\in C^{*0}} S_\xi(T,K). \tag{10}$$

Although the representation is stated in [10] for single-valued map, the statement and the proof are also valid when T is multivalued without any assumption on the values of T.

We now recall some definitions for set-valued monotone and pseudomonotone mapping.

Definition 1. A set-valued mapping $T : K \to 2^{L(X,Y)}$ is said to be as follows.

(i) C-monotone (resp. strictly C-monotone) on K if for all $x, y \in K$ (resp., $x \neq y$), for all $u^* \in T(x)$, $v^* \in T(y)$,

$$\langle v^* - u^*, y - x\rangle \in C \quad (\text{resp. } \operatorname{int} C). \tag{11}$$

(ii) C-pseudomonotone (resp., strictly C-monotone) on K if for all $x, y \in K$ (resp., $x \neq y$), for all $u^* \in T(x)$, $v^* \in T(y)$,

$$\langle u^*, y - x\rangle \notin -\operatorname{int} C \\ \Longrightarrow \langle v^*, y - x\rangle \in C \quad (\text{resp., } \operatorname{int} C). \tag{12}$$

Definition 2. A set-valued mapping $T : K \to 2^{L(X,Y)}$ is said to be scalar C-pseudomonotone (resp., strictly scalar C-pseudomonotone) on K if and only if for any $\xi \in C^* \setminus \{0\}$, for any $x, y \in K$ (resp., $x \neq y$), $u^* \in T(x)$, $v^* \in T(y)$, we have

$$\langle \xi(u^*), y - x\rangle \geq 0 \Longrightarrow \langle \xi(v^*), y - x\rangle \geq 0 \quad (\text{resp.,} > 0). \tag{13}$$

Example 3 is to clarify Definition 2.

Example 3. Let

$$K = R, \qquad C = R_+^2, \\ f_1(x) = [1, 2.5 + \sin x], \\ f_2(x) = [1, 2.5 + \cos x], \qquad T = (f_1, f_2). \tag{14}$$

Clearly, $T = (f_1, f_2)$ is strictly scalar C-pseudomonotone on K and so scalar C-pseudomonotone.

Remark 4. (i) If T is C-monotone (resp., strictly C-monotone) on K, then T is scalar C-pseudomonotone (resp., strictly scalar C-pseudomonotone) on K.

(ii) The scalar C-pseudomonotonicity in Definition 2 is weaker than C-pseudomonotonicity in Definition 1(ii). In fact, for any $\xi \in C^* \setminus \{0\}$, $x, y \in X$, if $\langle \xi(u^*), y - x\rangle \geq 0$, then we have $\langle u^*, y - x\rangle \notin -\operatorname{int} C$. Then, it follows from the C-pseudomonotonicity of T that $\langle v^*, y - x\rangle \in C$, which implies that $\langle \xi(v^*), y - x\rangle \geq 0$.

Definition 5. The topological space E is said to be connected if there do not exist nonempty open subsets V_i of E, $i = 1, 2$, such that $V_1 \cup V_2 = X$ and $V_1 \cap V_2 = \emptyset$.

Moreover, E is said to be path connected if for each pair of points x and y in E, there exists a continuous mapping $\phi : [0, 1] \to E$ such that $\phi(0) = x$ and $\phi(1) = y$.

Definition 6. Let E, F be two topological spaces. A set-valued mapping $G : E \to 2^F$ is said to be

(i) closed if graph $G = \{(x, y) \in E \times F : y \in G(x)\}$ is closed in $E \times F$,

(ii) upper semicontinuous, if for every $x \in E$ and every open set V satisfying $G(x) \subset V$, there exists a neighborhood U of x such that $G(U) \subset V$.

Lemma 7 follows directly from Theorem 3.1 of [14].

Lemma 7. *Let K be a closed convex subset of X and $T : K \to 2^{L(X,Y)}$ scalar pseudomonotone and upper semicontinuous with nonempty compact convex values. If for any $\xi \in C^{*0}$, $S_\xi(T, K)$ is nonempty and compact, then $S^w(T, K)$ is nonempty and compact.*

Remark 8. It is pointed out in [14] that if $K_\infty \cap (T(K))^{w\circ} = \{0\}$, then $S_\xi(T, K)$ is nonempty and compact for any $\xi \in C^{*0}$, and so $S^w(T, K)$ is nonempty and compact.

From Theorem 2 of [15], we have the following lemma.

Lemma 9. *Let K be a closed convex subset of X and $T : K \to 2^{L(X,Y)}$ scalar pseudomonotone and upper semicontinuous with nonempty compact convex values. Then, for any $\xi \in C^{*0}$, $S_\xi(T, K)$ is nonempty and compact if and only if $K_\infty \cap [\xi(T(K))]^- = \{0\}$.*

Lemma 10 (see [16]). *Let E, F be two topological spaces. If the set-valued mapping $G : E \to 2^F$ is closed and F is compact, then G is upper semicontinuous.*

Lemma 11 (see [17]). *Let E, F be two topological spaces. Assume that E is connected and the set-valued mapping $G : E \to 2^F$ is upper semicontinuous. If for every $x \in E$, the set $G(x)$ is nonempty and connected, then the set $G(X)$ is connected.*

Lemma 11 follows immediately from the definition of path connectedness.

Lemma 12. *Let E, F be two topological spaces. Assume that E is path connected and the mapping $A : E \to F$ is continuous. Then, the set $A(E)$ is path connected.*

Lemma 13. *Let Y be a normed space with its dual space of Y^*. Let $\{y_\beta\}$ a net in Y and $\{y^*_\beta\}$ be a net in Y^*. Suppose that y_β converges to y in the norm topology of Y and y^*_β weak*-converges to y^*. Then, $\langle y^*_\beta, y_\beta\rangle \to \langle y^*, y\rangle$.*

Proof. Indeed, we have

$$\left\langle y^*_\beta, y_\beta \right\rangle - \left\langle y^*, y \right\rangle = \left\langle y^*_\beta, y_\beta - y \right\rangle + \left\langle y^*_\beta - y^*, y \right\rangle. \quad (15)$$

Since y^*_β weak*-converges y^*, it holds that $\langle y^*_\beta - y^*, y\rangle \to 0$. Moreover, by the triangle inequality, we have $|\langle y^*_\beta, y_\beta - y\rangle| \leq \|y^*_\beta\|\|y_\beta - y\|$. Note that $\{y^*_\beta\}$ is bounded and y_β converges to y in the norm topology of Y, this yields that $\langle y^*_\beta, y_\beta - y\rangle \to 0$. Consequently, we obtain that $\langle y^*_\beta, y_\beta\rangle \to \langle y^*, y\rangle$. This completes the proof. □

3. Connectedness of Solution Sets for *WVVI*

In this section, we establish the connectedness of solution set for the set-valued *WVVI*, when the mapping is scalar C-pseudomonotone. Furthermore, when the mapping is strictly scalar C-pseudomonotone, we establish the path-connectedness result for the set-valued *WVVI*.

Theorem 14. *Let K be a closed convex subset of X and $T : K \to 2^{L(X,Y)}$ scalar C-pseudomonotone and upper semicontinuous with nonempty compact convex values, where $L(X, Y)$ is equipped with the norm topology. Suppose that $K_\infty \cap [\xi(T(K))]^- = \{0\}$ for any $\xi \in C^{*0}$. Then, $S^w(T, K)$ is connected.*

Proof. From Lemma 9, we know that $S_\xi(T, K)$ is nonempty and compact for any $\xi \in C^{*0}$. Then, it follows from Lemma 7 that $S^w(T, K)$ is nonempty and compact.

Now, we show that $S^w(T, K)$ is connected. First, we claim that $S_\xi(T, K)$ is convex. Indeed, by the scalar C-pseudomonotonicity of T, it follows from Proposition 1 of [15] that

$$S_\xi(T, K) = \{x_0 \in K : \langle \xi(x^*), x - x_0\rangle \geq 0, \quad \forall x \in K,\ x^* \in T(x)\}. \quad (16)$$

Clearly, the set $\{x_0 \in K : \langle \xi(x^*), x - x_0\rangle \geq 0, \forall x \in K, x^* \in T(x)\}$ is convex, and so $S_\xi(T, K)$ is convex.

Setting $F = S^w(T, K)$, then F is compact. Define a set-valued mapping $G : C^{*0} \to 2^F$ as follows:

$$G(\xi) := S_\xi(T, K), \quad \forall \xi \in C^{*0}. \quad (17)$$

We now show that $G : C^{*0} \to 2^F$ is closed, where C^{*0} is equipped with the weak* topology. Take $x^\alpha \in G(\xi^\alpha)$ with $x^\alpha \to x^0$ and $\xi^\alpha \to \xi^0$. The fact $x^\alpha \in G(\xi^\alpha)$ implies that there exists $u^\alpha \in T(x^\alpha)$ such that

$$\langle \xi^\alpha(u^\alpha), x - x^\alpha\rangle \geq 0, \quad \forall x \in K. \quad (18)$$

Since $F = S^w(T, K)$ is compact and $T : K \to 2^{L(X,Y)}$ is upper semicontinuous, we have $T(F)$ that is compact. Moreover, since $x^\alpha \in F$ and $u^\alpha \in T(x^\alpha) \subset T(F)$, there exists a subnet $\{u^\beta\}$ of $\{u^\alpha\}$ such that $u^\beta \to u^0$. Clearly, $u^0 \in T(x^0)$. From (18), we have

$$\left\langle \xi^\beta \left(u^\beta\right), x - x^\beta \right\rangle \geq 0, 1 \quad \forall x \in K. \tag{19}$$

Setting $y_\beta = \langle u^\beta, x - x^\beta \rangle$ and $y_\beta^* = \xi^\beta$, then it follows from Lemma 13 that

$$\langle \xi^0 \left(u^0\right), x - x^0 \rangle \geq 0, \quad \forall x \in K. \tag{20}$$

This implies that $x^0 \in G(\xi^0)$ and so $G : C^{*0} \to 2^F$ is closed. Then, from Lemma 10, we know that G is upper semicontinuous. This together with Lemma 11 yields that $S^w(T, K)$ is connected. This completes the proof. □

Remark 15. (i) If $C = R_+^P$, $T = (f_1, f_2, \ldots, f_p)$ and $f_i = M_i x + q_i$, where $M_i \in R^{n \times n}$ and $q_i \in X$, $i = 1, 2, \ldots, p$, then $(WVVI)$ reduces to an affine vector variational inequality. Lee and Bu [11] obtained the corresponding result of Theorem 14 for affine vector variational inequalities, see Theorems 2.1 and 2.2 of [11].

(ii) If K is compact, $C = R_+^P$ and f_i, $i = 1, 2, \ldots, p$, are single-valued and continuous; Cheng [6] obtained the corresponding result of Theorem 17, see Theorem 1 of [6]. Similar results can be founded in Gong [7] and Gong and Yao [8] for the connectedness of solution sets for vector equilibrium problems. Compared with these results, we do not need to assume that K is compact but only a unbounded close convex set.

Example 16 is to clarify Theorem 14.

Example 16. Consider problem $(WVVI)$, where

$$K = R_+^1, \qquad C = R_+^2, \qquad T = \left(f_1, f_2\right),$$
$$f_1(x) = f_2(x) = \begin{cases} 0, & x \in [0, 1), \\ [0, 1], & x = 1, \\ x^2, & x > 1. \end{cases} \tag{21}$$

Clearly, f_i, $i = 1, 2$, are upper semicontinuous, and $T = (f_1, f_2)$ is scalar C-pseudomonotone on K, $K_\infty = R_+^1$. Then, all the assumptions of Theorem 14 are satisfied. By a simple computation, we obtain that $S^w(f, K) = [0, 1]$ and so $S^w(f, K)$ is connected.

When T is strictly scalar C-pseudomonotone, we further obtain the following path-connectedness result for set-valued $WVVI$.

Theorem 17. *Let K be a closed convex subset of X and $T : K \to 2^{L(X,Y)}$ strictly scalar C-pseudomonotone and upper semicontinuous with nonempty compact convex values, where $L(X, Y)$ is equipped with the norm topology. Suppose that $K_\infty \cap [\xi(T(K))]^- = \{0\}$ for any $\xi \in C^{*0}$. Then, $S^w(T, K)$ is path connected.*

Proof. The nonemptiness of $S^w(T, K)$ is obvious. We now claim that for every $\xi \in C^{*0}$, $S_\xi(f, K)$ is unique. Let $x^*, y^* \in K$ with $x^* \neq y^*$ be the solutions of $(VI)_\xi$. Since x^* is a solution of $(VI)_\xi$, then there exists $u^* \in T(x^*)$ such that

$$\left\langle \xi\left(u^*\right), x - x^* \right\rangle \geq 0, \quad \forall x \in K. \tag{22}$$

Taking $x = y^*$ in the above inequality, we have

$$\left\langle \xi\left(u^*\right), y^* - x^* \right\rangle \geq 0. \tag{23}$$

By the strictly scalar C-pseudomonotonicity of T, it follows that

$$\left\langle \xi(v), y^* - x^* \right\rangle > 0, \quad \forall v \in T\left(y^*\right), \tag{24}$$

and so

$$\left\langle \xi(v), x^* - y^* \right\rangle < 0. \tag{25}$$

This contradicts the fact that y^* is a solution of $(VI)_\xi$. Thus, for every $\xi \in C^{*0}$, $(VI)_\xi$ has a unique solution $x(\xi)$ in K.

Then, from the proof of Theorem 14, we know that $x(\cdot)$ is a single-valued and upper semicontinuous mapping and so continuous. Since C^{*0} is path connected, from Lemma 12, the solution set $S^w(T, K) = \cup_{\xi \in C^{*0}}\{x(\xi)\} = x(C^{*0})$ is also path connected. This completes the proof. □

Remark 18. Note that a strongly monotone mapping is strictly monotone and hence strictly pseudomonotone; Theorem 17 generalized Theorem 4.2 of [10] from single-valued mappings to set-valued mappings under a weaker monotonicity assumption.

Example 19 is to clarify Theorem 17.

Example 19. Consider problems $(WVVI)$ and (VVI), where

$$K = R_+^1, \qquad C = R_+^2,$$
$$T = \left(f_1, f_2\right) \quad \text{with } f_1(x) = x + 1, \qquad f_2(x) \equiv 1. \tag{26}$$

Clearly, f_i, $i = 1, 2$, are upper semicontinuous, and $T = (f_1, f_2)$ is strictly C-pseudomonotone on K, $K_\infty = R_+^1$. Then, all the assumptions of Theorem 17 are satisfied. By a simple computation, we obtain that $S^w(f, K) = \{0\}$ and so $S^w(f, K)$ is path connected.

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References

[1] F. Giannessi, "Theorems of alternative, quadratic programs and complementarity problems," in *Variational Inequalities and Complementarity Problems*, pp. 151–186, Wiley, Chichester, UK, 1980.

[2] G. Y. Chen, "Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia theorem," *Journal of Optimization Theory and Applications*, vol. 74, no. 3, pp. 445–456, 1992.

[3] G. Y. Chen and S. J. Li, "Existence of solutions for a generalized vector quasivariational inequality," *Journal of Optimization Theory and Applications*, vol. 90, no. 2, pp. 321–334, 1996.

[4] A. Daniilidis and N. Hadjisavvas, "Existence theorems for vector variational inequalities," *Bulletin of the Australian Mathematical Society*, vol. 54, no. 3, pp. 473–481, 1996.

[5] F. Giannessi, G. Mastronei, and L. Pellegrini, "On the theory of vector optimization and variational inequalities," in *Image Space Analysis and Separation, Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, F. F. Giannessi, Ed., Kluwer Academic, Dordrecht, The Netherlands, 1999.

[6] Y. Cheng, "On the connectedness of the solution set for the weak vector variational inequality," *Journal of Mathematical Analysis and Applications*, vol. 260, no. 1, pp. 1–5, 2001.

[7] X. H. Gong, "Connectedness of the solution sets and scalarization for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 133, no. 2, pp. 151–161, 2007.

[8] X. H. Gong and J. C. Yao, "Connectedness of the set of efficient solutions for generalized systems," *Journal of Optimization Theory and Applications*, vol. 138, no. 2, pp. 189–196, 2008.

[9] B. Chen, Q.-y. Liu, Z.-b. Liu, and N.-j. Huang, "Connectedness of approximate solutions set for vector equilibrium problems in Hausdorff topological vector spaces," *Fixed Point Theory and Applications*, vol. 2011, article 36, 2011.

[10] G. M. Lee, D. S. Kim, B. S. Lee, and N. D. Yen, "Vector variational inequality as a tool for studying vector optimization problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 34, no. 5, pp. 745–765, 1998.

[11] G. M. Lee and I. J. Bu, "On solution sets for a±ne vector variational inequality," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, pp. 1847–1855, 2005.

[12] G. Jameson, *Ordered Linear Spaces*, vol. 141 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1970.

[13] R. Hu and Y.-P. Fang, "On the nonemptiness and compactness of the solution sets for vector variational inequalities," *Optimization*, vol. 59, no. 7, pp. 1107–1116, 2010.

[14] X. X. Huang, Y. P. Fang, and X. Q. Yang, "Characterizing the nonemptiness and compactness of the solution set of a vector variational inequality by scalarization," *Journal of Optimization Theory and Applications*, 2012.

[15] J.-P. Crouzeix, "Pseudomonotone variational inequality problems: existence of solutions," *Mathematical Programming*, vol. 78, no. 3, pp. 305–314, 1997.

[16] J.-P. Aubin, *Mathematical Methods of Game and Economic Theory*, vol. 7 of *Studies in Mathematics and its Applications*, North-Holland, Amsterdam, The Netherlands, 1979.

[17] A. R. Warburton, "Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives," *Journal of Optimization Theory and Applications*, vol. 40, no. 4, pp. 537–557, 1983.