Research Article

The Positive Properties of Green's Function for Fractional Differential Equations and Its Applications

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We consider the properties of Green's function for the nonlinear fractional differential equation boundary value problem: $\mathbf{D}_{0+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, 0 < t < 1, u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \beta u(\eta)$, where $n - 1 < \alpha \leq n, n \geq 3, 0 < \beta \leq 1, 0 \leq \eta \leq 1, \mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. Here our nonlinearity f may be singular at $u = 0$. As applications of Green's function, we give some multiple positive solutions for singular boundary value problems by means of Schauder fixed-point theorem.

1. Introduction

Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in selfsimilar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. For details, see [1–10].

It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions. Recently, there are some papers dealing with the existence and multiplicity of solution to the nonlinear fractional differential equations boundary value problems, see [11–17].

Bai [14] investigated the existence and uniqueness of positive solutions for a nonlocal boundary value problem of fractional differential equation

$$\mathbf{D}_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \beta u(\eta), \tag{1}$$

by contraction map principle and fixed-point index theory, where $1 < \alpha \leq 2, 0 < \beta \eta^{\alpha-1} < 1, 0 < \eta < 1, \mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. The function f is continuous on $[0, 1] \times [0, \infty)$.

Li et al. [17] investigated the the existence and multiplicity results of positive solutions for the nonlinear differential equation of fractional order

$$\mathbf{D}_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad \mathbf{D}_{0+}^{\beta} u(1) = a\mathbf{D}_{0+}^{\beta} u(\xi), \tag{2}$$

by using some fixed-point theorems, where $1 < \alpha \leq 2, 0 < \beta \leq 1, 0 \leq a \leq 1, \xi \in (0, 1), a\xi^{\alpha-\beta-2} \leq 1 - \beta, 0 \leq \alpha - \beta - 1, \mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative.

Xu and Fei [18] considered the properties of Green's function for the nonlinear fractional differential equation boundary value problem

$$\mathbf{D}_{0+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad \mathbf{D}_{0+}^{\beta} u(1) = a\mathbf{D}_{0+}^{\beta} u(\xi), \tag{3}$$

where $1 < \alpha \leq 2, 0 < \beta \leq 1, 0 \leq a \leq 1, 0 < \xi < 1, \alpha - \beta - 1 \geq 0, \mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. Here the nonlinearity f may be singular at $u = 0$.

As applications of Green's function, they give some existence of positive solutions for singular boundary value problems by means of Schauder fixed-point theorem. Here they consider the case: $\gamma_* = 0$, $\gamma_* \geq 0$, $\gamma^* \leq 0$.

In this paper, we consider the singular boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta),$$

$$(4)$$

where $n - 1 < \alpha \leq n$, $n \geq 3$, $0 < \beta$, $\eta < 1$ is a real constant, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. We will deduce a property of Green's function. The result we establish in Section 2 can be stated as follow.

Theorem 1. *The Function $G(t, s)$ defined by (12) is continuous and satisfies*

$$\frac{M t^{\alpha-1} s (1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta \eta^{\alpha-1})} \leq G(t, s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1 - \beta \eta^{\alpha-1})}, \quad (5)$$

$$for \ t, s \in [0, 1],$$

where $0 < M = \min\{1 - \beta \eta^{\alpha-1}, \beta \eta^{\alpha-2}(1 - \eta), \beta \eta^{\alpha-1}\} < 1$.

In this paper, we give some existence of positive solutions for singular boundary value problems by means of Schauder fixed-point theorem for the case: $\gamma_* = 0$, $\gamma_* \geq 0$, $\gamma^* \leq 0$, $\gamma_* < 0 < \gamma^*$.

The paper is organized as follows. In Section 2, we state some known results and give a property of Green's function. In Section 3, using Schauder fixed-point theorem, the existence of positive solutions to singular problems are obtained.

2. Background Materials

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory.

Definition 2 (see [7]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds, \quad (6)$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 3 (see [7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \, ds, \quad (7)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

From the definition of Riemann-Liouville's derivative, we can obtain the statement.

Lemma 4 (see [7]). *Let $\alpha > 0$, if one assumes that $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0 \quad (8)$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in R$, $i = 1, 2, \ldots, N$, *as unique solutions, where N is the smallest integer greater than or equal to α.*

Lemma 5 (see [7]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad (9)$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, N is the smallest integer greater than or equal to α.

Lemma 6. *Given $h \in C(0,1)$ the problem*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \ n-1 < \alpha \leq n, \ n \geq 3,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,$$

$$u(1) = \beta u(\eta), \quad 0 < \beta, \ \eta < 1$$

$$(10)$$

is equivalent to

$$u(t) = \int_0^1 G(t, s) h(s) \, ds, \quad (11)$$

where

$$G(t, s) = \begin{cases} \begin{aligned} &([t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta - s)^{\alpha-1} \\ &\quad - (t-s)^{\alpha-1}(1 - \beta \eta^{\alpha-1})) \\ &\quad \times ((1 - \beta \eta^{\alpha-1}) \Gamma(\alpha))^{-1}, \\ &\quad 0 \leq s \leq t \leq 1, \ s \leq \eta, \\ &([t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1 - \beta \eta^{\alpha-1})) \\ &\quad \times ((1 - \beta \eta^{\alpha-1}) \Gamma(\alpha))^{-1}, \\ &\quad 0 < \eta \leq s \leq t \leq 1, \\ &([t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta - s)^{\alpha-1}) \\ &\quad \times ((1 - \beta \eta^{\alpha-1}) \Gamma(\alpha))^{-1}, \\ &\quad 0 \leq t \leq s \leq \eta < 1, \\ &([t(1-s)]^{\alpha-1}) \\ &\quad \times ((1 - \beta \eta^{\alpha-1}) \Gamma(\alpha))^{-1}, \\ &\quad 0 \leq t \leq s \leq 1, \ \eta \leq s. \end{aligned} \end{cases} \quad (12)$$

Proof. We can apply Lemma 5 to reduce (10) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, \quad (13)$$

for some $C_1, C_2, \ldots, C_n \in R$. Consequently, the general solution of (10) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \qquad (14)$$
$$+ C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, this is $C_2 = C_3 = \cdots = C_n = 0$.

On the other hand, $u(1) = \beta u(\eta)$ combining with

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds + C_1,$$
$$u(\eta) = -\frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)\, ds + C_1 \eta^{\alpha-1} \qquad (15)$$

yields

$$C_1 = \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds$$
$$- \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds. \qquad (16)$$

Therefor, the unique solution of problem (10) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds$$
$$+ \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \qquad (17)$$
$$- \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds.$$

For $t \leq \eta$, we have

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds$$
$$+ \frac{1}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^t + \int_t^\eta + \int_\eta^1 \right) \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right]$$
$$- \frac{\beta}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^t + \int_t^\eta \right) \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right]$$
$$= \int_0^t \left(\left([t(1-s)]^{\alpha-1} - \beta[t(\eta-s)]^{\alpha-1} \right. \right.$$
$$\left. - (t-s)^{\alpha-1} \left(1 - \beta\eta^{\alpha-1}\right) \right)$$
$$\left. \times \left(\left(1-\beta\eta^{\alpha-1}\right) \Gamma(\alpha)\right)^{-1} \right) h(s)\, ds$$

$$+ \int_t^\eta \frac{[t(1-s)]^{\alpha-1} - \beta[t(\eta-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds$$
$$+ \int_\eta^1 \frac{[t(1-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds$$
$$= \int_0^1 G(t,s) h(s)\, ds. \qquad (18)$$

For $t \geq \eta$, we have

$$u(t) = -\left[\left(\int_0^\eta + \int_\eta^t \right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right]$$
$$- \frac{\beta}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(t(\eta-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds$$
$$+ \frac{1}{1-\beta\eta^{\alpha-1}} \left[\left(\int_0^\eta + \int_\eta^t + \int_t^1 \right) \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right]$$
$$= \int_0^\eta \left(\left([t(1-s)]^{\alpha-1} - \beta[t(\eta-s)]^{\alpha-1} \right. \right.$$
$$\left. - (t-s)^{\alpha-1} \left(1 - \beta\eta^{\alpha-1}\right) \right)$$
$$\left. \times \left(\left(1-\beta\eta^{\alpha-1}\right) \Gamma(\alpha)\right)^{-1} \right) h(s)\, ds$$
$$+ \int_\eta^t \frac{[t(1-s)]^{\alpha-1} - \left(1-\beta\eta^{\alpha-1}\right)(t-s)^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds$$
$$+ \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)} h(s)\, ds$$
$$= \int_0^1 G(t,s) h(s)\, ds. \qquad (19)$$

The proof is complete. $\square$

Proof of Theorem 1. It is easy to prove that $G(t,s)$ is continuous on $[0,1] \times [0,1]$, here we omit it. In the following, we consider $(1 - \beta\eta^{\alpha-1})\Gamma(\alpha)G(t,s)$. When $0 \leq s < t \leq 1, s < \eta$, let

$$g_1(t) = [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}$$
$$- (t-s)^{\alpha-1} \left(1 - \beta\eta^{\alpha-1}\right). \qquad (20)$$

We have

$$g_1(t) = [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}$$
$$- (t-s)^{\alpha-1} \left(1 - \beta\eta^{\alpha-1}\right)$$
$$= [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1} t^{\alpha-1} \left(1 - \frac{s}{\eta}\right)^{\alpha-1}$$
$$- t^{\alpha-1} \left(1 - \frac{s}{t}\right)^{\alpha-1} \left(1 - \beta\eta^{\alpha-1}\right)$$

$$\geq [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}\left(1-\frac{s}{\eta}\right)^{\alpha-1}$$

$$- t^{\alpha-1}(1-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)$$

$$= t^{\alpha-1}\beta\eta^{\alpha-1}\left[(1-s)^{\alpha-1} - \left(1-\frac{s}{\eta}\right)^{\alpha-1}\right]$$

$$= t^{\alpha-1}\beta\eta^{\alpha-1}\left[(1-s)^{\alpha-1} - \left(1-\frac{s}{\eta}\right)^{\alpha-2}\left(1-\frac{s}{\eta}\right)\right]$$

$$\geq t^{\alpha-1}\beta\eta^{\alpha-1}(1-s)^{\alpha-2}\left[1-s-\left(1-\frac{s}{\eta}\right)\right]$$

$$= t^{\alpha-1}\beta\eta^{\alpha-1}(1-s)^{\alpha-2}s\frac{1-\eta}{\eta}$$

$$\geq \beta\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}s\frac{1-\eta}{\eta}$$

$$\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s. \tag{21}$$

When $0 < \eta \leq s < t \leq 1$, let

$$g_2(t) = [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right). \tag{22}$$

We have

$$g_2(t) = [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)$$

$$= [t(1-s)]^{\alpha-1} - t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)$$

$$\geq [t(1-s)]^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right) \tag{23}$$

$$= \beta\eta^{\alpha-1}(t-ts)^{\alpha-1}$$

$$\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s.$$

When $0 \leq t \leq s \leq \eta < 1$, let $g_3(t) = [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta - s)^{\alpha-1}$, we have

$$g_3(t) = [t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}$$

$$= [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}\left(1-\frac{s}{\eta}\right)^{\alpha-1}$$

$$\geq [t(1-s)]^{\alpha-1} - \beta\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1} \tag{24}$$

$$= \left(1-\beta\eta^{\alpha-1}\right)t^{\alpha-1}(1-s)^{\alpha-1}$$

$$\geq Mt^{\alpha-1}(1-s)^{\alpha-1}s.$$

When $0 \leq t \leq s \leq 1, \eta \leq s$, we have

$$\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)G(t,s) = [t(1-s)]^{\alpha-1} \geq Mt^{\alpha-1}(1-s)^{\alpha-1}s. \tag{25}$$

It is easy to see that $G(t,s) \leq t^{\alpha-1}(1-s)^{\alpha-1}/\Gamma(\alpha)(1-\beta\eta^{\alpha-1})$. Thus, the proof is complete. □

Let us fix some notations to be used in the following. "For a.e." means "for almost every". Given $a \in L^1(0,1)$, we write $a \succ 0$ if $a \geq 0$ for a.e. $t \in [0,1]$, and it is positive in a set of positive measure, we write $f \in \text{Car}((0,1) \times (0,+\infty), (0,+\infty))$ if $f : (0,1) \times (0,+\infty) \to (0,+\infty)$ is a L^1-caratheodory function, that is, the map $x \mapsto f(t,x)$ is continuous for a.e. $t \in (0,1)$, and the map $t \mapsto f(t,x)$ is measurable for all $x \in (0,+\infty)$.

Let us define

$$\gamma^* = \sup_{t\in[0,1]}\int_0^1 \frac{G(t,s)}{t^{\alpha-1}}e(s)\,ds,$$

$$\gamma_* = \inf_{t\in[0,1]}\int_0^1 \frac{G(t,s)}{t^{\alpha-1}}e(s)\,ds. \tag{26}$$

Then,

$$t^{\alpha-1}\gamma_* \leq \int_0^1 G(t,s)e(s)\,ds \leq t^{\alpha-1}\gamma^*. \tag{27}$$

3. Main Results

In this section, we establish the existence of positive solutions for equation

$$\mathbf{D}_{0+}^{\alpha}u(t) + f(t,u(t)) + e(t) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \beta u(\eta), \tag{28}$$

where $n-1 < \alpha \leq n, n \geq 3, 0 < \beta \leq 1, 0 \leq \eta \leq 1$, $f \in \text{Car}((0,1) \times (0,+\infty), (0,+\infty))$, $e(t) \in L^1[0,1]$, $\mathbf{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. The following is the first main result in this section.

Theorem 7. *Suppose that the following conditions are satisfied.*

(H_1) *For each $L > 0$, there exists a function $\phi_L \succ 0$ such that $f(t, t^{\alpha-1}x) \geq \phi_L(t)$ for a.e. $t \in (0,1)$, all $x \in (0,L]$.*

(H_2) *There exist $g(x), h(x),$ and $k(t) \succ 0$, such that*

$$0 \leq f(t,x) \leq k(t)\{g(x) + h(x)\}$$

$$for\ a.e.\ t \in (0,1),\ all\ x \in (0,\infty), \tag{29}$$

here

$g : (0,+\infty)$

$\longrightarrow [0,+\infty)$ *is continuous and nonincreasing,*

$h : [0,+\infty)$

$\longrightarrow [0,+\infty)$ *is continuous, and $\frac{h}{g}$ is nondecreasing.* \tag{30}

(H$_3$) *There exist two positive constants $R > r > 0$ such that*

$$R > \Phi_{R1} + \gamma_* \geq r > 0,$$

$$\int_0^1 k(s) g\left(rs^{\alpha-1}\right) ds < +\infty,$$

$$R \geq \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s) g\left(rs^{\alpha-1}\right) ds + \gamma^*,$$

$$(31)$$

and here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s) ds. \qquad (32)$$

Then, (28) has at least one positive solution.

Proof. Let $E = (C[0,1], \|\cdot\|)$, and Ω is a closed convex set defined as

$$\Omega = \left\{x \in C[0,1] : t^{\alpha-1}r \leq x(t) \leq t^{\alpha-1}R \ \forall t \in [0,1]\right\}, \quad (33)$$

here $E = C[0,1]$ is the Banach space of continuous functions defined on $[0,1]$ with the norm

$$\|x\| := \max_{t \in [0,1]} |x(t)|, \qquad (34)$$

and $R > r > 0$ are positive constants to be given below.

Now, we define an operator $T : \Omega \to E$ by

$$(Tx)(t) := \int_0^1 G(t,s) \left[f(s, x(s)) + e(s)\right] ds. \qquad (35)$$

Then, (28) is equivalent to the fixed-point problem

$$x = Tx. \qquad (36)$$

Let R be the positive constant satisfying (H$_3$) and

$$\Phi_{R1} + \gamma_* \geq r. \qquad (37)$$

Then, we have $R > r > 0$. Now, we prove $T(\Omega) \subset \Omega$.

In fact, for each $x \in \Omega$ and for all $t \in (0,1)$, by (H$_1$) and (H$_3$)

$$(Tx)(t) \geq \int_0^1 t^{\alpha-1} \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s) ds$$

$$+ \int_0^1 G(t,s) e(s) ds \qquad (38)$$

$$\geq t^{\alpha-1} [\Phi_{R1} + \gamma_*] \geq t^{\alpha-1}r.$$

On the other hand, by conditions (H$_2$) and (H$_3$), we have

$$(Tx)(t)$$

$$\leq \int_0^1 t^{\alpha-1} \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1-a\xi^{\alpha-\beta-1}\right)} k(s)$$

$$\times \left[g\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right) + h\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right)\right] ds$$

$$+ \int_0^1 G(t,s) e(s) ds$$

$$\leq t^{\alpha-1} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s) g\left(s^{\alpha-1}\frac{x(s)}{s^{\alpha-1}}\right)\right.$$

$$\left. \times \left\{1 + \frac{h\left(s^{\alpha-1}\left(x(s)/s^{\alpha-1}\right)\right)}{g\left(s^{\alpha-1}\left(x(s)/s^{\alpha-1}\right)\right)}\right\} ds + \gamma^*\right]$$

$$\leq t^{\alpha-1}$$

$$\times \left[\left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s) g\left(rs^{\alpha-1}\right) ds + \gamma^*\right]$$

$$\leq t^{\alpha-1}R.$$

$$(39)$$

In conclusion, $T(\Omega) \subset \Omega$.

Finally, it is standard that $T : \Omega \to \Omega$ is a continuous and completely continuous operator. By a direct application of Schauder's fixed-point theorem, (28) has at least one positive solution $x(t) \in C[0,1]$, the proof is finished. $\square$

Case 1 ($\gamma_* = 0$). As an application of Theorem 7, we consider the case $\gamma_* = 0$. The following corollary is a direct result of Theorem 7 with $r = \Phi_{R1}$.

Corollary 8. *Suppose that $f(t,x)$ satisfies conditions (H$_1$)-(H$_2$). Furthermore, assume the following.*

(H$_3^*$) *There exists a positive constant $R > 0$ such that*

$$R > \Phi_{R1} > 0,$$

$$\int_0^1 k(s) g\left[\left(\Phi_{R1}\right) s^{\alpha-1}\right] ds < +\infty,$$

$$R \geq \left(1 + \frac{h(R)}{g(R)}\right)$$

$$\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} k(s) g\left[\left(\Phi_{R1}\right) s^{\alpha-1}\right] ds + \gamma^*,$$

$$(40)$$

and here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \phi_R(s) ds. \qquad (41)$$

Then, (28) has at least one positive solution.

From now on, let us define

$$\beta_1 = \int_0^1 \frac{M s^{1-\lambda(\alpha-1)}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s)\, ds,$$

$$\beta_2 = \int_0^1 \frac{s^{-\lambda(\alpha-1)}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s)\, ds. \tag{42}$$

Example 9. Suppose that the nonlinearity in (28) is

$$f(t,x) = \frac{k(t)}{x^\lambda}, \tag{43}$$

where $k > 0$, $0 < \lambda < 1$ and

$$\omega(\lambda) := \int_0^1 k(s) s^{-\lambda(\alpha-1)} ds < +\infty. \tag{44}$$

If $\gamma_* = 0$, then (28) has at least one positive solution.

Proof. We will apply Corollary 8. To this end, we take

$$\phi_L(t) = \frac{k(t)}{(t^{\alpha-1}L)^\lambda}, \qquad g(x) = \frac{1}{x^\lambda}, \qquad h(x) \equiv 0, \tag{45}$$

then (H_1) and (H_2) are satisfied since $\omega(\lambda) < +\infty$, and the existence condition (H_3^*) becomes

$$\left(\frac{R^\lambda}{\beta_1}\right)^\lambda \beta_2 + \gamma^* \le R, \quad R > \frac{\beta_1}{R^\lambda}, \ \omega(\lambda) < +\infty, \tag{46}$$

for some $R > 0$. Since $0 < \lambda < 1$, we can choose $R > 0$ large enough such that (46) is satisfied, and the proof is finished. $\qquad\square$

Example 10. Suppose that the nonlinearity in (28) is

$$f(t,x) = k(s)\left(x^{-\lambda} + \mu x^\nu\right), \tag{47}$$

where $0 < \lambda < 1$, $\nu \ge 0$ and $\mu \ge 0$ is a nonnegative parameter. For each $e(t)$ with $\gamma_* = 0$, $\omega(\lambda) < +\infty$,

(i) if $\lambda + \nu < 1 - \lambda^2$, then (28) has at least one positive solution for each $\mu \ge 0$.

(ii) If $\lambda + \nu \ge 1 - \lambda^2$, then (28) has at least one positive solution for each $0 \le \mu < \mu_1$, where μ_1 is some positive constant.

Proof. We will apply Corollary 8. To this end, we take

$$\phi_L(t) = \frac{k(t)}{(t^{\alpha-1}L)^\lambda}, \qquad g(x) = x^{-\lambda}, \qquad h(x) = \mu x^\nu. \tag{48}$$

Then, (H_1)-(H_2) are satisfied since $\omega(\lambda) < +\infty$. Now, the existence condition (H_3^*) becomes $\omega(\lambda) < +\infty$, and

$$\mu \le \frac{R^{1-\lambda^2}\beta_1^\lambda - \gamma^*\beta_1^\lambda R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{49}$$

for some $R > 0$ with $R^{1+\lambda} > \beta_1$. So, (28) has at least one positive solution for

$$0 < \mu < \mu_1 = \sup_{R > \beta_1^{1/1+\lambda}} \frac{R^{1-\lambda^2}\beta_1^\lambda - \gamma^*\beta_1^\lambda R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{50}$$

Note that $\mu_1 = \infty$ if $\lambda + \nu < 1 - \lambda^2$ and if $\lambda + \nu \ge 1 - \lambda^2$, set

$$l(R) := \frac{R^{1-\lambda^2}\beta_1^\lambda - \gamma^*\beta_1^\lambda R^{-\lambda^2} - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{51}$$

then, we have

$$l'(R) = \frac{1}{R^{\lambda+\nu+1}\beta_2}\left[\left(1 - \lambda^2 - \lambda - \nu\right) R^{1-\lambda^2}\beta_1^\lambda \right.$$
$$\left. + \left(\lambda^2 + \lambda + \nu\right)\gamma^*\beta_1^\lambda R^{-\lambda^2} + (\lambda + \nu)\beta_2\right]. \tag{52}$$

Let the function $l(R)$ possess a maximum at R_0, then

$$\left(\lambda^2 + \lambda + \nu - 1\right) R_0^{1-\lambda^2}\beta_1^\lambda$$
$$= \left(\lambda^2 + \lambda + \nu\lambda\right)\gamma^*\beta_1^\lambda R_0^{-\lambda^2} + (\lambda + \nu)\beta_2, \tag{53}$$

so we have

$$\left(\lambda + \nu - 1 + \lambda^2\right)\beta_1^\lambda R_0 \ge (\lambda + \nu)\beta_2 R_0^{\lambda^2}, \tag{54}$$

it is easy to find that $R_0 > (\beta_2/\beta_1^\lambda)^{1/(1-\lambda^2)}$ since $\lambda + \nu \ge 1 - \lambda^2$, and $0 < \lambda < 1$. Finally, it would remain to prove $R_0 > \beta_1^{1/1+\lambda}$. This is easily verified through elementary computations since $\beta_1 \le \beta_2$. We have the desired results (i) and (ii). $\qquad\square$

Case 2 ($\gamma_* > 0$). The next result explores the case when $\gamma_* > 0$. In this case $r = \gamma_*$.

Corollary 11. *Suppose that $f(t,x)$ satisfies (H_2). Furthermore, assume the following.*

(H_4) *There exists $R > 0$, $\int_0^1 k(s)g(\gamma_* s^{\alpha-1}) ds < +\infty$, such that*

$$\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} k(s) g\left(\gamma_* s^{\alpha-1}\right) ds + \gamma^* \le R. \tag{55}$$

If $\gamma_* > 0$, then (28) has at least one positive solution.

Example 12. Suppose that the nonlinearity in (28) be (43) with $k > 0$, $\lambda > 0$. If $\gamma_* > 0$, $\omega(\lambda) < +\infty$, then (28) has at least one positive solution.

Proof. We will apply Corollary 11. Take $k(t)$, $g(x)$, and $h(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_4) is satisfied if we take $R > 0$ with $R \ge \beta_2/\gamma_*^\lambda + \gamma^*$, and $\omega(\lambda) < +\infty$. $\qquad\square$

Example 13. Let the nonlinearity in (28) be (47) with $\lambda > 0$ and $\nu \ge 0$. For each $e(t)$ with $\gamma_* > 0$, $\omega(\lambda) < +\infty$,

(i) if $\lambda + \nu < 1$, then (28) has at least one positive solution for each $\mu \geq 0$.

(ii) If $\lambda + \nu \geq 1$, then (28) has at least one positive solution for each $0 \leq \mu < \mu_2$, where μ_2 is some positive constant.

Proof. We will apply Corollary 11. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then (H_2) is satisfied, and the existence condition (H_4) becomes $\omega(\lambda) < +\infty$,

$$\mu \leq \frac{R\gamma_*^\lambda - \gamma^* \gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}, \tag{56}$$

for some $R > 0$. So, (28) has at least one positive solution for

$$0 < \mu < \mu_2 = \sup_{R>0} \frac{R\gamma_*^\lambda - \gamma^* \gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{57}$$

Note that $\mu_2 = \infty$ if $\lambda + \nu < 1$ and $\mu_2 = \gamma_*^\lambda/\beta_2$ if $\lambda + \nu = 1$, and if $\lambda + \nu > 1$ set

$$l(R) := \frac{R\gamma_*^\lambda - \gamma^* \gamma_*^\lambda - \beta_2}{\beta_2 R^{\lambda+\nu}}. \tag{58}$$

The function $l(R)$ possesses a maximum at

$$R_0 := \frac{(\lambda + \nu)\left(\gamma^* \gamma_*^\lambda + \beta_2\right)}{(\lambda + \nu - 1)\gamma_*^\lambda}, \tag{59}$$

then $\mu_2 = l(R_0)$. We have the desired results (i) and (ii). $\square$

Case 3 ($\gamma^* \leq 0$). The next result considers the case $\gamma^* \leq 0$.

Corollary 14. *Suppose that $f(t, x)$ satisfies (H_1)-(H_2). Furthermore, assume the following.*

(H_5) *There exist two positive constants $R > r > 0$ such that*

$$R > \Phi_{R1} + \gamma_* \geq r > 0,$$

$$\int_0^1 k(s) g\left(rs^{\alpha-1}\right) ds < +\infty,$$

$$R \geq \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1 - \beta\eta^{\alpha-1}\right)} k(s) g\left(rs^{\alpha-1}\right) ds, \tag{60}$$

here

$$\Phi_{R1} = \int_0^1 \frac{Ms(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1 - \beta\eta^{\alpha-1}\right)} \phi_R(s) ds. \tag{61}$$

Then, (28) has at least one positive solution.

Example 15. Suppose that the nonlinearity in (28) be (43) with $k > 0$, $\lambda > 0$. If $\gamma^* \leq 0$, $\omega(\lambda) < +\infty$,

$$\gamma_* \geq \left[\frac{\beta_1 \lambda^2}{\beta_2^\lambda}\right]^{1/(1-\lambda^2)} \left(1 - \frac{1}{\lambda^2}\right), \tag{62}$$

then (28) has at least one positive solution.

Proof. We will apply Corollary 14. Take $k(t)$, $g(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_5) is satisfied if we take $R > r > 0$ with

$$\frac{\beta_1}{R^\lambda} + \gamma_* \geq r, \quad R \geq \frac{\beta_2}{r^\lambda}, \tag{63}$$

and $\omega(\lambda) < +\infty$. If we fix $R = \beta_2/r^\lambda$, then the first inequality holds if r satisfies

$$\frac{\beta_1}{\beta_2^\lambda} r^{\lambda^2} + \gamma_* \geq r, \tag{64}$$

or equivalently

$$\gamma_* \geq l(r) := r - \frac{\beta_1}{\beta_2^\lambda} r^{\lambda^2}. \tag{65}$$

The function $l(r)$ possesses a minimum at

$$r_0 := \left[\frac{\beta_1}{\beta_2^\lambda} \lambda^2\right]^{1/(1-\lambda^2)}. \tag{66}$$

Taking $r = r_0$, then the first inequality in (63) holds if $\gamma_* \geq l(r_0)$, which is just condition (62). The second inequality holds directly from the choice of R, so it remains to prove that $R = \beta_2/r^\lambda > r_0$. This is easily verified through elementary computations. $\square$

Example 16. Let the nonlinearity in (28) be (47) with $k > 0$, $\lambda > 0$ and $\nu \geq 0$. If $\gamma^* \leq 0$, $\omega(\lambda) < +\infty$,

$$\gamma_* \geq m_0^\lambda \left[\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} - \beta_1\right], \tag{67}$$

here m_0 is the unique solution of the equation

$$\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}$$
$$\times \left[m^{1-\lambda^2} + \mu(1-\lambda-\nu)m^{1-\lambda-\nu-\lambda^2}\right] = \lambda^2 \beta_1. \tag{68}$$

Then (28) has at least one positive solution.

Proof. We will apply Corollary 14. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then (H_2) is satisfied, and the existence condition (H_5) is satisfied if we take $R > r > 0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \quad \left(1 + \mu R^{\lambda+\nu}\right)\beta_2 r^{-\lambda} \leq R, \tag{69}$$

and $\omega(\lambda) < +\infty$. If we fix $R = (1 + \mu R^{\lambda+\nu})\beta_2 r^{-\lambda}$, then the first inequality holds if R satisfies

$$\gamma_* \geq \left[\frac{\left(1 + \mu R^{\lambda+\nu}\right)\beta_2}{R}\right]^{1/\lambda} - \beta_1 R^{-\lambda}. \tag{70}$$

Let $m = 1/R$, then

$$\gamma_* \geq m^\lambda \left[\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} - \beta_1\right] =: F(m). \tag{71}$$

Then, we have

$$
\begin{aligned}
F'(m) \\
&= \lambda m^{\lambda-1}\left[\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda}-\beta_1\right] \\
&\quad + m^\lambda\left[\frac{1}{\lambda}\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda-1}\right. \\
&\qquad\qquad \times\left(\left(1-\lambda^2\right)\beta_2 m^{-\lambda^2}\right. \\
&\qquad\qquad\quad \left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)\beta_2 m^{-\lambda-\nu-\lambda^2}\right)\right] \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\lambda^2\left[\left(\beta_2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right)^{1/\lambda}-\beta_1\right]\right. \\
&\qquad + \beta_2^{1/\lambda}\left[\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\qquad\qquad \times\left(\left(1-\lambda^2\right)m^{1-\lambda^2}\right. \\
&\qquad\qquad\quad \left.\left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)m^{1-\lambda-\nu-\lambda^2}\right)\right]\right\} \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\qquad \times\left[\lambda^2\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)\right. \\
&\qquad\quad +\left(\left(1-\lambda^2\right)m^{1-\lambda^2}\right. \\
&\qquad\qquad \left.\left.\left.+\mu\left(1-\lambda-\nu-\lambda^2\right)m^{1-\lambda-\nu-\lambda^2}\right)\right]-\lambda^2\beta_1\right\} \\
&= \frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\qquad \left. \times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right]-\lambda^2\beta_1\right\}.
\end{aligned}
\tag{72}
$$

Let $F'(m)=0$, then we have

$$
\begin{aligned}
\frac{1}{\lambda}m^{\lambda-1}&\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right. \\
&\left. \times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right]-\lambda^2\beta_1\right\}=0.
\end{aligned}
\tag{73}
$$

Now, let us define $\Phi(m)$ by

$$
\begin{aligned}
\Phi(m) :=&\ \beta_2^{1/\lambda}\left(m^{1-\lambda^2}+\mu m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\times\left[m^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m^{1-\lambda-\nu-\lambda^2}\right].
\end{aligned}
\tag{74}
$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m \in [0,+\infty)$ and $\Phi(m) \to +\infty$, as $m \to +\infty$. Thus, $\Phi(m) = \lambda^2\beta_1$ has a unique solution m_0 such that

$$
\begin{aligned}
\beta_2^{1/\lambda}&\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\times\left[m_0^{1-\lambda^2}+\mu\left(1-\lambda-\nu\right)m_0^{1-\lambda-\nu-\lambda^2}\right]=\lambda^2\beta_1,
\end{aligned}
\tag{75}
$$

and $F(m_0)=\inf_{m>0}F(m)$.

So, it remains to prove that $R > r = [(1+\mu R^{\lambda+\nu})\beta_2/R]^{1/\lambda}$, that is,

$$
m_0^{\lambda+1}\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}<1.
\tag{76}
$$

In fact, by (75), we have

$$
\lambda^2\beta_1 \geq \left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda-1}\left(\beta_2 m_0^{1-\lambda^2}\right)=\left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda},
\tag{77}
$$

that is,

$$
m_0 \leq \left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}.
\tag{78}
$$

Also we have

$$
\begin{aligned}
\lambda^2\beta_1 \geq &\ \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1} \\
&\times\left[(1-\lambda-\nu)m_0^{1-\lambda^2}+\mu(1-\lambda-\nu)m_0^{1-\lambda-\nu-\lambda^2}\right] \\
= &\ (1-\lambda-\nu)\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda},
\end{aligned}
\tag{79}
$$

that is,

$$
\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \leq \frac{\lambda^2\beta_1}{1-\lambda-\nu}.
\tag{80}
$$

Thus, we have

$$
\begin{aligned}
m_0^{\lambda+1}&\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2}+\mu m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \\
&< \left(\left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}\right)^{\lambda+1}\frac{\lambda^2\beta_1}{1-\lambda-\nu} \\
&= \left(\frac{\lambda^2\beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda)}\frac{\lambda^2\beta_1}{1-\lambda-\nu} \\
&= \lambda^{2\lambda/(1-\lambda)}\left(\frac{\beta_1}{\beta_2}\right)^{1/(1-\lambda)}\frac{\lambda^2}{1-\lambda-\nu}<1
\end{aligned}
\tag{81}
$$

since $0<\lambda,\nu<1$, and $1-\lambda-\nu-\lambda^2>0$.

We have the desired results. $\qquad\square$

Case 4 ($\gamma_* < 0 < \gamma^*$).

Example 17. Suppose that the nonlinearity in (28) be (43) with $k > 0, \lambda > 0$. If $\gamma_* < 0 < \gamma^*, \omega(\lambda) < +\infty$,

$$\gamma_* \geq m_0^\lambda \left[\beta_2^{1/\lambda} m_0^{(1-\lambda^2)/\lambda} - \beta_1 (1 + \gamma^* m_0)^{-\lambda} \right], \qquad (82)$$

and here m_0 is the unique solution of the equation

$$\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} \left(1 + \gamma^* m \right)^{\lambda+1} = \lambda^2 \beta_1, \qquad (83)$$

then (28) has at least one positive solution.

Proof. We will apply Theorem 7. Take $k(t), g(x)$ as the same in the proof of Example 9. Then, (H_2) is satisfied, and the existence condition (H_3) is satisfied if we take $R > r > 0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \qquad \beta_2 r^{-\lambda} + \gamma^* \leq R, \qquad (84)$$

and $\omega(\lambda) < +\infty$. If we fix $R = \beta_2/r^\lambda + \gamma^*$, then the first inequality holds if R satisfies

$$\gamma_* \geq \left(\frac{\beta_2}{R - \gamma^*} \right)^{1/\lambda} - \beta_1 R^{-\lambda}. \qquad (85)$$

Taking $m = 1/(R - \gamma^*)$, then for $m \in (0, +\infty)$, we have

$$\gamma_* \geq (\beta_2 m)^{1/\lambda} - \frac{\beta_1 m^\lambda}{(1 + \gamma^* m)^\lambda}$$

$$= m^\lambda \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1 (1 + \gamma^* m)^{-\lambda} \right] =: F(m). \qquad (86)$$

Then, we have

$$F'(m)$$

$$= \lambda m^{\lambda-1} \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1 (1 + \gamma^* m)^{-\lambda} \right]$$

$$+ m^\lambda \left[\frac{1-\lambda^2}{\lambda} \beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda-1} + \lambda \beta_1 \gamma^* (1 + \gamma^* m)^{-\lambda-1} \right]$$

$$= \frac{1}{\lambda} m^{\lambda-1} \left\{ \lambda^2 \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \beta_1 (1 + \gamma^* m)^{-\lambda} \right] \right.$$

$$+ \lambda m \left[\frac{1-\lambda^2}{\lambda} \beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda-1} \right.$$

$$\left. \left. + \lambda \beta_1 \gamma^* (1 + \gamma^* m)^{-\lambda-1} \right] \right\}$$

$$= \frac{1}{\lambda} m^{\lambda-1} \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} - \lambda^2 \beta_1 (1 + \gamma^* m)^{-\lambda-1} \right]$$

$$= \frac{1}{\lambda} m^{\lambda-1} (1+\gamma^* m)^{-\lambda-1} \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} (1+\gamma^* m)^{\lambda+1} - \lambda^2 \beta_1 \right]. \qquad (87)$$

Let $F'(m) = 0$, then we have

$$\frac{1}{\lambda} m^{\lambda-1} (1+\gamma^* m)^{-\lambda-1} \left[\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} (1 + \gamma^* m)^{\lambda+1} - \lambda^2 \beta_1 \right] = 0. \qquad (88)$$

Now, let us define $\Phi(m)$ by

$$\Phi(m) =: \beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} (1 + \gamma^* m)^{\lambda+1}. \qquad (89)$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m \in [0, +\infty)$, and $\Phi(m) \to +\infty$, as $m \to +\infty$. Thus, $\Phi(m) = \lambda^2 \beta_1$ has a unique solution m_0 such that

$$\beta_2^{1/\lambda} m^{(1-\lambda^2)/\lambda} (1 + \gamma^* m)^{\lambda+1} = \lambda^2 \beta_1, \qquad (90)$$

and $F(m_0) = \inf_{m>0} F(m)$.

So, it remains to prove that $R > r = [\beta_2/(R - \gamma^*)]^{1/\lambda}$, that is,

$$\beta_2^{1/\lambda} m_0^{1+1/\lambda} \leq 1 + \gamma^* m_0. \qquad (91)$$

In fact, by (90), we have

$$\beta_2^{1/\lambda} m_0^{(1-\lambda^2)/\lambda} \leq \lambda^2 \beta_1, \qquad (92)$$

that is,

$$m_0 \leq \left(\frac{\lambda^2 \beta_1}{\beta_2^{1/\lambda}} \right)^{\lambda/(1-\lambda^2)}. \qquad (93)$$

Thus, we have

$$\beta_2^{1/\lambda} m_0^{1+1/\lambda} < \beta_2^{1/\lambda} \left(\left(\frac{\lambda^2 \beta_1}{\beta_2^{1/\lambda}} \right)^{\lambda/(1-\lambda^2)} \right)^{(1+\lambda)/\lambda} m_0^{1+1/\lambda}$$

$$= \lambda^{2/(1-\lambda)} \left(\frac{\beta_1}{\beta_2} \right)^{1/(1-\lambda)} < 1 < 1 + \gamma^* m_0. \qquad (94)$$

The proof is complete. □

Example 18. Let the nonlinearity in (28) be (47) with $k > 0, \lambda > 0$ and $\nu \geq 0$. If $\gamma_* < 0 < \gamma^*, \omega(\lambda) < +\infty$,

$$\gamma_* \geq m_0^\lambda \left[\beta_2^{1/\lambda} \left(m_0^{1-\lambda^2} + \mu (1 + \gamma^* m_0)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2} \right)^{1/\lambda} \right.$$

$$\left. - \beta_1 (1 + \gamma^* m_0)^{-\lambda} \right], \qquad (95)$$

here m_0 is the unique solution of the equation

$$\beta_2^{1/\lambda} \left(m^{1-\lambda^2} + \mu (1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2} \right)^{1/\lambda-1}$$

$$\times \left[m^{1-\lambda^2} (1 + \gamma^* m)^{\lambda+1} + \mu (1 - \lambda - \nu + \gamma^* m) m^{1-\lambda-\nu-\lambda^2} \right.$$

$$\left. \times (1 + \gamma^* m)^{2\lambda+\nu} \right] = \lambda^2 \beta_1. \qquad (96)$$

Then (28) has at least one positive solution.

Proof. We will apply Theorem 7. To this end, we take $g(x)$, $h(x)$, and $k(t)$ as the same in the proof of Example 10, then

(H_2) is satisfied, and the existence condition (H_3) is satisfied if we take $R > r > 0$ with

$$\gamma_* \geq r - \beta_1 R^{-\lambda}, \qquad \left(1 + \mu R^{\lambda+\nu}\right)\beta_2 r^{-\lambda} + \gamma^* \leq R, \quad (97)$$

and $\omega(\lambda) < +\infty$. If we fix $R = (1 + \mu R^{\lambda+\nu})\beta_2 r^{-\lambda} + \gamma^*$, then the first inequality holds if

$$\gamma_* \geq \left[\frac{\left(1 + \mu R^{\lambda+\nu}\right)\beta_2}{R - \gamma^*}\right]^{1/\lambda} - \beta_1 R^{-\lambda}. \quad (98)$$

Let $m = 1/(R - \gamma^*)$, then

$$\gamma_* \geq m^{\lambda}\left[\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\right.$$
$$\left. - \beta_1(1 + \gamma^*)^{-\lambda}\right] =: F(m). \quad (99)$$

Then, we have

$$F'(m)$$
$$= \lambda m^{\lambda-1}\left[\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\right.$$
$$\left. - \beta_1(1 + \gamma^* m)^{-\lambda}\right]$$
$$+ m^{\lambda}\left[\frac{1}{\lambda}\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right.$$
$$\times\left(\left(1-\lambda^2\right)m^{-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu}\right.$$
$$\times\left(1 - \lambda - \nu - \lambda^2\right)m^{-\lambda-\nu-\lambda^2}$$
$$\left.+ \mu(\lambda + \nu)(1 + \gamma^* m)^{\lambda+\nu-1} m^{1-\lambda-\nu-\lambda^2}\gamma^*\right)$$
$$\left.+ \beta_1\lambda(1 + \gamma^* m)^{-\lambda-1}\gamma^*\right]$$
$$= \frac{1}{\lambda}m^{\lambda-1}\left\{\lambda^2\beta_2^{1/\lambda}\left[\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}\right.\right.$$
$$\left.- \lambda^2\beta_1(1 + \gamma^* m)^{-\lambda}\right.$$
$$\left.+ \beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu}\right.\right.$$
$$\left.\left.\times m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right.$$
$$\times\left[\left(1-\lambda^2\right)m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu}\right.$$
$$\times\left(1 - \lambda - \nu - \lambda^2\right)m^{1-\lambda-\nu-\lambda^2}$$
$$+ \mu(\lambda + \nu)(1 + \gamma^* m)^{\lambda+\nu-1}$$

$$\times m^{1-\lambda-\nu-\lambda^2}m\gamma^*\right]\bigg]$$
$$+ \lambda^2\beta_1(1 + \gamma^* m)^{-\lambda-1}m\gamma^*\bigg\}$$
$$= \frac{1}{\lambda}m^{\lambda-1}\left\{\beta_2^{1/\lambda}\left[\left(m^{1-\lambda^2} + \mu(1+\gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)\right]^{1/\lambda-1}\right.$$
$$\times\left[\lambda^2\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)\right.$$
$$+ \left(1 - \lambda^2\right)m^{1-\lambda^2}$$
$$+ \mu(1 + \gamma^* m)^{\lambda+\nu}\left(1 - \lambda - \nu - \lambda^2\right)m^{1-\lambda-\nu-\lambda^2}$$
$$\left.+ \mu(\lambda + \nu)(1 + \gamma^* m)^{\lambda+\nu-1} m^{1-\lambda-\nu-\lambda^2}m\gamma^*\right]$$
$$\times \lambda^2\beta_1(1+\gamma^* m)^{-\lambda-1}m\gamma^* - \lambda^2\beta_1(1+\gamma^* m)^{-\lambda}\bigg\}$$
$$= \frac{1}{\lambda}m^{\lambda-1}(1 + \gamma^* m)^{-\lambda-1}$$
$$\times\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right.$$
$$\times\left[m^{1-\lambda^2}(1 + \gamma^* m)^{\lambda+1} + \mu(1 - \lambda - \nu + \gamma^* m)\right.$$
$$\left.\times m^{1-\lambda-\nu-\lambda^2}(1 + \gamma^* m)^{2\lambda+\nu}\right] - \lambda^2\beta_1\bigg\}. \quad (100)$$

Let $F'(m) = 0$, then we have

$$\frac{1}{\lambda}m^{\lambda-1}(1 + \gamma^* m)^{-\lambda-1}$$
$$\times\left\{\beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}\right.$$
$$\times\left[m^{1-\lambda^2}(1 + \gamma^* m)^{\lambda+1} + \mu(1 - \lambda - \nu + \gamma^* m) m^{1-\lambda-\nu-\lambda^2}\right.$$
$$\left.\times(1 + \gamma^* m)^{2\nu+\nu}\right] - \lambda^2\beta_1\bigg\} = 0. \quad (101)$$

Now, let us define $\Phi(m)$ by

$$\Phi(m)$$
$$=: \beta_2^{1/\lambda}\left(m^{1-\lambda^2} + \mu(1 + \gamma^* m)^{\lambda+\nu} m^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}$$
$$\times\left[m^{1-\lambda^2}(1 + m\gamma^* m)^{\lambda+1}\right.$$
$$\left.+ \mu(1 - \lambda - \nu + \gamma^* m) m^{1-\lambda-\nu-\lambda^2}(1 + \gamma^* m)^{2\lambda+\nu}\right]. \quad (102)$$

It is easy to see that $\Phi(m)$ is a nondecreasing function for $m \in [0, +\infty)$, and $\Phi(m) \to +\infty$, as $m \to +\infty$. Thus, $\Phi(m) = \lambda^2 \beta_1$ has a unique solution m_0 such that

$$
\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}
$$
$$
\times \left[m_0^{1-\lambda^2}\left(1 + \gamma^* m\right)^{\lambda+1}\right.
$$
$$
\left. + \mu\left(1 - \lambda - \nu + \gamma^* m\right) m_0^{1-\lambda-\nu-\lambda^2}\left(1 + \gamma^* m_0\right)^{2\lambda+\nu}\right]
$$
$$
= \lambda^2 \beta_1, \tag{103}
$$

and $F(m_0) = \inf_{m>0} F(m)$.

So, it remains to prove that $R > r = [(1 + \mu R^{\lambda+\nu})\beta_2/(R - \gamma^*)]^{1/\lambda}$, that is,

$$
m_0^{\lambda+1} \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} < 1 + \gamma^* m_0. \tag{104}
$$

In fact, by (103), we have

$$
\lambda^2 \beta_1 \geq \left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda-1}\left(\beta_2 m_0^{1-\lambda^2}\right) = \left(\beta_2 m_0^{1-\lambda^2}\right)^{1/\lambda}, \tag{105}
$$

that is,

$$
m_0 \leq \left(\frac{\lambda^2 \beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}. \tag{106}
$$

Also we have

$$
\lambda^2 \beta_1
$$
$$
\geq \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda-1}
$$
$$
\times \left[(1 - \lambda - \nu) m_0^{1-\lambda^2}\right.
$$
$$
\left. + \mu\left(1 - \lambda - \nu\right) m_0^{1-\lambda-\nu-\lambda^2}\left(1 + \gamma^* m\right)^{\lambda+\nu}\right]
$$
$$
= (1 - \lambda - \nu) \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}, \tag{107}
$$

that is,

$$
\beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda} \leq \frac{\lambda^2 \beta_1}{1 - \lambda - \nu}. \tag{108}
$$

Thus, we have

$$
m_0^{\lambda+1} \beta_2^{1/\lambda}\left(m_0^{1-\lambda^2} + \mu\left(1 + \gamma^* m\right)^{\lambda+\nu} m_0^{1-\lambda-\nu-\lambda^2}\right)^{1/\lambda}
$$
$$
< \left(\left(\frac{\lambda^2 \beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda^2)}\right)^{\lambda+1} \frac{\lambda^2 \beta_1}{1 - \lambda - \nu}
$$

$$
= \left(\frac{\lambda^2 \beta_1}{\beta_2^{1/\lambda}}\right)^{\lambda/(1-\lambda)} \frac{\lambda^2 \beta_1}{1 - \lambda - \nu}
$$
$$
= \lambda^{2\lambda/(1-\lambda)}\left(\frac{\beta_1}{\beta_2}\right)^{1/(1-\lambda)} \frac{\lambda^2}{1 - \lambda - \nu} < 1 < 1 + \gamma^* m, \tag{109}
$$

since $0 < \lambda, \nu < 1$ and $1 - \lambda - \nu - \lambda^2 > 0$.

Thus, the proof is complete. $\qquad\square$

Remark 19. It is easy to find that analogous results to Examples 10, 13, 16, and 18 for the general equation with the nonlinearity in (28) are

$$
f(t, x) = \frac{b(t)}{x^\lambda} + \mu c(t) x^\nu + e(t), \tag{110}
$$

with $b, c > 0$, but the notation becomes cumbersome. Here we consider the nonlinearity (47) only for the simplicity.

Acknowledgments

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