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Research Article

Synchronization of Chaotic Delayed Fuzzy Neural Networks under Impulsive and Stochastic Perturbations

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The synchronization problem of chaotic fuzzy cellular neural networks with mixed delays is investigated. By an impulsive integrodifferential inequality and the Itô's formula, some sufficient criteria to synchronize the networks under both impulsive and stochastic perturbations are obtained. The example and simulations are given to demonstrate the efficiency and advantages of the proposed results.

1. Introduction

Fuzzy cellular neural network (FCNN), which integrated fuzzy logic into the structure of a traditional cellular neural networks (CNNs) and maintained local connectivity among cells, was first introduced by T. Yang and L. Yang [1] to deal with some complexity, uncertainty, or vagueness in CNNs. Lots of studies have illustrated that FCNNs are a useful paradigm for image processing and pattern recognition [2]. So far, many important results on stability analysis and state estimation of FCNNs have been reported (see [3–12] and the references therein).

Recently, it has been revealed that if the network's parameters and time delays are appropriately chosen, then neural networks can exhibit some complicated dynamics and even chaotic behaviors [13, 14]. The chaotic system exhibits unpredictable and irregular dynamics, and it has been found in many fields. Since the drive-response concept was proposed by Pecora and Carroll [15] in 1990 for constructing the synchronization of coupled chaotic systems, the control and synchronization problems of chaotic systems have been extensively investigated. In recent years, various synchronization schemes for chaotic neural networks have derived and demonstrated potential applications in many areas such as secure communication, image processing and harmonic oscillation generation; see [16–32].

Although there have been many results which can be applied to synchronization problems of a broad class of FCNNs [25–32], there are some disadvantages that need attention.

(1) Synchronization procedures and schemes are rather sensitive to the unavoidable channel disturbances which are usually presented in two forms: impulse and random noise. However, in [25–27], authors provided some new schemes to synchronize the chaotic systems without considering both impulse and random noise. In [28, 29], under the condition of no channel disturbance, Yu et al. and Xing and Peng studied the lag synchronization problems of FCNNs, respectively. In [30, 31], authors studied the synchronization of impulsive fuzzy cellular neural networks (IFCNNs) with delays. In [32], authors derived some synchronization schemes for FCNNs with random noise. In fact, in real system, it is more reasonable that the two perturbations coexist simultaneously.

(2) The criteria proposed in [25–32] are valid only for FCNNs with discrete delays. For example, in [25, 28, 30, 31], the involved delays are constants. In [26, 27, 32], the involved delays are time-varying delays which are continuously differentiable, and the corresponding derivatives are required to be finite or not greater than 1. In [29], Xing and Peng provided some new criteria on lag synchronization problem of FCNNs but they only considered the case for bounded time-varying delays. In fact, time delays may occur in an irregular fashion,

and sometimes they may be not continuously differentiable. Besides this, distribution delays may also exist when neural networks have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths.

(3) Some conditions imposed on the impulsive perturbations are too strong. For instance, Feng et al. [31] required the magnitude of jumps not to be smaller than 0 and not greater than 2. However, the disturbance in the real environment may be very intense.

Therefore, it is of great theoretical and practical significance to investigate synchronization problems of IFCNNs with mixed delays and random noise. However, up to now, to the best of our knowledge, no result for synchronization of IFNNs with mixed delays and random noise has been reported.

Inspired by the above discussion, this paper addresses the exponential synchronization problem of IFCNNs with mixed delays and random noise. Based on the properties of nonsingular $\mathscr{M}$-matrix and the Itô's formula, we design some synchronization schemes with a state feedback controller to ensure the exponential synchronization control. Our method does not resort to complicated Lyapunov-Krasovskii functional which is widely used. The proposed synchronization schemes are novel and improve some of the previous literature.

This paper is organized as follows. In Section 2, we introduce the drive-response models and some preliminaries. In Section 3, some synchronization criteria for FCNNs with mixed delays are derived. In Section 4, an example and its simulations are given to illustrate the effectiveness of theoretical results. Finally, conclusions are drawn in Section 5.

2. Model Description and Preliminaries

Let $\mathbb{R}^n$ be the space of n-dimensional real column vectors, and let $\mathbb{R}^{m\times n}$ represent the class of $m\times n$ matrices with real components. $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The inequality "$\leq$" ("$>$") between matrices or vectors such as $A \leq B$ ($A > B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\leq$" ("$>$"). $A \in \mathbb{R}^{m\times n}$ is called a nonnegative matrix if $A \geq 0$, and $x \in \mathbb{R}^n$ is called a positive vector if $x > 0$. The transpose of $A \in \mathbb{R}^{m\times n}$ or $x \in \mathbb{R}^n$ is denoted by A^T or x^T. Let E denote the unit matrix with appropriate dimensions. $\mathscr{N} := \{1, 2, \ldots, n\}$, and $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{R}_+ := [0, +\infty)$.

$\mathbb{PC}[J, \mathbb{R}^n] = \{\psi : J \to \mathbb{R}^n \mid \psi(s)$ is continuous and bounded for all but at most countable points $s \in J$ and at these points, $\psi(s^+)$ and $\psi(s^-)$ exist, $\psi(s) = \psi(s^+)\}$. Here, $J \subset \mathbb{R}$ is an interval; $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$, respectively. Especially $\mathbb{PC} := \mathbb{PC}[(-\infty, 0], \mathbb{R}^n]$ with the norm $\|\psi\| = \sup_{-\infty<s\leq 0} |\psi(s)|$ for $\psi \in \mathbb{PC}$.

$\mathbb{L}^e = \{\psi : \mathbb{R}_+ \to \mathbb{R} \mid \psi(s)$ is piecewise continuous and satisfies $\int_0^{+\infty} |\psi(s)| e^{\sigma_0 s} ds < +\infty$ for some constant $\sigma_0 > 0\}$.

For $A, B \in \mathbb{R}^{n\times n}$ and $\phi : \mathbb{R} \to \mathbb{R}^n$, we denote that

$$\begin{aligned}
[A]^+ &= \left(\left|a_{ij}\right|\right)_{n\times n}, \\
A \circ B &= \left(a_{ij} b_{ij}\right)_{n\times n}, \\
[\phi]^+ &= \left(|\phi_1|, \ldots, |\phi_n|\right)^T, \\
[\phi(t)]_\tau &= \left([\phi_1(t)]_\tau, \ldots, [\phi_n(t)]_\tau\right)^T, \\
\text{where } [\phi_i(t)]_\tau &= \sup_{-\tau\leq s\leq 0} \phi_i(t+s), \quad i \in \mathscr{N},
\end{aligned} \tag{1}$$

and $D^+\phi(t)$ denotes the upper-right derivative of $\phi(t)$ at time t.

Consider IFCNNs with mixed delays as follows:

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -c_i x_i + \sum_{j=1}^n a_{ij} f_j(x_j) + \sum_{j=1}^n b_{ij} \nu_j + J_i \\
&\quad + \bigwedge_{j=1}^n T_{ij} \mu_j + \bigwedge_{j=1}^n \alpha_{ij} f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) \\
&\quad + \bigwedge_{j=1}^n \gamma_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(x_j(t-s)\right) ds \\
&\quad + \bigvee_{j=1}^n S_{ij} \mu_j + \bigvee_{j=1}^n \beta_{ij} f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) \\
&\quad + \bigvee_{j=1}^n \theta_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(x_j(t-s)\right) ds, \\
&\qquad\qquad t \geq t_0,\ t \neq t_k, \\
\Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) \\
&= I_{ik}\left(x_i(t_k^-)\right), \quad k \in \mathbb{N}, \\
x_i(t_0 + s) &= \phi_i(s), \quad -\infty < s \leq 0,
\end{aligned} \tag{2}$$

where $i = 1, 2, \ldots, n$, n denotes the number of units in the neural network. $x(t) = (x_1(t), \ldots, x_n(t))^T$ represents the state variable. $f_j(\cdot)$ is the activation function of the jth neuron. c_i represents the passive decay rate to the state of ith neuron at time t. α_{ij} and γ_{ij} are elements of the fuzzy feedback MIN template. β_{ij} and θ_{ij} are elements of the fuzzy feedback MAX template. T_{ij} and S_{ij} are elements of fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively. a_{ij} and b_{ij} are elements of feedback and feed-forward template, respectively. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively. ν_i and J_i denote input and bias of the ith neuron, respectively. For any $i, j \in \mathscr{N}$, $\tau_{ij}(t)$ corresponding to the transmission delay satisfies $0 \leq \tau_{ij}(t) \leq \tau$, and $k_{ij} \in \mathbb{L}^e$ is the feedback kernel. For any $k \in \mathbb{N}$, $I_k(\cdot)$ represents the impulsive perturbation, and t_k denotes impulsive moment satisfying $t_k < t_{k+1}$, $\lim_{k\to+\infty} t_k = +\infty$.

We make the following assumptions throughout this paper.

(A_1) f_i is globally Lipschitz continuous, that is, for any $i \in \mathcal{N}$, there exists nonnegative constant L_i such that

$$\left| f_i(u) - f_i(v) \right| \leq L_i |u - v| \quad \text{for } u, v \in \mathbb{R}. \tag{3}$$

(A_2) For any $k \in \mathbb{N}$, there is a nonnegative constant η_k such that

$$\left| u + I_{ik}(u) - v - I_{ik}(v) \right| \leq \eta_k |u - v| \quad \text{for } u, v \in \mathbb{R},\ i \in \mathcal{N}. \tag{4}$$

Let (2) be the drive system, and let the response system with random noise be described by

$$\begin{aligned}
dy_i(t) = \Bigg[& -c_i y_i + \sum_{j=1}^{n} a_{ij} f_j(y_j) + \sum_{j=1}^{n} b_{ij} \nu_j \\
& + J_i + \bigwedge_{j=1}^{n} T_{ij} \mu_j + \bigwedge_{j=1}^{n} \alpha_{ij} f_j\left(y_j\left(t - \tau_{ij}(t)\right)\right) \\
& + \bigwedge_{j=1}^{n} \gamma_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(y_j(t-s)\right) ds \\
& + \bigvee_{j=1}^{n} S_{ij} \mu_j + \bigvee_{j=1}^{n} \beta_{ij} f_j\left(y_j\left(t - \tau_{ij}(t)\right)\right) \\
& + \bigvee_{j=1}^{n} \theta_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(y_j(t-s)\right) ds + U_i(t) \Bigg] dt \\
& + \sum_{j=1}^{n} \sigma_{ij}\left(t, x_j(t) - y_j(t), x_j\left(t - \tau_{ij}(t)\right)\right. \\
& \left. - y_j\left(t - \tau_{ij}(t)\right)\right) dw_j(t), \\
& \qquad t \geq t_0, t \neq t_k, \\
\Delta y_i(t_k) = {} & y_i(t_k^+) - y_i(t_k^-) = I_{ik}\left(y_i(t_k^-)\right), \quad k \in \mathbb{N}, \\
y_i(t_0 + s) = {} & \psi_i(s), \quad -\infty < s \leq 0,
\end{aligned} \tag{5}$$

where $w(t) = (w_1(t), \ldots, w_n(t))^T$ is an n-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$ and satisfying the usual conditions (i.e., it is right continuous, and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). The initial value $\psi = (\psi_1(s), \ldots, \psi_n(s))^T \in \mathbb{PC}^b_{\mathcal{F}_0}[(-\infty, 0], \mathbb{R}^n]$ which denotes the family of all bounded $\mathcal{F}_0$-measurable and $\mathbb{PC}$-valued random variables ψ with the norm $\|\psi\|^p_{\mathcal{F}} = \sup_{-\infty < s \leq 0} \mathbf{E}|\psi(s)|^p$, where $\mathbf{E}$ denotes the expectation of stochastic process. $U(t) = (U_1(t), \ldots, U_n(t))^T$ is the state feedback controller designed by

$$\begin{aligned}
U_i(t) = {} & \sum_{j=1}^{n} M_{ij}\left(y_j(t) - x_j(t)\right) \\
& + \sum_{j=1}^{n} N_{ij}\left(y_j\left(t - \tau_{ij}(t)\right) - x_j\left(t - \tau_{ij}(t)\right)\right),
\end{aligned} \tag{6}$$

where $M = (M_{ij})_{n \times n}$, $N = (N_{ij})_{n \times n}$ are the controller gain matrices to be scheduled. The diffusion coefficient matrix (or noise intensity matrix) $\sigma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ satisfies the local Lipschitz condition and the linear growth condition (see [33]). In addition,

(A_3) for $i \in \mathcal{N}$, there exist nonnegative constants c_{ij}, d_{ij} such that

$$\begin{aligned}
\sigma_i \sigma_i^T \leq {} & \sum_{j=1}^{n} c_{ij} \left| x_j - y_j \right|^2 \\
& + \sum_{j=1}^{n} d_{ij} \left| x_j\left(t - \tau_{ij}(t)\right) - y_j\left(t - \tau_{ij}(t)\right) \right|^2,
\end{aligned} \tag{7}$$

where $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{in})$.

Let $e(t) = (e_1(t), \ldots, e_n(t))^T$, where $e_i(t) = y_i(t) - x_i(t)$, be the synchronization error. Then, the error dynamical system between (2) and (5) is given by

$$\begin{aligned}
de_i(t) = \Bigg[& -c_i e_i + \sum_{j=1}^{n} a_{ij}\left(f_j(y_j) - f_j(x_j)\right) \\
& + \sum_{j=1}^{n} M_{ij} e_j(t) + \sum_{j=1}^{n} N_{ij} e_j\left(t - \tau_{ij}(t)\right) \\
& + \bigwedge_{j=1}^{n} \alpha_{ij} f_j\left(y_j\left(t - \tau_{ij}(t)\right)\right) \\
& - \bigwedge_{j=1}^{n} \alpha_{ij} f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) \\
& + \bigvee_{j=1}^{n} \beta_{ij} f_j\left(y_j\left(t - \tau_{ij}(t)\right)\right) \\
& - \bigvee_{j=1}^{n} \beta_{ij} f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) \\
& + \bigwedge_{j=1}^{n} \gamma_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(y_j(t-s)\right) ds \\
& - \bigwedge_{j=1}^{n} \gamma_{ij} \int_0^{+\infty} k_{ij}(s) f_j\left(x_j(t-s)\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\bigvee_{j=1}^{n}\theta_{ij}\int_{0}^{+\infty}k_{ij}(s)\,f_j\left(y_j(t-s)\right)ds\\
&-\bigvee_{j=1}^{n}\theta_{ij}\int_{0}^{+\infty}k_{ij}(s)\,f_j\left(x_j(t-s)\right)ds\Bigg]\,dt\\
&+\sum_{j=1}^{n}\sigma_{ij}\left(t,e_j(t),e_j\left(t-\tau_{ij}(t)\right)\right)dw_j(t),\\
&\qquad t\geq t_0,\ t\neq t_k,\\
\Delta e_i(t_k)&=e_i(t_k^+)-e_i(t_k^-)\\
&=I_{ik}\left(y_i(t_k^-)\right)-I_{ik}\left(x_i(t_k^-)\right),\quad k\in\mathbb{N},\\
e_i(t_0+s)&=\psi_i(s)-\phi_i(s),\quad -\infty<s\leq 0.
\end{aligned}
\tag{8}
$$

For convenience, we use the following notations: $D_1 = \operatorname{diag}\{-c_1,\ldots,-c_n\}$, $L = \operatorname{diag}\{L_1,\ldots,L_n\}$, $K(s) = (k_{ij}(s))_{n\times n}$, $A = (a_{ij})_{n\times n}$, $\overline{M} = (\overline{M}_{ij})_{n\times n}$ with $\overline{M}_{ii} = M_{ii}$, $\overline{M}_{ij} = |M_{ij}|$ for $i\neq j$, $\alpha = (\alpha_{ij})_{n\times n}$, $\beta = (\beta_{ij})_{n\times n}$, $\Gamma = (\gamma_{ij})_{n\times n}$, $\Theta = (\theta_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, and $D = (d_{ij})_{n\times n}$.

The following definition and lemmas will be employed.

Definition 1. The systems (2) and (5) are called to be globally exponentially synchronized in p-moment, if there exist positive constants λ, K such that

$$
\mathbf{E}|e(t)|^p\leq K\|\psi-\phi\|_{\mathcal{F}}^{p}e^{-\lambda(t-t_0)},\quad t\geq t_0.\tag{9}
$$

It is said especially to be globally exponentially synchronized in mean square when $p = 2$.

For any nonsingular $\mathcal{M}$-matrix A (see [34]), we define that

$$
\mathcal{M}_A=\left\{z\in\mathbb{R}^n\mid Az>0,\ z>0\right\}.\tag{10}
$$

Lemma 2 (see [35]). *For a nonsingular $\mathcal{M}$-matrix A, $\mathcal{M}_A$ is a nonempty cone without conical surface.*

Lemma 3 (see [36]). *For $x_i\geq 0$, $\alpha_i>0$, and $\sum_{i=1}^{n}\alpha_i = 1$,*

$$
\prod_{i=1}^{n}x_i^{\alpha_i}\leq\sum_{i=1}^{n}\alpha_i x_i.\tag{11}
$$

The sign of equality holds if and only if $x_i = x_j$ for all $i, j\in\mathcal{N}$.

Lemma 4 (see [1]). *Let α_{ij}, $\beta_{ij}\in\mathbb{R}$ and x, $y\in\mathbb{R}^n$ be the two states of the system* (2). *Then, one has*

$$
\begin{aligned}
&\left|\bigwedge_{j=1}^{n}\alpha_{ij}f_j(x_j)-\bigwedge_{j=1}^{n}\alpha_{ij}f_j(y_j)\right|\\
&\qquad\leq\sum_{j=1}^{n}\left|\alpha_{ij}\right|\left|f_j(x_j)-f_j(y_j)\right|,\\
&\left|\bigvee_{j=1}^{n}\beta_{ij}f_j(x_j)-\bigvee_{j=1}^{n}\beta_{ij}f_j(y_j)\right|\\
&\qquad\leq\sum_{j=1}^{n}\left|\beta_{ij}\right|\left|f_j(x_j)-f_j(y_j)\right|.
\end{aligned}
\tag{12}
$$

Lemma 5 (see [36]). *For the integer $p\geq 2$ and $x = (x_1,\ldots,x_n)^T\in\mathbb{R}^n$, there exists a positive constant $e_p(n)$ such that*

$$
e_p(n)\left(\sum_{i=1}^{n}x_i^2\right)^{p/2}\leq\sum_{i=1}^{n}|x_i|^p.\tag{13}
$$

Lemma 6. *For $k\in\mathbb{N}$, assume that $v = (v_1(t),\ldots,v_n(t))^T\in\mathbb{PC}[(-\infty,+\infty),\mathbb{R}^n]$ satisfies*

$$
\begin{aligned}
&D^+v(t)\leq A_0v(t)+Pv(t)+Q\left[v(t)\right]_\tau\\
&\qquad+\int_0^{+\infty}\Upsilon(s)\,v(t-s)\,ds,\quad t\geq t_0,\ t\neq t_k,\\
&v(t_k^+)\leq\rho_k v(t_k^-),\quad \rho_k\geq 0,\\
&v_{t_0}\in\mathbb{PC},\quad \text{where } v_{t_0}=v(t_0+s),\quad s\in(-\infty,0],
\end{aligned}
\tag{14}
$$

in which

- (C_1) *$A_0 = \operatorname{diag}\{a_1,\ldots,a_n\}$, $P = (p_{ij})_{n\times n}$ with $p_{ij}\geq 0$ for $i\neq j$, $Q = (q_{ij})_{n\times n}\geq 0$, $\Upsilon(s) = (v_{ij}(s))_{n\times n}\geq 0$, and $v_{ij}\in\mathbb{L}^e$, $i, j\in\mathcal{N}$.*
- (C_2) *$\Pi = -(A_0 + P + Q + \int_0^{+\infty}\Upsilon(s)ds)$ is a nonsingular $\mathcal{M}$-matrix.*

Then, there must exist $z = (z_1,\ldots,z_n)^T\in\mathcal{M}_\Pi$ and $\lambda\in(0,\sigma_0]$ such that

$$
v(t)\leq\left(\prod_{j=0}^{k-1}\varrho_j\right)ze^{-\lambda(t-t_0)},\quad t_{k-1}\leq t<t_k,\ k\in\mathbb{N},\tag{15}
$$

provided that the initial value v_{t_0} satisfies

$$
v(t)\leq ze^{-\lambda(t-t_0)},\quad -\infty<t\leq t_0,\tag{16}
$$

where $\varrho_0 = 1$, $\varrho_k = \max\{1,\rho_k\}$, and z, λ can be determined by

$$
\left(\lambda E+A_0+P+Qe^{\lambda\tau}+\int_0^{+\infty}\Upsilon(s)\,e^{\lambda s}ds\right)z<0.\tag{17}
$$

Proof. By condition (C_2) and Lemma 2, we can find $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n)^T \in \mathscr{M}_\Pi$ such that $\Pi\tilde{z} > 0$, namely, $(A_0 + P + Q + \int_0^{+\infty} \Upsilon(s)ds)\,\tilde{z} < 0$. By the continuity, there must be some positive constant $\lambda \in (0, \sigma_0]$ satisfying

$$\left(\lambda E + A_0 + P + Qe^{\lambda\tau} + \int_0^{+\infty} \Upsilon(s) e^{\lambda s} ds\right)\tilde{z} < 0. \tag{18}$$

Noting that $\upsilon_{t_0} \in \mathbb{PC}$, we can find a constant $B > 0$ such that $\|\upsilon_{t_0}\| \le B$. Denote that $z := (B/\min_{i\in\mathcal{N}}\tilde{z}_i)\tilde{z} = (z_1, \dots, z_n)^T$. Obviously, z and λ satisfy (16) and (17).

Let $w_i(t) := z_i e^{-\lambda(t-t_0)}$ for $t \in \mathbb{R}$, $i \in \mathcal{N}$. For any small enough $\epsilon > 0$, (16) implies that $\upsilon_i(t) \le w_i(t) < (1+\epsilon)w_i(t)$, $t \in (-\infty, t_0]$. Next, we claim that for any $t \in [t_0, t_1)$,

$$\upsilon_i(t) < (1+\epsilon) w_i(t), \quad i \in \mathcal{N}. \tag{19}$$

If inequality (19) is not true, then there must exist some $m \in \mathcal{N}$ and $t^* \in (t_0, t_1)$ such that

$$\begin{aligned} \upsilon_m(t^*) = (1+\epsilon) w_m(t^*), \quad & \upsilon_i(t) < (1+\epsilon) w_i(t), \\ & t \in (-\infty, t^*), \ i \in \mathcal{N}, \end{aligned} \tag{20}$$

$$D^+\upsilon_m(t^*) \ge (1+\epsilon) w_m'(t^*). \tag{21}$$

On the other hand, (14) together with (17) and (20) leads to

$$\begin{aligned} D^+\upsilon_m(t^*) \le{} & a_m\upsilon_m(t^*) + \sum_{j=1}^n p_{mj}\upsilon_j(t^*) \\ & + \sum_{j=1}^n q_{mj}\left[\upsilon_j(t^*)\right]_\tau \\ & + \sum_{j=1}^n \int_0^{+\infty} \upsilon_{mj}(s)\,\upsilon_j(t^*-s)\,ds \\ \le{} & (1+\epsilon) e^{-\lambda(t^*-t_0)} a_m z_m \\ & + (1+\epsilon) e^{-\lambda(t^*-t_0)} \sum_{j=1}^n p_{mj} z_j \\ & + (1+\epsilon) e^{-\lambda(t^*-t_0)} \sum_{j=1}^n q_{mj} z_j e^{\lambda\tau} \\ & + (1+\epsilon) e^{-\lambda(t^*-t_0)} \\ & \times \sum_{j=1}^n \int_0^{+\infty} \upsilon_{mj}(s) e^{\lambda s} ds z_j \\ <{} & (1+\epsilon) e^{-\lambda(t^*-t_0)} (-\lambda z_m) \\ ={} & (1+\epsilon) w_m'(t^*), \end{aligned} \tag{22}$$

which contradicts (21). Therefore, (19) holds. Letting $\epsilon \to 0^+$ in (19), we get

$$\upsilon_i(t) \le w_i(t) = \varrho_0 w_i(t), \quad t \in [t_0, t_1), \ i \in \mathcal{N}. \tag{23}$$

Suppose that for $\nu = 1, 2, \dots, k$, the following inequalities hold

$$\upsilon_i(t) \le \left(\prod_{m=0}^{\nu-1} \varrho_m\right) w_i(t), \quad t \in [t_{\nu-1}, t_\nu), \ i \in \mathcal{N}. \tag{24}$$

For $t = t_k$, from (14) and (24), we have

$$\begin{aligned} \upsilon_i(t_k) &\le \rho_k \upsilon_i(t_k^-) \le \varrho_k \left(\prod_{m=0}^{k-1} \varrho_m\right) w_i(t_k) \\ &\le \left(\prod_{m=0}^{k} \varrho_m\right) w_i(t_k), \quad i \in \mathcal{N}. \end{aligned} \tag{25}$$

Recalling $\rho_k \ge 1$, it follows from (24) and (25) that

$$\upsilon_i(t) \le \left(\prod_{m=0}^{k} \varrho_m\right) w_i(t), \quad t \in (-\infty, t_k], \ i \in \mathcal{N}. \tag{26}$$

Repeating the proof similar to (19) can yield

$$\upsilon_i(t) \le \left(\prod_{m=0}^{k} \varrho_m\right) w_i(t), \quad t \in [t_k, t_{k+1}), \ i \in \mathcal{N}. \tag{27}$$

By the mathematical induction, we derive that for any $k \in \mathbb{N}$,

$$\begin{aligned} \upsilon_i(t) &\le \left(\prod_{j=0}^{k-1} \varrho_j\right) w_i(t) \\ &\le \left(\prod_{j=0}^{k-1} \varrho_j\right) z_i e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ i \in \mathcal{N}. \end{aligned} \tag{28}$$

The proof is completed. □

3. Exponential Synchronization

In this section, by using Lemma 6, we will obtain some sufficient criteria to synchronize the drive-response systems (2) and (5).

Theorem 7. *Assume that (A_1)–(A_3) hold and*

(A_4) *for $p \ge 2$, $\widetilde{D} = -(D_0 + \overline{P} + \overline{Q} + \int_0^{+\infty} \overline{\Upsilon}(s)ds)$ is a nonsingular $\mathscr{M}$-matrix, where $\overline{P} = [A]^+L + \overline{M} + (p-1)C := (\overline{p}_{ij})_{n\times n}$, $\overline{Q} = ([\alpha]^+ + [\beta]^+)L + [N]^+ + (p-1)D := (\overline{q}_{ij})_{n\times n}$, $\overline{\Upsilon}(s) = ([\Gamma]^+L + [\Theta]^+L)\circ[K(s)]^+ := (\overline{\upsilon}_{ij}(s))_{n\times n}$, $D_0 = \operatorname{diag}\{d_1, \dots, d_n\}$ with*

$$\begin{aligned} d_i = -pc_i + (p-1)\sum_{j=1}^n \Bigg[\Bigg(& \left|a_{ij}\right| + \left|\alpha_{ij}\right| + \left|\beta_{ij}\right| \\ & + \int_0^{+\infty} \left(\left|\gamma_{ij}\right| + \left|\theta_{ij}\right|\right)\left|k_{ij}(s)\right| ds \Bigg) L_j \\ & + \frac{p-2}{2}\left(c_{ij} + d_{ij}\right) + \overline{M}_{ij} + \left|N_{ij}\right| \Bigg], \\ & i \in \mathcal{N}, \end{aligned} \tag{29}$$

(A_5) *the impulsive perturbations satisfy*

$$\sup_{k\in\mathbb{N}}\frac{\ln\zeta_k}{t_k-t_{k-1}}<\lambda, \tag{30}$$

where $\zeta_k=\max\{1,\eta_k^p\}$, *and* $\lambda\in(0,\sigma_0]$ *is determined by*

$$\left(\lambda E+D_0+\overline{P}+\overline{Q}e^{\lambda\tau}+\int_0^{+\infty}\overline{\Upsilon}(s)e^{\lambda s}ds\right)z<0, \quad \text{for a given } z\in\mathscr{M}_{\widetilde{D}}. \tag{31}$$

Then, drive-response systems (2) *and* (5) *are globally exponential synchronization in p-moment.*

Proof. Since $\widetilde{D}=-(D_0+\overline{P}+\overline{Q}+\int_0^{+\infty}\overline{\Upsilon}(s)ds)$ is a nonsingular $\mathscr{M}$-matrix, by Lemma 2 and the continuity, there must be a constant vector $z=(z_1,\ldots,z_n)^T\in\mathscr{M}_{\widetilde{D}}$ and a constant $\lambda\in(0,\sigma_0]$ such that (31) holds.

We denote by $e=(e_1,\ldots,e_n)^T$ the solution of error dynamical system (8) with the initial value $\psi-\phi\in\mathbb{PC}^b_{\mathscr{F}_0}[(-\infty,0],\mathbb{R}^n]$ and let

$$V(e)=(V_1(e),\ldots,V_n(e))^T, \qquad V_i(e)=|e_i|^p,\ i\in\mathscr{N}. \tag{32}$$

Calculating the time derivative of $V_i(e(t))$ along the trajectory of error system (8) and by the Itô's formula [33], we get for any $k\in\mathbb{N}$,

$$dV_i(e(t))=\mathscr{L}V_i(e(t))\,dt+\frac{\partial V_i(e)}{\partial e}\sigma dw(t),\quad t\ge t_0,\ t\neq t_k, \tag{33}$$

where $\mathscr{L}V_i(e(t))$ is given by

$$\begin{aligned}
\mathscr{L}V_i(e(t))= {}& p|e_i(t)|^{p-2}e_i(t)\\
&\times\Bigg[-c_ie_i+\sum_{j=1}^n a_{ij}\\
&\times\left(f_j(y_j)-f_j(x_j)\right)+\sum_{j=1}^n M_{ij}e_j(t)\\
&+\sum_{j=1}^n N_{ij}e_j(t-\tau_{ij}(t))\\
&+\bigwedge_{j=1}^n\alpha_{ij}f_j\left(y_j\left(t-\tau_{ij}(t)\right)\right)\\
&-\bigwedge_{j=1}^n\alpha_{ij}f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)\\
&+\bigvee_{j=1}^n\beta_{ij}f_j\left(y_j\left(t-\tau_{ij}(t)\right)\right)\\
&-\bigvee_{j=1}^n\beta_{ij}f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)+\bigwedge_{j=1}^n\gamma_{ij}\\
&\times\int_0^{+\infty}k_{ij}(s)f_j\left(y_j(t-s)\right)ds-\bigwedge_{j=1}^n\gamma_{ij}\\
&\times\int_0^{+\infty}k_{ij}(s)f_j\left(x_j(t-s)\right)ds\\
&+\bigvee_{j=1}^n\theta_{ij}\int_0^{+\infty}k_{ij}(s)f_j\left(y_j(t-s)\right)ds\\
&-\bigvee_{j=1}^n\theta_{ij}\int_0^{+\infty}k_{ij}(s)f_j\left(x_j(t-s)\right)ds\Bigg]\\
&+\frac{1}{2}p(p-1)|e_i(t)|^{p-2}\sigma_i\sigma_i^T.
\end{aligned} \tag{34}$$

By (A_1) and Lemma 4, we have

$$\begin{aligned}
\mathscr{L}V_i(e(t))\le{}& -pc_i|e_i(t)|^p+p|e_i(t)|^{p-1}\\
&\times\sum_{j=1}^n|a_{ij}|L_j|e_j(t)|+p|e_i(t)|^{p-1}\\
&\times\sum_{j=1}^n\overline{M}_{ij}|e_j(t)|+p|e_i(t)|^{p-1}\\
&\times\sum_{j=1}^n|N_{ij}||e_j(t-\tau_{ij}(t))|\\
&+p|e_i(t)|^{p-1}\sum_{j=1}^n|\alpha_{ij}|\\
&\times L_j|e_j(t-\tau_{ij}(t))|+p|e_i(t)|^{p-1}\\
&\times\sum_{j=1}^n|\beta_{ij}|L_j|e_j(t-\tau_{ij}(t))|+p|e_i(t)|^{p-1}\\
&\times\sum_{j=1}^n|\gamma_{ij}|L_j\int_0^{+\infty}|k_{ij}(s)||(t-s)|ds\\
&+p|e_i(t)|^{p-1}\sum_{j=1}^n|\theta_{ij}|L_j\int_0^{+\infty}|k_{ij}(s)|\\
&\times|e_j(t-s)|ds+\frac{1}{2}p(p-1)|e_j(t)|^{p-2}\sigma_i\sigma_i^T\\
:={}&-pc_i|e_i(t)|^p+I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned} \tag{35}$$

Using Lemma 3 and (A_3), it is easy to get

$$\begin{aligned}
I_1 \leq & \left(\sum_{j=1}^{n}\left|a_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|a_{ij}\right| L_j\left|e_j(t)\right|^p, \\
I_2 \leq & \left(\sum_{j=1}^{n}\overline{M}_{ij}\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\overline{M}_{ij}\left|e_j(t)\right|^p, \\
I_3 \leq & \left(\sum_{j=1}^{n}\left|N_{ij}\right|\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|N_{ij}\right|\left[\left|e_j(t)\right|^p\right]_\tau, \\
I_4 \leq & \left(\sum_{j=1}^{n}\left|\alpha_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\alpha_{ij}\right| L_j\left[\left|e_j(t)\right|^p\right]_\tau, \\
I_5 \leq & \left(\sum_{j=1}^{n}\left|\beta_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\beta_{ij}\right| L_j\left[\left|e_j(t)\right|^p\right]_\tau, \\
I_6 \leq & \left(\sum_{j=1}^{n}\left|\gamma_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right| ds\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\gamma_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right|\left|e_j(t-s)\right|^p ds, \\
I_7 \leq & \left(\sum_{j=1}^{n}\left|\theta_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right| ds\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\theta_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right|\left|e_j(t-s)\right|^p ds, \\
I_8 \leq & \frac{1}{2}p(p-1)\left|e_i(t)\right|^{p-2}\sum_{j=1}^{n}c_{ij}\left|e_j(t)\right|^2 \\
& + \frac{1}{2}p(p-1)\left|e_i(t)\right|^{p-2}\sum_{j=1}^{n}d_{ij}\left|e_j\left(t-\tau_{ij}(t)\right)\right|^2 \\
\leq & \frac{(p-1)(p-2)}{2}\left(\sum_{j=1}^{n}c_{ij}\right)\left|e_i(t)\right|^p \\
& + (p-1)\sum_{j=1}^{n}c_{ij}\left|e_j(t)\right|^p \\
& + \frac{(p-1)(p-2)}{2}\left(\sum_{j=1}^{n}d_{ij}\right)\left|e_i(t)\right|^p \\
& + (p-1)\sum_{j=1}^{n}d_{ij}\left[\left|e_j(t)\right|^p\right]_\tau.
\end{aligned} \tag{36}$$

Thus, we have

$$\begin{aligned}
\mathscr{L}V_i(e(t)) \leq & -pc_i\left|e_i(t)\right|^p + \left(\sum_{j=1}^{n}\left|a_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|a_{ij}\right| L_j\left|e_j(t)\right|^p \\
& + \left(\sum_{j=1}^{n}\overline{M}_{ij}\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\overline{M}_{ij}\left|e_j(t)\right|^p \\
& + \left(\sum_{j=1}^{n}\left|N_{ij}\right|\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|N_{ij}\right|\left[\left|e_j(t)\right|^p\right]_\tau \\
& + \left(\sum_{j=1}^{n}\left|\alpha_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\alpha_{ij}\right| L_j\left[\left|e_j(t)\right|^p\right]_\tau \\
& + \left(\sum_{j=1}^{n}\left|\beta_{ij}\right| L_j\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\beta_{ij}\right| L_j\left[\left|e_j(t)\right|^p\right]_\tau \\
& + \left(\sum_{j=1}^{n}\left|\gamma_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right| ds\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\gamma_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right|\left|e_j(t-s)\right|^p ds \\
& + \left(\sum_{j=1}^{n}\left|\theta_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right| ds\right)(p-1)\left|e_i(t)\right|^p \\
& + \sum_{j=1}^{n}\left|\theta_{ij}\right| L_j\int_0^{+\infty}\left|k_{ij}(s)\right|\left|e_j(t-s)\right|^p ds
\end{aligned}$$

$$+\frac{(p-1)(p-2)}{2}\left(\sum_{j=1}^{n}c_{ij}\right)|e_i(t)|^p+(p-1)\times\sum_{j=1}^{n}c_{ij}\left|e_j(t)\right|^p+\frac{(p-1)(p-2)}{2}\times\left(\sum_{j=1}^{n}d_{ij}\right)|e_i(t)|^p+(p-1)\sum_{j=1}^{n}d_{ij}\left[\left|e_j(t)\right|^p\right]_{\tau}. \tag{37}$$

It follows from (A_4) and (32) that

$$\mathscr{L}V_i(e(t))\le d_iV_i(e(t))+\sum_{j=1}^{n}\left(\overline{p}_{ij}V_j(e(t))+\overline{q}_{ij}\left[V_j(e(t))\right]_{\tau}+\int_0^{+\infty}\overline{v}_{ij}(s)V_j(e(t-s))\,ds\right). \tag{38}$$

Substituting (38) into (33) gives

$$dV_i(e(t))\le\left[d_iV_i(e(t))+\sum_{j=1}^{n}\left(\overline{p}_{ij}V_j(e(t))+\overline{q}_{ij}\left[V_j(e(t))\right]_{\tau}+\int_0^{+\infty}\overline{v}_{ij}(s)V_j(e(t-s))\,ds\right)\right]dt+\frac{\partial V_i(e)}{\partial e}\sigma\,dw(t),\quad t\ge t_0,\ t\ne t_k,\ k\in\mathbb{N}. \tag{39}$$

Integrating and taking the expectations on both sides of (39) lead to

$$\mathbf{E}V_i(e(t+\delta))-\mathbf{E}V_i(e(t))\le\int_t^{t+\delta}\left[d_i\mathbf{E}V_i(e(u))+\sum_{j=1}^{n}\left(\overline{p}_{ij}\mathbf{E}V_j(e(u))+\overline{q}_{ij}\mathbf{E}\left[V_j(e(u))\right]_{\tau}+\int_0^{+\infty}\overline{v}_{ij}(s)\times\mathbf{E}V_j(e(u-s))\,ds\right)\right]du, \tag{40}$$

where $\delta>0$ is small enough such that $t,t+\delta\in[t_{k-1},t_k)$ for $k\in\mathbb{N}$.

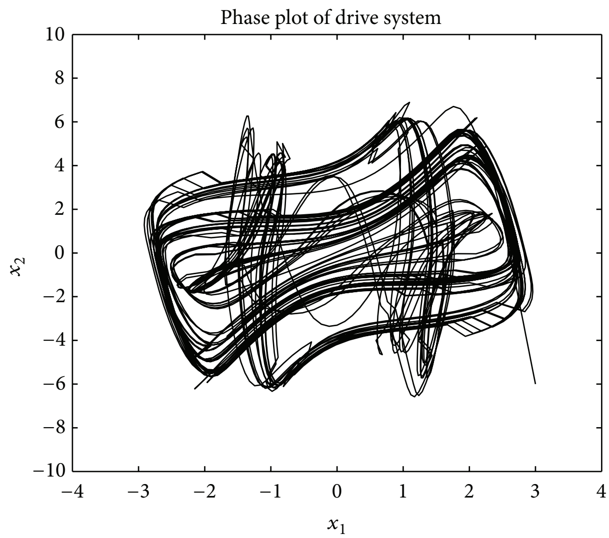

Figure 1: Chaos behavior of drive system.

By the continuity of $\mathbf{E}V_i(e(t))$, we conclude that

$$D^{+}\mathbf{E}V_i(e(t))\le d_i\mathbf{E}V_i(e(t))+\sum_{j=1}^{n}\left(\overline{p}_{ij}\mathbf{E}V_j(e(t))+\overline{q}_{ij}\mathbf{E}\left[V_j(e(t))\right]_{\tau}+\int_0^{+\infty}\overline{v}_{ij}(s)\,\mathbf{E}V_j(e(t-s))\,ds\right), \tag{41}$$

which implies that for $t\ne t_k,\ k\in\mathbb{N}$,

$$D^{+}\mathbf{E}V(e(t))\le D_0\mathbf{E}V(e(t))+\overline{P}\mathbf{E}V(e(t))+\overline{Q}\mathbf{E}[V(e(t))]_{\tau}+\int_0^{+\infty}\overline{\Upsilon}(s)\,\mathbf{E}V(e(t-s))\,ds. \tag{42}$$

Meanwhile, it follows from (A_2) and (32) that

$$\mathbf{E}V_i(e(t_k))=\mathbf{E}\left|I_{ik}\left(y_i(t_k^-)\right)-I_{ik}\left(x_i(t_k^-)\right)+e_i(t_k^-)\right|^p\le\eta_k^p\mathbf{E}V_i\left(e(t_k^-)\right) \tag{43}$$

for $i\in\mathcal{N}$ and $k\in\mathbb{N}$, which means that

$$\mathbf{E}V\left(e(t_k)\right)\le\eta_k^p\mathbf{E}V\left(e(t_k^-)\right). \tag{44}$$

Obviously, (42) and (44) indicate that $\mathbf{E}V(e(t))$ satisfies inequality (14) in Lemma 6.

On the other hand, noting that $e(t_0+s)=\psi(s)-\phi(s)$ in (8) and by a simple calculation, we get

$$\mathbf{E}\left|e_i(t_0+s)\right|^p=\mathbf{E}\left|\psi_i(s)-\phi_i(s)\right|^p\le\|\psi-\phi\|_{\mathscr{F}}^p \tag{45}$$

for any $s \in (-\infty, 0]$, which means $\mathbf{E}|e_i(t)|^p \le \|\psi - \phi\|_{\mathcal{F}}^p$ for $t \in (-\infty, t_0]$ and $i \in \mathcal{N}$. Recalling the definition of $V(e(t))$ in (32) and $z > 0$, we conclude that for $t \in (-\infty, t_0]$

$$\begin{aligned}\mathbf{E}V(e(t)) &\le \|\psi - \phi\|_{\mathcal{F}}^p (1, \dots, 1)^T \\ &\le \|\psi - \phi\|_{\mathcal{F}}^p \left(\frac{z_1}{\min_{i\in\mathcal{N}}\{z_i\}}, \dots, \frac{z_n}{\min_{i\in\mathcal{N}}\{z_i\}} \right)^T \\ &\le \frac{\|\psi - \phi\|_{\mathcal{F}}^p}{\min_{i\in\mathcal{N}}\{z_i\}} z,\end{aligned} \tag{46}$$

which further indicates that

$$\mathbf{E}V(e(t)) \le \bar{z} e^{-\lambda(t-t_0)}, \quad t \in (-\infty, t_0], \tag{47}$$

where $\bar{z} = (\|\psi - \phi\|_{\mathcal{F}}^p / \min_{i\in\mathcal{N}}\{z_i\}) z$. This implies that condition (16) in Lemma 6 holds.

Therefore, by Lemma 6, we derive that

$$\mathbf{E}V(e(t)) \le \left(\prod_{j=0}^{k-1} \zeta_j \right) \bar{z} e^{-\lambda(t-t_0)}, \quad t_{k-1} \le t < t_k,\ k \in \mathbb{N}, \tag{48}$$

with $\zeta_0 = 1$. Meanwhile, (30) implies that there is a small enough constant ϵ $(0 < \epsilon < \lambda)$ such that

$$\zeta_k \le e^{(\lambda-\epsilon)(t_k - t_{k-1})}, \quad k \in \mathbb{N}. \tag{49}$$

Thus, inequality (48) together with (49) shows that for $k = 1$

$$\mathbf{E}V(e(t)) \le \zeta_0 \bar{z} e^{-\lambda(t-t_0)} \le \bar{z} e^{-\epsilon(t-t_0)}, \quad t_0 \le t < t_1, \tag{50}$$

and for any $k \ge 2$,

$$\begin{aligned}\mathbf{E}V(e(t)) &\le \bar{z}\zeta_0\zeta_1 \cdots \zeta_{k-1} e^{-\lambda(t-t_0)} \\ &\le \bar{z} e^{(\lambda-\epsilon)(t_1-t_0)} \cdots e^{(\lambda-\epsilon)(t_{k-1}-t_{k-2})} e^{-\lambda(t-t_0)} \\ &= \bar{z} e^{(\lambda-\epsilon)(t_{k-1}-t_0)} e^{-\lambda(t-t_0)} \\ &\le \bar{z} e^{(\lambda-\epsilon)(t-t_0)} e^{-\lambda(t-t_0)} \\ &\le \bar{z} e^{-\epsilon(t-t_0)}, \quad t_{k-1} \le t < t_k.\end{aligned} \tag{51}$$

By Lemma 5, we get

$$\mathbf{E}|e(t)|^p \le K \|\psi - \phi\|_{\mathcal{F}}^p e^{-\epsilon(t-t_0)}, \quad t \ge t_0, \tag{52}$$

where $K = (\sum_{i=1}^n z_i)/(e_p(n) \min_{i\in\mathcal{N}}\{z_i\})$. The proof is complete. □

Remark 8. In [25–32], the authors established some useful criteria for ensuring synchronization of FCNNs with delays, respectively. However, once the unbounded distributed delays are involved, all results in [25–32] will be invalid. Hence, in this sense, the proposed Theorem 7 has a wider range of applications than those in previous papers.

Remark 9. In synchronization scheme, we take both impulsive perturbations and random noise into account. Comparing with the results in [25–32], Theorem 7 can reflect a more realistic dynamical behavior and synchronization procedure.

If the random noise has not been considered, which means $\sigma \equiv 0$ in (5), then the response system reduces to

$$\begin{aligned}\frac{dy_i(t)}{dt} &= -c_i y_i + \sum_{j=1}^n a_{ij} f_j(y_j) \\ &\quad + \sum_{j=1}^n b_{ij}\nu_j + J_i + \bigwedge_{j=1}^n T_{ij}\mu_j \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij} f_j(y_j(t - \tau_{ij}(t))) \\ &\quad + \bigwedge_{j=1}^n \gamma_{ij} \int_0^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds \\ &\quad + \bigvee_{j=1}^n S_{ij}\mu_j + \bigvee_{j=1}^n \beta_{ij} f_j(y_j(t - \tau_{ij}(t))) \\ &\quad + \bigvee_{j=1}^n \theta_{ij} \int_0^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds \\ &\quad + U_i(t), \quad t \ge t_0,\ t \ne t_k, \\ \Delta y_i(t_k) &= y_i(t_k^+) - y_i(t_k^-) = I_{ik}(y_i(t_k^-)), \quad k \in \mathbb{N}, \\ y_i(t_0 + s) &= \psi_i(s), \quad -\infty < s \le 0.\end{aligned} \tag{53}$$

In this case, the following globally exponential synchronization scheme for drive-response IFCNNs (2) and (53) can be derived.

Theorem 10. *Assume that* (A_1) *and* (A_2) *hold and*

(A_6) $\widehat{D} = -(D_1 + \overline{P}_1 + \overline{Q}_1 + \int_0^{+\infty} \overline{\Upsilon}(s) ds)$ *is a nonsingular* $\mathcal{M}$*-matrix, where* $\overline{P}_1 = [A]^+ L + \overline{M}$, $\overline{Q}_1 = ([\alpha]^+ + [\beta]^+) L + [N]^+$,

(A_7) *the impulsive perturbations satisfy*

$$\sup_{k\in N} \frac{\ln \xi_k}{t_k - t_{k-1}} < \lambda, \tag{54}$$

where $\xi_k = \max\{1, \eta_k\}$, *and* $\lambda \in (0, \sigma_0]$ *is determined by*

$$\left(\lambda E + D_1 + \overline{P}_1 + \overline{Q}_1 e^{\lambda\tau} + \int_0^{+\infty} \overline{\Upsilon}(s) e^{\lambda s} ds \right) z < 0, \quad \text{for a given } z \in \mathcal{M}_{\widehat{D}}. \tag{55}$$

Then, the drive-response systems (2) *and* (53) *are globally exponential synchronization.*

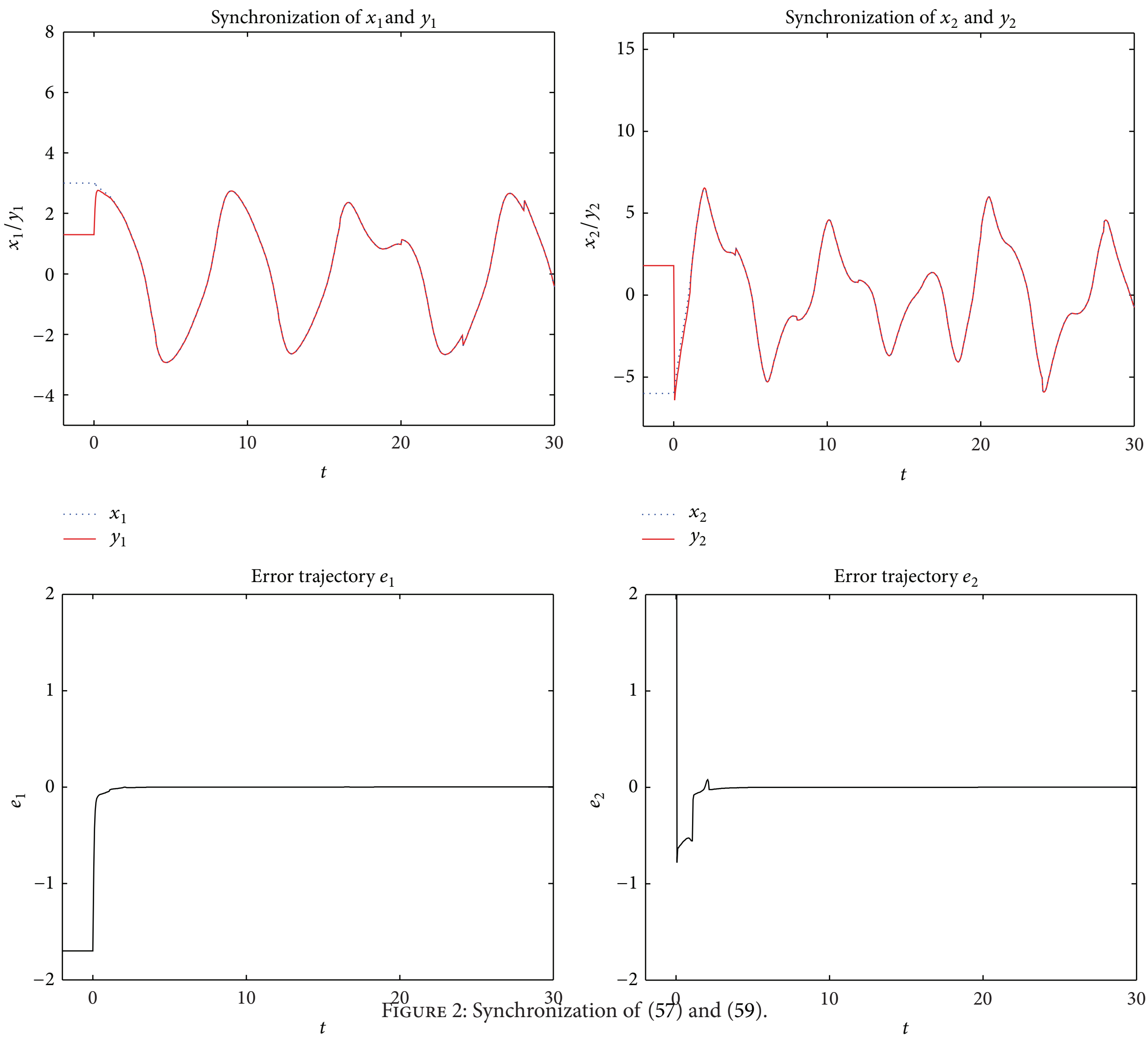

FIGURE 2: Synchronization of (57) and (59).

Proof. Let $V(e(t)) = [e(t)]^{+} = (|e_1(t)|, \ldots, |e_n(t)|)^T$. Calculating the time derivative of $V(e(t))$ along with the trajectory of error system can give

$$\begin{aligned} D^{+}V(e(t)) \le\ & D_1 V(e(t)) \\ & + \overline{P}_1 V(e(t)) + \overline{Q}_1 [V(e(t))]_\tau \\ & + \int_0^{+\infty} \overline{\Upsilon}(s) V(e(t-s))\, ds. \end{aligned} \tag{56}$$

The rest proof is similar to Theorem 7 and omitted here. We complete the proof. □

Remark 11. In [31], Feng et al. derived some criteria on the globally exponential synchronization for a special case of (2) and (53) with $\gamma_{ij} = \theta_{ij} = 0$, $\tau_{ij}(t) = \tau_{ij}$ and $I_{ik}(x_i(t_k^-)) = -\eta_{ik} x_i(t_k^-)$ for $i, j \in \mathcal{N}$, $k \in \mathbb{N}$. In order to achieve synchronous control, the conditions as $0 \le \eta_{ik} \le 2$ for $i \in \mathcal{N}$, $k \in \mathbb{N}$ have been imposed on the impulsive perturbations. However, Theorem 10 drops these restrictions.

4. Illustrative Example

In this section, a numerical example and its simulations are given to illustrate the effectiveness of our results.

Example 1. Consider the following 2-dimensional IFCNNs with mixed delays as the drive system

$$\begin{aligned} \frac{dx_i(t)}{dt} =\ & -c_i x_i + \sum_{j=1}^{2} a_{ij} f_j(x_j) \\ & + \bigwedge_{j=1}^{2} \alpha_{ij} f_j(x_j(t-1)) \end{aligned}$$

$$
\begin{aligned}
&+\bigvee_{j=1}^{2}\beta_{ij}f_j\left(x_j(t-1)\right)\\
&+\bigwedge_{j=1}^{2}\gamma_{ij}\int_0^{+\infty}k_{ij}(s)\,f_j\left(x_j(t-s)\right)ds\\
&+\bigvee_{j=1}^{2}\theta_{ij}\int_0^{+\infty}k_{ij}(s)\,f_j\left(x_j(t-s)\right)ds,\\
&\qquad t\neq t_k,\\
&\Delta x_i(t_k)=x_i(t_k^+)-x_i(t_k^-)=-J_kx_i(t_k^-),
\end{aligned}
\tag{57}
$$

where $i,j=1,2$, $f_j(u)=\tanh(u)$. $J_k=e^{0.15}+1, t_k=t_{k-1}+4$ for $k\in\mathbb{N}$. For the simplicity of computer simulations, we choose $k_{ij}(s)=e^{-s}$ for $s\in[0,20]$, $k_{ij}(s)=0$ for $s\in[20,+\infty)$. The system parameters are as follows:

$$
\begin{gathered}
D_1=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\qquad A=\begin{pmatrix}2&-0.1\\-4&3.2\end{pmatrix},\\
\alpha=\beta=\begin{pmatrix}-1.3&-0.2\\-0.2&-4.2\end{pmatrix},\qquad \Gamma=\Theta=\begin{pmatrix}-0.5&-0.5\\5&-2.5\end{pmatrix}.
\end{gathered}
\tag{58}
$$

We can choose $\sigma_0=0.8$ and $L_1=L_2=1$ such that $k_{ij}\in\mathbb{L}^e$ and (A_1) holds, respectively. Obviously, (A_2) holds with $\eta_k=e^{0.15}, k\in\mathbb{N}$.

Choosing the initial value $\phi(s)=(3,-6)^T$ for $s\in(-\infty,0]$, the drive system (57) possesses a chaotic behavior as shown in Figure 1.

Case 1. The response system without random noise is given by

$$
\begin{aligned}
\frac{dy_i(t)}{dt}&=-c_iy_i+\sum_{j=1}^{2}a_{ij}f_j\left(y_j\right)\\
&+\bigwedge_{j=1}^{2}\alpha_{ij}f_j\left(y_j(t-1)\right)\\
&+\bigvee_{j=1}^{2}\beta_{ij}f_j\left(y_j(t-1)\right)\\
&+\bigwedge_{j=1}^{2}\gamma_{ij}\int_0^{+\infty}k_{ij}(s)\,f_j\left(y_j(t-s)\right)ds\\
&+\bigvee_{j=1}^{2}\theta_{ij}\int_0^{+\infty}k_{ij}(s)\,f_j\\
&\times\left(y_j(t-s)\right)ds+U_i(t),\quad t\neq t_k,\\
\Delta y_i(t_k)&=y_i(t_k^+)-y_i(t_k^-)=-J_ky_i(t_k^-).
\end{aligned}
\tag{59}
$$

The control gain matrices M and N can be chosen as

$$
\begin{gathered}
M=\begin{pmatrix}-10.6&0\\0&-20.6\end{pmatrix},\\
N=\begin{pmatrix}e^{-20}&e^{-20}\\10e^{-20}&5e^{-20}\end{pmatrix}.
\end{gathered}
\tag{60}
$$

By simple calculation, we get that

$$
\widehat{D}=\begin{pmatrix}-6&1.5\\14.4&-5\end{pmatrix}
\tag{61}
$$

is a nonsingular $\mathscr{M}$-matrix, which implies that (A_6) holds. Moreover, we can choose $\lambda=0.2$ and $z=(1,3.5)^T\in\mathscr{M}_{\widehat{D}}$ such that (A_7) holds. Therefore, by Theorem 10, the drive-response systems (57) and (59) are globally exponentially synchronized. The simulation result with $\psi(s)=(1.3,1.8)^T$ is shown in Figure 2.

Remark 12. It is worth noting that the impulsive perturbations $J_k>2$, which are not a satisfied condition $(H2)$ in [31]. That is to say, even in the absence of the unbounded distributed delays, the results in [31] still cannot be applied to the synchronization problem of (57) and (59).

Case 2. Consider the response system with random noise as follows:

$$
\begin{aligned}
dy_i(t)=&\left[-c_iy_i+\sum_{j=1}^{2}a_{ij}f_j\left(y_j\right)\right.\\
&+\bigwedge_{j=1}^{2}\alpha_{ij}f_j\left(y_j(t-1)\right)\\
&+\bigvee_{j=1}^{2}\beta_{ij}f_j\left(y_j(t-1)\right)\\
&+\bigwedge_{j=1}^{2}\gamma_{ij}\int_0^{+\infty}k_{ij}(s)\,f_j\left(y_j(t-s)\right)ds\\
&+\bigvee_{j=1}^{2}\theta_{ij}\int_0^{+\infty}k_{ij}(s)\\
&\left.\times f_j\left(y_j(t-s)\right)ds+U_i(t)\right]dt\\
&+\sum_{j=1}^{2}\sigma_{ij}\left(t,x_j(t)-y_j(t),\right.\\
&\qquad\left.x_j\left(t-\tau_{ij}(t)\right)-y_j\left(t-\tau_{ij}(t)\right)\right)dw_j(t),\\
&\qquad t\neq t_k,\\
\Delta y_i(t_k)=&\ y_i(t_k^+)-y_i(t_k^-)=-J_ky_i(t_k^-),
\end{aligned}
\tag{62}
$$

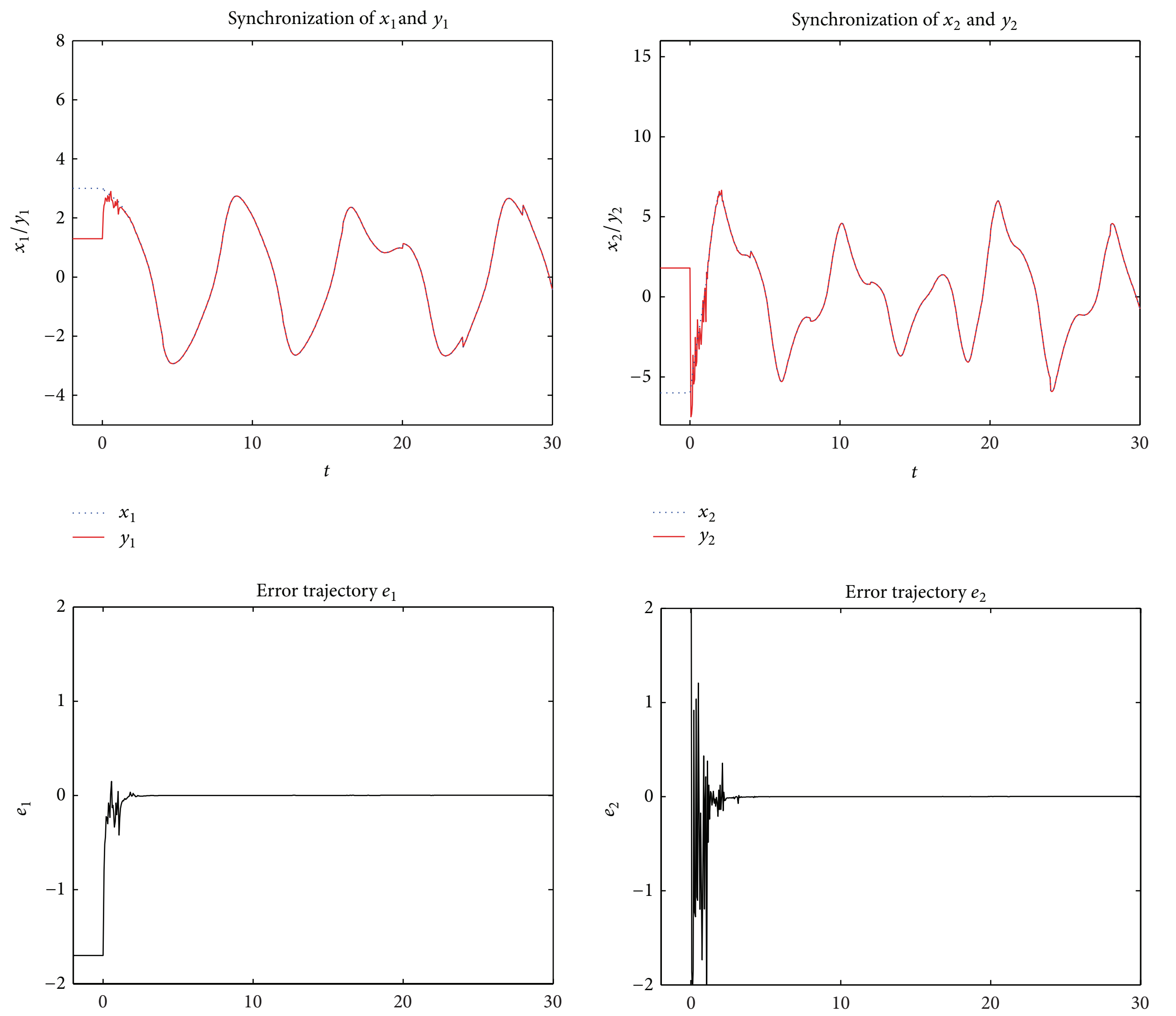

FIGURE 3: Synchronization of (57) and (62).

and the noise intensity matrix is

$$\sigma = \begin{pmatrix} -x_1(t) & 0.5x_1(t-1) \\ x_2(t) & -0.5x_2(t-1) \end{pmatrix}. \tag{63}$$

Clearly, we can choose

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix} \tag{64}$$

such that (A_3) holds.

For $p = 2$, let control gain matrices be

$$M = \begin{pmatrix} -8.475 & 0 \\ 0 & -25.925 \end{pmatrix},$$
$$N = \begin{pmatrix} e^{-20} & e^{-20} \\ 10e^{-20} & 5e^{-20} \end{pmatrix}. \tag{65}$$

It is easy to deduce that

$$\widetilde{D} = \begin{pmatrix} -5 & 1.5 \\ 14.4 & -5 \end{pmatrix} \tag{66}$$

is a nonsingular $\mathscr{M}$-matrix, which implies that (A_4) holds. Meanwhile, we can choose $\lambda = 0.1$ and $z = (1,3)^T \in \mathscr{M}_{\widetilde{D}}$ such that (A_5) holds. Hence, by Theorem 7, the drive-response systems (57) and (62) are globally exponentially synchronized in mean square. The simulation result based on Euler-Maruyama method is illustrated in Figure 3.

Remark 13. The schemes proposed in [25–29] cannot solve the synchronization problem of (57) and (62) due to the impulsive perturbations and random noise. Besides this, the distributed delays make those methods in [30–32] cannot be applied to synchronization problem of (57) and (62).

5. Conclusion

In this paper, we investigate the synchronization problem of IFCNNs with mixed delays. Based on the properties of nonsingular $\mathscr{M}$-matrix and some stochastic analysis approaches, some useful synchronization criteria under both impulse and random noise are obtained. The methods used in this paper are novel and can be extended to many other types of neural networks. These problems will be considered in near future.

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