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Research Article

Triple Positive Solutions of a Nonlocal Boundary Value Problem for Singular Differential Equations with p-Laplacian

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We establish the existence of triple positive solutions of an m-point boundary value problem for the nonlinear singular second-order differential equations of mixed type with a p-Laplacian operator by Leggett-William fixed point theorem. At last, we give an example to demonstrate the use of the main result of this paper. The conclusions in this paper essentially extend and improve the known results.

1. Introduction

The existence and multiplicity of positive solutions for differential equations boundary value problems (BVPs) with the p-Laplacian operator subject to Dirichlet, Sturm-Liouville, or nonlinear boundary value conditions have been extensively investigated in recent years; see [1–10] and the references therein. Particularly, the following differential equations with one-dimensional p-Laplacian

$$\left(\phi_p\left(u'\right)\right)' + q(t) f(t,u) = 0, \quad 0 \le t \le 1 \tag{1}$$

have been studied subject to different kinds of boundary conditions; see [1–4] and the references therein. The methods mainly depend on Kransnosel'skii fixed point theorem, upper and lower solution technique, Leggett-Williams fixed point theorem, and some new fixed point theorems in cones, and so forth.

Recently, in [9], Kong et al. have studied the existence of triple positive solutions for the following BVP:

$$\begin{aligned}&\left(\phi_p\left(u'\right)\right)'(t) + q(t) f \\ &\quad\times\left(u(t), u'(t), (Tu)(t), (Su)(t)\right) = 0, \quad 0 \le t \le 1, \\ &u'(0) = 0, \qquad u(1) = g\left(u'(1)\right).\end{aligned} \tag{2}$$

More recently, in [10], Hu and Ma have pointed out that the equivalent integral equation of BVP (2) is wrong in [9] and studied the existence of triple positive solutions for the following BVP:

$$\begin{aligned}&\left(\phi_p\left(u'\right)\right)'(t) + q(t) f \\ &\quad\times\left(u(t), u'(t), (Tu)(t), (Su)(t)\right) = 0, \quad 0 \le t \le 1, \\ &u'(0) = \beta u'(\eta), \qquad u(1) = g\left(u'(1)\right).\end{aligned} \tag{3}$$

Firstly, we confirm that the mistakes which have been pointed out in [10] exist. At the same time, we think that the value of M designed in Theorem 3.1 in [10] is not suitable, since the proof needs the condition $\phi_q(M) \le M$, but in fact this condition does not always hold.

Motivated by the work above, in this paper, we will study the following more extensive second-order m-point BVP:

$$\begin{aligned}&\left(\phi_p\left(u'\right)\right)'(t) + q(t) f \\ &\quad\times\left(u(t), u'(t), (Tu)(t), (Su)(t)\right) = 0, \quad 0 \le t \le 1, \\ &\phi_p\left(u'(0)\right) = \sum_{i=1}^{m-2} \beta_i \phi_p\left(u'(\eta_i)\right), \qquad u(1) = g\left(u'(1)\right),\end{aligned} \tag{4}$$

where $\phi_p(s) = |s|^{p-2}s (p > 1)$ is an increasing function, $(\phi_p)^{-1}(s) = \phi_q(s)$, $1/p + 1/q = 1$; $0 \le \beta_i < 1, i = 1, 2, \ldots, m-2$, $\sum_{i=1}^{m-2} \beta_i < 1$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2}$; T and S are two linear operators defined by

$$Tu(t) = \int_0^t k(t,s)\, u(s)\, ds,$$
$$Su(t) = \int_0^1 h(t,s)\, u(s)\, ds, \tag{5}$$
$$u \in C^1[0,1],$$

in which $k \in C[D, R^+]$, $h \in C[D_0, R^+]$, $D = \{(t,s) \in R^2 : 0 \le s \le t \le 1\}$, $D_0 = \{(t,s) \in R^2 : 0 \le s, t \le 1\}$, $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, $k_0 = \max\{k(t,s) : (t,s) \in D\}$, and $h_0 = \max\{h(t,s) : (t,s) \in D_0\}$.

Obviously, when $m = 3$, BVP (4) reduces to BVP (3), and when $m = 3$, $\beta_1 = 0$, BVP (4) reduces to BVP (2), so BVP (2) and BVP (3) are special cases of BVPs (4).

Throughout this paper, we always suppose the following conditions hold:

(C_1) $f \in C(R^+ \times R \times R^+ \times R^+, (0, +\infty))$;

(C_2) $q(t) \in C([0,1], R^+)$ may be singular at $t = 0, 1$ and $0 < \int_0^1 q(t)dt < +\infty$, so it is easy to see that there exists a constant $M > 0$ such that $0 < \int_0^1 q(t)dt < \phi_p(M)$;

(C_3) $g : R \to R^+$ is nonincreasing and continuous, and $0 \le g(v) \le |v|$ for $v \in R$.

2. Preliminary Results

In this section, we firstly present some definitions, theorems, and lemmas, which will be needed in the proof of the main result.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \ge 0$ implies $\lambda x \in P$;

(ii) $x \in P, -x \in P$ implies $x = 0$.

Definition 2. Given a cone P in a real Banach space E, a continuous map α is called a concave (resp., convex) functional on P if and only if, for all $x, y \in P$ and $0 \le t \le 1$, it holds:

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y),$$
$$(\text{resp., } \alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y)).$$

We consider the Banach space $E = C^1[0,1]$ equipped with norm $\|u\| = \max_{0\le t\le 1}\{\|u(t)\|_0, \|u'(t)\|_0\}$, where $\|u\|_0 = \max_{0\le t\le 1}|u(t)|$.

We denote, for any fixed constants a, b, r,

$$C^+[0,1] = \{u \in C[0,1] : u(t) \ge 0, t \in [0,1]\},$$
$$P = \{u \in E \mid u(t) \text{ is concave and nonincreasing on } [0,1]\},$$
$$P_r = \{u \in P : \|u\| < r\},$$
$$P(\alpha, a, b) = \{u \in P : a \le \alpha(u), \|u\| < b\}.$$

It's easy to see that P is a cone in E.

Theorem 3 (Leggett-William). *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous map and let α be a nonnegative continuous concave functional on P with $\alpha(u) \le \|u\|$ for any $u \in \overline{P}_c$. Suppose there exist constants a, b, and d with $0 < a < b < d \le c$ such that*

(i) *$\{u \in P(\alpha, b, d) : \alpha(u) > b\} \ne \phi$ and $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$;*

(ii) *$\|Au\| < a$ for all $u \in \overline{P}_a$;*

(iii) *$\alpha(Au) > b$ for all $u \in P(\alpha, b, c)$ with $\|Au\| > d$.*

Then A has at least three fixed points u_1, u_2, and u_3 satisfying

$$\|u_1\| < a, \qquad b < \alpha(u_2),$$
$$\|u_3\| > a, \qquad \alpha(u_3) < b. \tag{6}$$

Lemma 4. *Suppose $y \in C^1[0,1]$ with $(\phi_p(y'))' \in L^1[0,1]$ satisfies*

$$\left(\phi_p\left(y'\right)\right)'(t) \le 0, \quad 0 \le t \le 1,$$
$$\phi_p\left(y'(0)\right) = \sum_{i=1}^{m-2} \beta_i \phi_p\left(y'(\eta_i)\right), \quad y(1) = g\left(y'(1)\right). \tag{7}$$

Then, $y(t) \ge 0$ is concave and nonincreasing on $[0,1]$, that is, $y \in P$.

Proof. Since $(\phi_p(y'))'(t) \le 0$, we know that $\phi_p(y')$ is nonincreasing, that is, $y'(t)$ is nonincreasing, which means $y(t)$ is concave. At the same time, we have $y'(t) \le y'(0)$, so $y'(0) = \phi_q(\sum_{i=1}^{m-2} \beta_i \phi_p(y'(\eta_i))) \le \phi_q(\sum_{i=1}^{m-2} \beta_i \phi_p(y'(0))) = \phi_q(\sum_{i=1}^{m-2} \beta_i) y'(0)$, namely $y'(0) \le 0$. Then, $y'(t) \le 0$; that is to say, $y(t)$ is nonincreasing. So $y(t) \ge y(1) = g(y'(1)) \ge 0$. Above all, $y \in P$. This completes the proof. □

Lemma 5. *Let $y \in C[0,1]$ and $(\phi_p(u'))' \in L^1[0,1]$, then, BVP*

$$\left(\phi_p\left(u'\right)\right)'(t) = -y(t), \quad 0 \le t \le 1,$$
$$\phi_p\left(u'(0)\right) = \sum_{i=1}^{m-2} \beta_i \phi_p\left(u'(\eta_i)\right), \quad u(1) = g\left(u'(1)\right), \tag{8}$$

has a unique solution

$$u(t) = \int_t^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i \int_0^{\eta_i} y(r)\, dr + \int_0^s y(r)\, dr\right) ds + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i \int_0^{\eta_i} y(r)\, dr + \int_0^1 y(r)\, dr\right)\right), \tag{9}$$

where $\beta = \sum_{i=1}^{m-2} \beta_i$.

Define the operator $A : P \to E$ by

$$(Au)(t) = \int_t^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) \times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right)\right). \tag{10}$$

Obviously, A is well defined and $u \in E$ is a solution of BVP (4) if and only if u is a fixed point of A.

Lemma 6. *$A : P \to P$ is completely continuous.*

Proof. It is similar to the proof of Lemma 2.2 in [9]. □

Lemma 7. *For any $u \in P$, one has $\|Au\|_0 \le 2\|(Au)'\|_0$, $\|Au\| \le 2\|(Au)'\|_0$.*

Proof. From (10), we obtain

$$\begin{aligned}\|(Au)\|_0 &= \int_0^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds \\ &\quad + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) \times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right)\right) \\ &\le \int_0^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds \\ &\quad + \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\ &= 2\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\ &= 2\left|(Au)'(1)\right| \\ &= 2\left\|(Au)'\right\|_0.\end{aligned} \tag{11}$$

Since $\|Au\| = \max\{\|Au\|_0, \|(Au)'\|_0\}$, so we have $\|Au\| \le 2\|(Au)'\|_0$, which completes the proof. □

3. Main Results

For any $\delta \in (0, \min\{\eta_1, 1/2\})$, we define a nonnegative continuous concave function $\alpha : P \to R^+$ by $\alpha(u) = \min_{\delta\le t\le(1-\delta)} u(t)$. Obviously, the following two conclusions hold:

$$\alpha(u) = u(1-\delta) \le \|u\|_0, \qquad \alpha(Au) = Au(1-\delta), \quad \forall u \in P. \tag{12}$$

The main result of this paper is following.

Theorem 8. *Let $m_0 = \min_{0\le t\le 1} q(t)$ and $\beta = \sum_{i=1}^{m-2}\beta_i < 1$. Suppose (C_1), (C_2), and (C_3) hold. Suppose further that there exist numbers $\delta \in (0, \min\{\eta_1, 1/2\})$, a, b, c, and d such that $0 < a < b \le m_0\delta d/M < d \le c$, and*

- (H_1) *$f(u,v,w,l) \le (1-\beta)\phi_p(a/2M)$, for $(u,v,w,l) \in [0,a]\times[-a,0]\times[0,k_0a]\times[0,h_0a]$;*
- (H_2) *$f(u,v,w,l) \le (1-\beta)\phi_p(c/2M)$, for $(u,v,w,l) \in [0,c]\times[-c,0]\times[0,k_0c]\times[0,h_0c]$;*
- (H_3) *$f(u,v,w,l) > \phi_p(b/\delta L)$, for $(u,v,w,l) \in [b,d]\times[-d,0]\times[0,k_0d]\times[0,h_0d]$, where $L = \phi_q(\int_0^{1-\delta} q(t)dt)$;*
- (H_4) *$\min_{(u,v,w,l)\in J} f(u,v,w,l)\phi_p(M/(2m_0))\int_0^{1-\delta} q(t)dt \ge \max_{(u,v,w,l)\in J} f(u,v,w,l)\int_0^1 q(t)dt$, where $J = [0,c]\times[-c,0]\times[0,k_0c]\times[0,h_0c]$.*

Then, BVP (4) has at least three positive solutions u_1, u_2, and u_3 such that

$$\|u_1\| < a, \qquad b < \min_{\delta\le t<(1-\delta)} u_2(t), \qquad \|u_3\| > a, \qquad \min_{\delta\le t<(1-\delta)} u_3(t) < b. \tag{13}$$

Proof. We divide the proof into three steps.

Step 1. We prove $A\overline{P}_c \subset \overline{P}_c$, $A\overline{P}_a \subset \overline{P}_a$; that is, (ii) of Theorem 3. By Lemma 6, we have $A\overline{P}_c \subset P$, so $\forall u \in \overline{P}_c$, we get $0 \le u(t) \le c$, $-c \le u'(t) \le 0$, $0 \le (Tu)(t) \le k_0 c$, $0 \le (Su)(t) \le h_0 c$. For $t \in [0,1]$ and by (H_2)

$$\begin{aligned}
&\|(Au)\|_0 \\
&= \int_0^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) \times f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds \\
&\quad + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) \times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right)\right) \\
&\le \int_0^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds \\
&\quad + \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&= 2\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\le 2\phi_q\left(\frac{1}{1-\beta}\int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\le 2\phi_q\left(\phi_p\left(\frac{c}{2M}\right)\right)\phi_q\left(\int_0^1 q(r)\,dr\right) \\
&\le c.
\end{aligned}$$

$$\begin{aligned}
\left\|(Au)'\right\|_0 &= \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\le \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\le \phi_q\left(\frac{1}{1-\beta}\int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\le \phi_q\left(\phi_p\left(\frac{c}{2M}\right)\right)\phi_q\left(\int_0^1 q(r)\,dr\right) \\
&\le c.
\end{aligned} \tag{14}$$

Hence, $\|Au\| < c$ and $A\overline{P}_c \subset \overline{P}_c$. Similarly, we obtain $A\overline{P}_a \subset \overline{P}_a$.

Step 2. We show

$$\{u \in P(\alpha,b,d) : \alpha(u) > b\} \ne \phi, \tag{15}$$

$$\alpha(Au) > b, \quad \forall u \in P(\alpha,b,d), \tag{16}$$

that is, (i) of Theorem 3.

Let $u = (b+d)/2$, then $u \in P(\alpha,b,d)$, $\alpha(u) = (b+d)/2 > b$. Hence, (15) holds. For any $u \in P(\alpha,b,d)$, we have $b \le u(t) \le d$, $-d \le u'(t) \le 0$, $0 \le (Tu)(t) \le k_0 d$, $0 \le (Su)(t) \le h_0 d$, $t \in [0, 1-\delta]$, so by (H_3), we have

$$\begin{aligned}
\alpha(Au) &= \min_{t\in[\delta,1-\delta]}(Au)(t) = (Au)(1-\delta) \\
&= \int_{1-\delta}^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^s q(r) f\left(u,u',Tu,Su\right)dr\right)ds \\
&\quad + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right)\right) \\
&\ge \delta\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^{1-\delta} q(r) f\left(u,u',Tu,Su\right)dr\right) \\
&\quad + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r) f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r) f\left(u,u',Tu,Su\right)dr\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \delta\phi_q\left(\int_0^{1-\delta} q(r)\, f\left(u,u',Tu,Su\right)dr\right)\\
&\geq \delta\phi_q\left(\phi_p\left(\frac{b}{\delta L}\right)\right)\phi_q\left(\int_0^{1-\delta} q(r)\,dr\right)\\
&= b.
\end{aligned} \tag{17}$$

Hence (16) holds.

Step 3. We show that $\alpha(Au) > b$ for all $u \in P(\alpha,b,c)$ with $\|Au\| > d$, that is, (iii) of Theorem 3.

If $u \in P(\alpha,b,c)$ with $\|Au\| > d$, we obtain $0 \leq u(t) \leq c$, $-c \leq u'(t) \leq 0$, $0 \leq (Tu)(t) \leq k_0 c$, $0 \leq (Su)(t) \leq h_0 c$, for any $t \in [0,1]$, and so by (H_4), we have

$$\begin{aligned}
&\phi_p\left(\frac{M}{2m_0}\right)\int_0^{1-\delta} q(r)\, f\left(u,u',Tu,Su\right)dr\\
&\qquad \geq \int_0^1 q(r)\, f\left(u,u',Tu,Su\right)dr.
\end{aligned} \tag{18}$$

Furthermore, we have

$$\begin{aligned}
&\phi_p\left(\frac{M}{2m_0}\right)\int_0^{1-\delta} q(r)\, f\left(u,u',Tu,Su\right)dr\\
&\quad + \phi_p\left(\frac{M}{2m_0}\right)\frac{1}{1-\beta}\\
&\quad \times \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\, f\left(u,u',Tu,Su\right)dr\\
&\geq \int_0^1 q(r)\, f\left(u,u',Tu,Su\right)dr\\
&\quad + \frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\, f\left(u,u',Tu,Su\right)dr.
\end{aligned} \tag{19}$$

Therefore, by Lemma 7, we have

$$\begin{aligned}
\alpha(Au) &= (Au)(1-\delta)\\
&= \int_{1-\delta}^1 \phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\right.\\
&\qquad \left.\times f\left(u,u',Tu,Su\right)dr + \int_0^s q(r)\, f\left(u,u',Tu,Su\right)dr\right)ds\\
&\quad + g\left(-\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\right.\right.\\
&\qquad \left.\left.\times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r)\, f\left(u,u',Tu,Su\right)dr\right)\right)\\
&\geq \delta\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\, f\left(u,u',Tu,Su\right)dr\right.\\
&\qquad \left.+\int_0^{1-\delta} q(r)\, f\left(u,u',Tu,Su\right)dr\right)\\
&\geq \delta\phi_q\left(\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\right.\right.\\
&\qquad \left.\times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r)\, f\left(u,u',Tu,Su\right)dr\right)\\
&\qquad \left.\times\left(\phi_p\left(\frac{M}{2m_0}\right)\right)^{-1}\right)\\
&= \frac{2\delta m_0}{M}\phi_q\left(\frac{1}{1-\beta}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i} q(r)\right.\\
&\qquad \left.\times f\left(u,u',Tu,Su\right)dr + \int_0^1 q(r)\, f\left(u,u',Tu,Su\right)dr\right)\\
&= \frac{2\delta m_0}{M}\left|(Au)'(1)\right|\\
&= \frac{2\delta m_0}{M}\left\|(Au)'\right\|_0\\
&\geq \frac{\delta m_0}{M}\left\|(Au)'\right\|\\
&> \frac{\delta m_0}{M}d\\
&\geq b.
\end{aligned} \tag{20}$$

Hence, by Theorem 3, the results of Theorem 8 hold. This completes the proof of Theorem 8. □

4. Example

Consider the following BVP:

$$\begin{aligned}
&\left(\left|u'\right|u'\right)'(t) + q(t)\\
&\quad \times f\left(u(t),u'(t),(Tu)(t),(Su)(t)\right) = 0,\\
&\qquad 0 \leq t \leq 1,
\end{aligned}$$

$$\left|u'(0)\right| u'(0) = \frac{1}{4}\left|u'\left(\frac{2}{5}\right)\right| u'\left(\frac{2}{5}\right) + \frac{1}{4}\left|u'\left(\frac{1}{2}\right)\right| u'\left(\frac{1}{2}\right), \quad u(1) = g\left(u'(1)\right), \tag{21}$$

where

$$q(t) = \begin{cases} t^{-1/2}, & 0 \le t \le \frac{16}{25}, \\ -\frac{187}{54}t + \frac{9359}{2700}, & \frac{16}{25} \le t \le 1. \end{cases}$$

$$f(u,v,w,l) = \begin{cases} 10u^{10} + \frac{2+\sin v}{1000} + \frac{\sqrt[3]{w}}{1000} + \frac{\sqrt{l}}{1000}, \\ \qquad 0 \le u \le 1,\ v \le 0,\ w,l \ge 0, \\ 10\sqrt[10]{u} + \frac{2+\sin v}{1000} + \frac{\sqrt[3]{w}}{1000} + \frac{\sqrt{l}}{1000}, \\ \qquad 1 \le u,\ v \le 0,\ w,l \ge 0. \end{cases} \tag{22}$$

$$g(v) = \begin{cases} v^{1/2}, & |v| \ge 1, \\ v^2, & |v| < 1. \end{cases}$$

Proof. Since $M = \sqrt{2}$ and $m_0 = 1/300$, $\beta = 1/2$, $\delta = 9/25$, $a = 1/2$, $b = 1$, $d = 1296$, $c = 1600$, $k(t,s) = 1$ and $h(t,s) = 1$, then we can obtain $0 < a < b \le (m_0\delta d)/M < d \le c$, and

$$L = \sqrt{\int_0^{1-9/25} q(t)\,dt} = \sqrt{\int_0^{16/25} t^{-1/2}dt} = \sqrt{\frac{8}{5}}, \quad \phi_p\left(\frac{a}{2M}\right) = \phi_p\left(\frac{1/2}{2\sqrt{2}}\right) = \frac{1}{32}, \quad \phi_p\left(\frac{c}{2M}\right) = \phi_p\left(\frac{1600}{2\sqrt{2}}\right) = 320000, \quad \phi_p\left(\frac{b}{\delta L}\right) = \phi_p\left(\frac{1}{(9/25)\sqrt{8/5}}\right) = \frac{25^2}{9^2}\frac{5}{8} = \frac{3125}{648}. \tag{23}$$

Next, we show that (H_1)–(H_4) are satisfied.

If $0 \le u \le 1/2,\ -1/2 \le v \le 0,\ 0 \le w,\ l \le 1/2$, then

$$f(u,v,w,l) < 10\frac{1}{2}^{10} + \frac{3}{1000} + \frac{2}{1000} < (1-\beta)\phi_p\left(\frac{a}{2M}\right) = \frac{1}{64}. \tag{24}$$

So (H_1) is satisfied.

If $0 \le u \le 1600,\ -1600 \le v \le 0,\ 0 \le w,\ l \le 1600$, then

$$f(u,v,w,l) < 10\sqrt[10]{1600} + \frac{3}{1000} + \frac{40\times 2}{1000} < (1-\beta)\phi_p\left(\frac{c}{2M}\right) = 32000. \tag{25}$$

So (H_2) is satisfied.

If $1 \le u \le 1296,\ -1296 \le v \le 0,\ 0 \le w,\ l \le 1296$, then

$$f(u,v,w,l) > 10\sqrt[10]{1} + \frac{1}{1000} > \phi_p\left(\frac{b}{\delta L}\right) = \frac{3125}{648}. \tag{26}$$

So (H_3) is satisfied.

For any $(u,v,w,l) \in [0,1600]\times[-1600,0]\times[0,1600]\times[0,1600]$, we have

$$\min f(u,v,w,l) \ge \frac{1}{1000}, \quad \max f(u,v,w,l) \le 10\sqrt[10]{1600} + \frac{3}{1000} + \frac{40\times 2}{1000}, \quad \phi_p\left(\frac{M}{2m_0}\right) = 2\cdot 150^2, \quad \int_0^{1-\delta} q(t)\,dt = \frac{8}{5}, \quad \int_0^1 q(t)\,dt = \frac{1141}{625}. \tag{27}$$

Hence, it's easy to know that (H_4) is satisfied.

So by Theorem 8, we conclude that the BVP (21) has three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\| < \frac{1}{2}, \quad 1 < \min_{\delta\le t\le(1-\delta)} u_2(t), \quad \|u_3\| > \frac{1}{2}, \quad \min_{\delta\le t\le(1-\delta)} u_3(t) < 1. \tag{28}$$

□

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