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Research Article

Complex Oscillation of Higher-Order Linear Differential Equations with Coefficients Being Lacunary Series of Finite Iterated Order

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The authors introduce the lacunary series of finite iterated order and use them to investigate the growth of solutions of higher-order linear differential equations with entire coefficients of finite iterated order and obtain some results which improve and extend some previous results of Belaidi, 2006, Cao and Yi, 2007, Kinnunen, 1998, Laine and Wu, 2000, Tu and Chen, 2009, Tu and Deng, 2008, Tu and Deng, 2010, Tu and Liu, 2009, and Tu and Long, 2009.

1. Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and standard notations of the Nevanlinna theory of meromorphic functions (see [1–3]). In order to describe the growth of order of entire functions or meromorphic functions more precisely, we first introduce some notations about finite iterated order. Let us define inductively, for $r \in (0,+\infty)$, $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in \mathbb{N}$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $i \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the logarithmic measure of a set $E \subset (0,+\infty)$ by $m_l E = \int_E dt/t$, and the upper logarithmic density of $E \subset (0,+\infty)$ is defined by

$$\overline{\log\ \text{dens}}\, E = \overline{\lim_{r\to\infty}} \frac{m_l (E \bigcap [1,r])}{\log r}. \tag{1}$$

Throughout this paper, we use p to denote a positive integer. In the following, we recall some definitions of entire functions or meromorphic functions of finite iterated order (see [4–10]).

Definition 1. The p-iterated order of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r}. \tag{2}$$

Remark 2. If $f(z)$ is an entire function, then the p-iterated order of $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log r}. \tag{3}$$

If $p = 1$, the classical growth of order of $f(z)$ is defined by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_2 M(r,f)}{\log r}. \tag{4}$$

If $p = 2$, the hyperorder of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_3 M(r,f)}{\log r}. \tag{5}$$

Definition 3. If $f(z)$ is an entire function with $0 < \sigma_p(f) = \sigma < \infty$, then the p-iterated type of $f(z)$ is defined by

$$\tau_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p M(r,f)}{r^\sigma}. \quad (6)$$

Definition 4. The p-iterated lower order of an entire function $f(z)$ is defined by

$$\mu_p(f) = \underline{\lim}_{r\to\infty} \frac{\log_p T(r,f)}{\log r} = \underline{\lim}_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r}. \quad (7)$$

Definition 5. The finiteness degree of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{for } f(z) \text{ rational,} \\ \min\{p \in \mathbb{N} : \sigma_p(f) < \infty\} & \text{for } f \text{ transcendental for which some } p \in \mathbb{N} \text{ with } \sigma_p(f) < \infty \text{ exists,} \\ \infty & \text{or } f(z) \text{ with } \sigma_p(f) = \infty\ \forall p \in \mathbb{N}. \end{cases} \quad (8)$$

Definition 6. The p-iterated exponent of convergence of a-point of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,a)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,a)}{\log r}. \quad (9)$$

If $a = 0$, the p-iterated exponent of convergence of zero-sequence and distinct zero-sequence of a meromorphic function $f(z)$ are defined, respectively, by

$$\lambda_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,1/f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,1/f)}{\log r},$$
$$\overline{\lambda}_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r,1/f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r,1/f)}{\log r}. \quad (10)$$

The p-iterated lower exponent of convergence of zero-sequence and distinct zero-sequence of a meromorphic function $f(z)$ are defined, respectively, by

$$\underline{\lambda}_p(f) = \underline{\lim}_{r\to\infty} \frac{\log_p \overline{n}(r,1/f)}{\log r} = \underline{\lim}_{r\to\infty} \frac{\log_p \overline{N}(r,1/f)}{\log r},$$
$$\underline{\overline{\lambda}}_p(f) = \underline{\lim}_{r\to\infty} \frac{\log_p \overline{n}(r,1/f)}{\log r} = \underline{\lim}_{r\to\infty} \frac{\log_p \overline{N}(r,1/f)}{\log r}. \quad (11)$$

2. Introductions and Main Results

In the past ten years, many authors have investigated the complex oscillation properties of the higher-order linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \quad (12)$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F(z) \quad (13)$$

with $A_j(z)$ $(j = 0,\dots,k-1)$, $F(z)$ being entire functions or meromorphic functions of fast growing (e.g., see [4–12]), and obtained the following results.

Theorem A (see [8]). *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions, if $i(A_j) \le p$ $(j = 0,\dots,k-1)$, then $i(f) \le p+1$ and $\sigma_{p+1}(f) \le \max\{\sigma_p(A_j), j = 0,\dots,k-1\}$ hold for all solutions of* (12).

Theorem B (see [8]). *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions and let $i(A_0) = p$. If $i(A_j) < p$ or $\sigma_p(A_j) < \sigma_p(A_0)$ for all $j = 1,\dots,k-1$, then $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$ hold for all nontrivial solutions of* (12).

Theorem C (see [4, 12]). *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions and let $i(A_0) = p$. Assume that $\max\{\sigma_p(A_j) : j = 1,\dots,k-1\} \le \sigma_p(A_0)$ (> 0) and $\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0)$. Then, every solution $f(z) \not\equiv 0$ of* (12) *satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

Theorem D (see [10]). *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions of finite iterated order satisfying $i(A_0) = p$, $\sigma_p(A_0) = \sigma$, and $\overline{\lim}_{r\to\infty}(\sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0)) < 1$. Then, every nontrivial solution $f(z)$ of* (12) *satisfies $\sigma_{p+1}(f) = \sigma_p(A_0) = \sigma$.*

Theorem E (see [10]). *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions of finite iterated order satisfying $\max\{\sigma_p(A_j), j \ne 0\} < \mu_p(A_0) = \sigma_p(A_0)$ and $\overline{\lim}_{r\to\infty}(\sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0)) < 1$ $(r \notin E)$, where E is a set of r of finite linear measure, then every nontrivial solution $f(z)$ of* (12) *satisfies $\sigma_{p+1}(f) = \mu_p(A_0) = \sigma_p(A_0)$.*

Theorem F (see [5]). *Let $A_j(z)$ $(j = 0,\dots,k-1)$ be entire functions of finite iterated order such that there exists one transcendental A_d $(0 \le d \le k-1)$ satisfying $\sigma_p(A_j) \le \sigma_p(A_d) < \infty$ for all $j \ne s$, then* (12) *has at least one solution $f(z)$ that satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma_p(A_d)$.*

Remark 7. Theorems B–E are investigating the growth of solutions of (12) when the coefficients are of finite iterated order and $A_0(z)$ grows faster than other coefficients in (12). What can we have if there exists one middle coefficient $A_d(z)$ $(1 \le d \le k-1)$ such that $A_d(z)$ grows faster than other coefficients in (12) or (13)? Many authors have investigated this question when $A_d(z)$ is of finite order and obtained many results (e.g., see [13–15]). Here, our question is that under what conditions can we obtain similar results with Theorems B-C if $A_d(z)$ $(1 \le d \le k-1)$ is of finite iterated order and grows faster than other coefficients in (12) or (13).

In 2009, Tu and Liu make use of the proposition of lacunary power series to investigate the above question in the case $p = 1$ and obtain the following result.

Theorem G (see [15]). *Let $A_j(z)$ $(j = 0,\dots,k-1)$, $F(z)$ be entire functions satisfying $\max\{\sigma(A_j), j \ne d, \sigma(F)\} < \sigma(A_d) < \infty$ $(1 \le d \le k-1)$. Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function of regular growth such that the sequence of exponents $\{\lambda_n\}$ satisfies Fabry gap condition*

$$\frac{\lambda_n}{n} \longrightarrow \infty \quad (n \longrightarrow \infty), \quad (14)$$

then one has

(i) *if* $F(z) \equiv 0$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\sigma_2(f) = \sigma(A_d)$;

(ii) *if* $F(z) \not\equiv 0$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_d)$.

In this paper, we continue our research in this area and obtain the following results.

Theorem 8. *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ *be entire functions of finite iterated order and satisfying* $0 < \max\{\sigma_p(A_j), j \neq d\} \leq \sigma_p(A_d) < \infty$ *and* $\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_d)\} < \tau_p(A_d)$ $(0 \leq d \leq k-1)$. *Suppose that* $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *is an entire function such that the sequence of exponents* $\{\lambda_n\}$ *satisfies*

$$\frac{\lambda_n}{n} > (\log n)^{2+\eta} \quad (\eta > 0,\ n \in \mathbb{N}), \tag{15}$$

for some $\eta > 0$, *then one has*

(i) *if* $\sigma_p(F) < \sigma_p(A_d)$ *or* $\sigma_p(F) = \sigma_p(A_d)$ *and* $\tau_p(F) < \tau_p(A_d)$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\sigma_{p+1}(f) = \sigma_p(A_d)$; *furthermore if* $F(z) \not\equiv 0$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d)$;

(ii) *if* $\sigma_p(F) > \sigma_p(A_d)$ *and* $\sigma_{p+1}(F) \leq \sigma_p(A_d)$, *then all solutions of* (13) *satisfy* $\sigma_p(f) \geq \sigma_p(F)$ *and* $\sigma_{p+1}(f) \leq \sigma_p(A_d)$;

(iii) *if* $\sigma_{p+1}(F) > \sigma_p(A_d)$, *then all solutions of* (13) *satisfy* $\sigma_{p+1}(f) = \sigma_{p+1}(F)$, *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(F)$ *holds for all solutions of* (13) *with at most one exceptional solution* f_0 *satisfying* $\lambda_{p+1}(f_0) < \sigma_{p+1}(F)$.

Remark 9. If $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function of finite order and the sequence of exponents $\{\lambda_n\}$ satisfies (14), then (18) in Lemma 15 holds for $f(z)$, but for entire functions of infinite order, (14) certainly does not imply (18) in Lemma 15 (see [8]); therefore, we need more stringent gap condition (15) which is sufficient and unnecessary for Theorem 8.

Theorem 10. *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ *be entire functions of finite iterated order satisfying* $\max\{\sigma_p(A_j), j \neq d, \sigma_p(F)\} < \mu_p(A_d) = \sigma_p(A_d) = \sigma < \infty$ $(0 \leq d \leq k-1)$. *Suppose that* $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *is an entire function such that the sequence of exponents* $\{\lambda_n\}$ *satisfies gap condition* (15), *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\mu_{p+1}(f) = \sigma_{p+1}(f) = \sigma$. *Furthermore if* $F(z) \not\equiv 0$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\underline{\overline{\lambda}}_{p+1}(f) = \underline{\lambda}_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma$.

Theorem 11. *Let* $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ *be entire functions satisfying* $\max\{\sigma_p(A_j), j \neq d\} < \sigma_p(A_d) < \infty$ $(0 \leq d \leq k-1)$. *Suppose that* $T(r, A_d) \sim \log M(r, A_d)$ *as* $r \to \infty$ *outside a set of* r *of finite logarithmic measure, then one has*

(i) *if* $\sigma_p(F) < \sigma_p(A_d)$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\sigma_{p+1}(f) = \sigma_p(A_d)$; *furthermore, if* $F(z) \not\equiv 0$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d)$;

(ii) *if* $\sigma_p(F) > \sigma_p(A_d)$ *and* $\sigma_{p+1}(F) \leq \sigma_p(A_d)$, *then all solutions of* (13) *satisfy* $\sigma_p(f) \geq \sigma_p(F)$ *and* $\sigma_{p+1}(f) \leq \sigma_p(A_d)$;

(iii) *if* $\sigma_{p+1}(F) > \sigma_p(A_d)$, *then all solutions of* (13) *satisfy* $\sigma_{p+1}(f) = \sigma_{p+1}(F)$, *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(F)$ *holds for all solutions of* (13) *with at most one exceptional solution* f_0 *satisfying* $\lambda_{p+1}(f_0) < \sigma_{p+1}(F)$;

(iv) *if* $\mu_p(A_d) = \sigma_p(A_d) = \sigma$ *and* $F(z) \not\equiv 0$ *and* $\sigma_p(F) < \sigma$, *then every transcendental solution* $f(z)$ *of* (13) *satisfies* $\underline{\overline{\lambda}}_{p+1}(f) = \underline{\lambda}_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma$.

Remark 12. Theorem 10 implies that all the solutions of (13) are of regular growth if A_d is of regular growth under some conditions; Theorem 11 is an improvement of the Theorem in [14, page 2694] and Theorems 1-2 in [16, page 624]. In fact, by Lemma 15, the gap condition (15) in Theorem 8 implies that $T(r, A_d) \sim \log M(r, A_d)$ as $r \to \infty$ outside a set of r of finite logarithmic measure; therefore, Theorem 11 is a generalization of Theorem 8 in a sense, but the condition on A_d in Theorem 8 is more stringent than that in Theorem 11. In addition, Theorems 8–11 may have polynomial solutions of degree $< d$ if $d > 1$.

3. Preliminary Lemmas

Lemma 13 (see [17]). *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant, for any given constant and for any given* $\varepsilon > 0$,

(i) *there exist a constant* $B > 0$ *and a set* $E_1 \subset (0, +\infty)$ *having finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin E_1$, *one has*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\right]^{j-i} \quad (0 \leq i < j). \tag{16}$$

(ii) *There exists a set* $H_1 \subset [0, 2\pi)$ *that has linear measure zero a constant* $B > 0$ *that depends only on* α, *for any* $\theta \in [0, 2\pi) \setminus H_1$, *there exists a constant* $R_0 = R_0(\theta) > 1$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| = r > R_0$, *one has*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B[T(\alpha r, f) \log T(\alpha r, f)]^{j-i} \quad (0 \leq i < j). \tag{17}$$

Remark 14. Throughout this paper, we use $E_1 \subset (0, \infty)$ to denote a set having finite logarithmic measure or finite linear measure, not always the same at each occurrence.

Lemma 15 (see [18]). *Let $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function and the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (15). Then for any given $\varepsilon > 0$,*

$$\log L(r, f) > (1-\varepsilon)\log M(r, f) \tag{18}$$

holds outside a set of finite logarithmic measure, where $M(r, f) = \sup_{|z|=r} |f(z)|$, $L(r, f) = \inf_{|z|=r} |f(z)|$.

Lemma 16 (see [4]). *Let $f(z)$ be an entire function of finite iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) = \tau > 0$, then for any given $\beta < \tau$, there exists a set $E_2 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, one has*

$$M(r, f) > \exp_p\{\beta r^{\sigma}\}. \tag{19}$$

Lemma 17. *Let $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of finite iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) = \tau > 0$ such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (15). Then, for any given $\beta > \tau$, there exists a set $E_3 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_3$, one has*

$$|f(z)| > \exp_p\{\beta r^{\sigma}\}. \tag{20}$$

Proof. By Lemma 15, for any given $\varepsilon > 0$, we have

$$|f(z)| \geq L(r, f) > M(r, f)^{(1-\varepsilon)}, \quad (r \notin E_1). \tag{21}$$

For any given $\beta < \tau$, we can choose β_1 and sufficiently small $\varepsilon > 0$ such that $\beta < \beta_1 < \tau$ and $\beta < (1-\varepsilon)\beta_1 < \tau$; by Lemma 16, there exists a set $E_2 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_2 \setminus E_1$, we have

$$\begin{aligned} |f(z)| &> M(r, f)^{(1-\varepsilon)} \\ &> \left(\exp_p\{\beta_1 r^{\sigma}\}\right)^{(1-\varepsilon)} > \exp_p\{\beta r^{\sigma}\}, \end{aligned} \tag{22}$$

where $E_3 = E_2 \setminus E_1$ is a set having infinite logarithmic measure. □

Lemma 18 (see [13]). *Let $f(z)$ be a transcendental entire function. Then, there is a set $E_1 \subset (0, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$, one has*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j, \quad (j \in \mathbb{N}). \tag{23}$$

Lemma 19 (see [7, 9, 10]). *Let $A_0(z), \ldots, A_{k-1}(z)$, $F(z) \not\equiv 0$ be meromorphic functions, and let $f(z)$ be a meromorphic solution of (13) satisfying one of the following conditions:*

(i) $\max\{i(F) = q, i(A_j)(j = 0, \ldots, k-1)\} < i(f) = p + 1$ $(p, q \in \mathbb{N})$;

(ii) $b = \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, \ldots, k-1)\} < \sigma_{p+1}(f)$,

then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f)$.

Lemma 20 (see [8]). *Let $A_0(z), \ldots, A_{k-1}(z)$, $F(z)$ be entire functions of finite iterated order, if $i(A_j) \leq p$, $i(F) \leq p$ $(j = 0, \ldots, k-1)$. Then $\sigma_{p+1}(f) \leq \max\{\sigma_p(A_j), \sigma_p(F), j = 0, \ldots, k-1\}$ holds for all solutions of (13).*

Lemma 21 (see [2]). *Let $g : (0, +\infty) \to R, h : (0, +\infty) \to R$ be monotone increasing functions such that*

(i) *$g(r) \leq h(r)$ outside of an exceptional set of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

(ii) *$g(r) \leq h(r)$ outside of an exceptional set of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^{\alpha})$ for all $r > r_0$.*

Lemma 22 (see [19]). *Let $f(z)$ be an entire function of finite iterated order satisfying $\mu_p(f) = \mu < \infty$. Then, for any given $\varepsilon > 0$, there exists a set $E_4 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, one has*

$$M(r, f) < \exp_p\{r^{\mu+\varepsilon}\}. \tag{24}$$

Lemma 23 (see [2, 20]). *Let $f(z)$ be a transcendental entire function, let $0 < \eta_1 < 1/4$ and z_r a point such that $|z_r| = r$ and that $|f(z_r)| > M(r, f)\nu_f(r)^{-(1/4)+\eta_1}$ holds. Then, there exists a set $E_1 \subset (0, +\infty)$ of finite logarithmic measure such that*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu_f(r)}{z_r}\right)^j (1 + o(1)) \tag{25}$$

holds for all $j \in \mathbb{N}$ and all $r \notin E_1$, where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 24 (see [7, 9]). *Let $f(z)$ be an entire function of finite iterated order satisfying $\sigma_p(f) = \sigma$, $\mu_q(f) = \mu$, $p, q \in \mathbb{N}$. Then, one has*

$$\begin{aligned} \overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log r} &= \sigma, \\ \underline{\lim_{r\to\infty}} \frac{\log_q \nu_f(r)}{\log r} &= \mu. \end{aligned} \tag{26}$$

Lemma 25. *Let $A_j(z)$ $(j = 0, \ldots, k-1)$, $F(z)$ be entire functions of finite iterated order satisfying $\max\{\sigma_p(A_j), j \neq d, \sigma_p(F)\} \leq \mu_p(A_d) < \infty$. Then, every solution $f(z)$ of (13) satisfies $\mu_{p+1}(f) \leq \mu_p(A_d)$.*

Proof. By (13), we have

$$\begin{aligned} \left|\frac{f^{(k)}}{f}\right| \leq &\left|A_{k-1}\frac{f^{(k-1)}}{f}\right| \\ &+ \cdots + \left|A_1\frac{f'}{f}\right| + |A_0| + \left|\frac{F(z)}{f(z)}\right|. \end{aligned} \tag{27}$$

By Lemma 23, there exists a set E_1 having finite logarithmic measure such that for all $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \quad (j \in \mathbb{N}). \tag{28}$$

By Lemma 22, for any given $\varepsilon > 0$, there exists a set E_4 having infinite logarithmic measure such that for all $|z| = r \in E_4$ and $|f(z)| = M(r, f) > 1$, we have

$$\begin{aligned} \left|A_d(z)\right| &\le \exp_p\left\{r^{\mu_p(A_d)+\varepsilon}\right\}, \\ \left|A_j(z)\right| &\le \exp_p\left\{r^{\mu_p(A_d)+\varepsilon}\right\}, \\ \left|\frac{F(z)}{f(z)}\right| &\le |F(z)| < \exp_p\left\{r^{\mu_p(A_d)+\varepsilon}\right\}. \end{aligned} \tag{29}$$

Hence from (27)–(29), for any given $\varepsilon > 0$ and for all z satisfying $|z| = r \in E_4 \setminus E_1$ and $|f(z)| = M(r, f)$, we have

$$\begin{aligned} &\left(\frac{\nu_f(r)}{r}\right)^k (1 + o(1)) \\ &\quad \le (k+1)\left(\frac{\nu_f(r)}{r}\right)^{k-1} (1 + o(1)) \exp_p\left\{r^{\mu_p(A_d)+\varepsilon}\right\}. \end{aligned} \tag{30}$$

By (30) and Lemma 24, we have $\mu_{p+1}(f) \le \mu_p(A_d)$. Therefore, we complete the proof of Lemma 25. □

Lemma 26. *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions of finite iterated order; if f is a meromorphic solution of the* (13) *and satisfies $b = \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j), j = 0, \ldots, k-1\} < \mu_{p+1}(f)$, then $\overline{\lambda}_{p+1}(f) = \underline{\lambda}_{p+1}(f) = \mu_{p+1}(f)$.*

Proof. By (13), we have

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right). \tag{31}$$

By (31), we get

$$N\left(r, \frac{1}{f}\right) \le k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N\left(r, A_j\right). \tag{32}$$

By the lemma of logarithmic derivative and (31), we have

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\le m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m\left(r, A_j\right) \\ &\quad + O\{\log(rT(r, f))\} \quad (r \notin E_1). \end{aligned} \tag{33}$$

By (32) and (33), we have

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\le k\overline{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{F}\right) \\ &\quad + \sum_{j=0}^{k-1} T\left(r, A_j\right) + O\{\log(rT(r, f))\} \\ &= k\overline{N}\left(r, \frac{1}{f}\right) + T(r, F) + \sum_{j=0}^{k-1} T\left(r, A_j\right) \\ &\quad + O\{\log(rT(r, f))\}, \quad (r \notin E_1). \end{aligned} \tag{34}$$

Since $\max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j), j = 0, 1, \ldots, k-1\} < \mu_{p+1}(f)$, for sufficiently large r, we have

$$\begin{aligned} T(r, F) &= o\{T(r, f)\}, \\ T\left(r, A_j\right) &= o\{T(r, f)\}, \quad j = 0, \ldots, k-1, \\ \log(rT(r, f)) &= o\{T(r, f)\}. \end{aligned} \tag{35}$$

By (34)-(35), we have

$$(1 - o(1))\, T(r, f) \le k\overline{N}\left(r, \frac{1}{f}\right), \quad (r \notin E_1). \tag{36}$$

By Lemma 21 (i) and by (36), we have $\overline{\lambda}_{p+1}(f) = \underline{\lambda}_{p+1}(f) = \mu_{p+1}(f)$. □

Lemma 27. *Let $f(z)$ be a transcendental entire function, for each sufficiently large $|z| = r$, and let $z_r = re^{i\theta_r}$ be a point satisfying $|f(z_r)| = M(r, f)$. Then, there exists a constant δ_r (> 0) such that for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, one has*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \quad (j \in \mathbb{N}). \tag{37}$$

Proof. If $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r, f)$, since $|f(z)|$ is continuous in $|z| = r$, then there exists a constant δ_r (> 0) such that for all z satisfying $|z| = r$ (large enough) and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\begin{aligned} &\left|\left|f(z)\right| - \left|f(z_r)\right|\right| < \varepsilon; \\ &\text{that is, } \left|f(z)\right| > \frac{1}{2}\left|f(z_r)\right| \\ &\qquad = \frac{1}{2} M(r, f) \\ &\qquad > M(r, f)\, \nu_f(r)^{-(1/4)+\eta_1}. \end{aligned} \tag{38}$$

By Lemma 23, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \quad (j \in \mathbb{N}) \tag{39}$$

holds for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. □

Lemma 28. *Let $f(z)$ be a transcendental entire function, for each sufficiently large $|z| = r$, and let $z_r = re^{i\theta_r}$ be a point satisfying $|f(z_r)| = M(r, f)$. Then, there exists a constant δ_r (> 0) such that for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, one has*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \le 2r^j \quad (j \in \mathbb{N}). \tag{40}$$

Proof. If $z_r = re^{i\theta_r}$ is a point satisfying $|f(z_r)| = M(r, f)$, then by Lemma 27 there exists a constant δ_r (> 0) such that for all

z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)). \quad (41)$$

Since $f(z)$ is transcendental, we have $\nu_f(r) \to \infty$ $(r \to \infty)$. Hence by (41), for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\begin{gathered}\left|\frac{f^{(j)}(z)}{f(z)}\right| \geq \frac{1}{2} r^{-j}; \\ \text{that is, } \left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j \quad (j \in \mathbb{N}).\end{gathered} \quad (42)$$

Therefore, we complete the proof of Lemma 28. □

Lemma 29. *Let $f(z)$ be an entire function of order $0 < \sigma_p(f) = \sigma < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_5 \subset (0, +\infty)$ with positive upper logarithmic density such that for all $|z| = r \in E_5$, one has*

$$M(r, f) > \exp_p\{r^{\sigma-\varepsilon}\}. \quad (43)$$

Proof. Since $0 < \sigma_p(f) = \sigma < \infty$, then for any given $\varepsilon > 0$, there exists an increasing sequence $\{r_n\}$ tending to ∞ such that for $n \geq n_4$, we have

$$M(r_n, f) > \exp_p\left\{r_n^{\sigma-(\varepsilon/2)}\right\}. \quad (44)$$

Since $\varepsilon > 0$, we can choose α to satisfy $1 < \alpha < (\sigma-(\varepsilon/2))/(\sigma-\varepsilon)$. Then for all $r \in [r_n, r_n^\alpha]$ $(n \geq n_4)$, we have

$$\begin{aligned} M(r, f) &\geq M(r_n, f) > \exp_p\left\{r_n^{\sigma-(\varepsilon/2)}\right\} \\ &\geq \exp_p\left\{r^{(\sigma-(\varepsilon/2))/\alpha}\right\} > \exp_p\{r^{\sigma-\varepsilon}\}. \end{aligned} \quad (45)$$

Setting $E_5 = \bigcup_{n=n_4}^{\infty} [r_n, r_n^\alpha]$, we have

$$\begin{aligned} \overline{\log \operatorname{dens}} E_5 &= \overline{\lim_{r\to\infty}} \frac{m_l(E_5 \cap [1, r])}{\log r} \\ &\geq \overline{\lim_{n\to\infty}} \frac{m_l(E_5 \cap [1, r_n^\alpha])}{\log r_n^\alpha} \\ &\geq \lim_{n\to\infty} \frac{m_l([r_n, r_n^\alpha])}{\log r_n^\alpha} = \frac{\alpha - 1}{\alpha} > 0. \end{aligned} \quad (46)$$

Thus, Lemma 29 is proved. □

Lemma 30. *Let $f(z)$ be a transcendental entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $T(r, f) \sim \log M(r, f)$ as $r \to \infty$ outside a set r of finite logarithmic measure. Then for any given $\varepsilon > 0$, there exists a set $E_6 \subset (0, +\infty)$ with positive upper logarithmic density and a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that for all z satisfying $r \in E_6$ and $\arg z = \theta \in [0, 2\pi) \setminus H_2$, one has*

$$\left|f\left(re^{i\theta}\right)\right| > \exp_p\{r^{\sigma-\varepsilon}\}. \quad (47)$$

Proof. Since $m(r, f) \sim \log M(r, f)$ as $r \to \infty$ $(r \notin E_1)$, then by the definition of $m(r, f)$, there exists a set $H_2 \subset [0, 2\pi)$ having linear measure zero such that for all z satisfying $\arg z = \theta \in [0, 2\pi) \setminus H_2$, we have

$$\left|f\left(re^{i\theta}\right)\right| > M(r, f)^{1-\varepsilon} \quad (r \notin E_1). \quad (48)$$

By Lemma 29, for any given $\varepsilon > 0$, there exists a set $E_6 \subset (0, +\infty)$ with positive upper logarithmic density, we have

$$M(r, f) > \exp_p\left\{r^{\sigma-(\varepsilon/2)}\right\}. \quad (49)$$

By (48) and (49), for any given $\varepsilon > 0$ and for all z satisfying $|z| = r \in E_6 \setminus E_1$ and $\arg z = \theta \in [0, 2\pi) \setminus H_2$, we have

$$\left|f\left(re^{i\theta}\right)\right| > M(r, f)^{1-\varepsilon} > \left(\exp_p\left\{r^{\sigma-(\varepsilon/2)}\right\}\right)^{1-\varepsilon} > \exp_p\{r^{\sigma-\varepsilon}\}. \quad (50)$$

Therefore, we complete the proof of Lemma 30. □

4. Proofs of Theorems 8–11

Proof of Theorem 8. (i) Assume that $f(z)$ is a transcendental solution of (13). By (13), we have

$$\begin{aligned} |A_d| \leq{}& \left|\frac{f^{(k)}}{f^{(d)}}\right| + \cdots + |A_{d+1}|\left|\frac{f^{(d+1)}}{f^{(d)}}\right| \\ &+ \left|\frac{f}{f^{(d)}}\right|\left(|A_{d-1}|\left|\frac{f^{(d-1)}}{f}\right| + \cdots + |A_0| + \left|\frac{F}{f}\right|\right). \end{aligned} \quad (51)$$

By Lemma 13 (i), there exists a set $E_1 \subset [1, \infty)$ having finite logarithmic measure and a constant $B > 0$ such that

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B(T(2r, f))^{2k}, \quad (0 \leq i < j \leq k) \quad (52)$$

holds for all $|z| = r \notin E_1$ and for sufficiently large r. Since $\max\{\sigma_p(A_j), j \neq d, \sigma_p(F)\} \leq \sigma_p(A_d)$ and $\max\{\tau_p(A_j), \tau_p(F) : \sigma_p(A_j) = \sigma_p(A_d) = \sigma_p(F)\} < \tau_p(A_d)$, we choose α_1, β_1 to satisfy $\max\{\tau_p(A_j), j \neq d, \tau_p(F)\} < \alpha_1 < \beta_1 < \tau_p(A_d)$; by Lemma 17, there exists a set $E_3 \subset (0, +\infty)$ having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_3$ and for sufficiently large r, we have

$$\begin{gathered} |A_d(z)| > \exp_p\left\{\beta_1 r^{\sigma_p(A_d)}\right\}, \\ |A_j(z)| < \exp_p\left\{\alpha_1 r^{\sigma_p(A_d)}\right\}, \quad j \neq d. \end{gathered} \quad (53)$$

By Lemma 18, there exists a set $E_1 \subset (0, +\infty)$ having finite logarithmic measure such that for all z satisfying $|f(z)| = M(r, f) > 1$ and $|z| = r \notin E_1$, we have

$$\begin{gathered} \left|\frac{f(z)}{f^{(d)}(z)}\right| \leq 2r^d, \\ \left|\frac{F(z)}{f(z)}\right| \leq |F(z)| < \exp_p\left\{\alpha_1 r^{\sigma_p(A_d)}\right\}. \end{gathered} \quad (54)$$

Hence from (51)–(54), for all z satisfying $|z| = r \in E_3 \setminus E_1$ and $|f(z)| = M(r, f)$, we have

$$\begin{aligned} &\exp_p \left\{\beta_1 r^{\sigma_p(A_d)}\right\} \\ &\quad \leq 2B(k+1) r^d \exp\left\{\alpha_1 r^{\sigma_p(A_d)}\right\} \left[T(2r, f)\right]^{2k}. \end{aligned} \tag{55}$$

By (55), we have

$$\sigma_{p+1}(f) = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} T(r,f)}{\log r} \geq \sigma_p(A_d). \tag{56}$$

On the other hand, by Lemma 20, we have $\sigma_{p+1}(f) \leq \sigma_p(A_d)$. Therefore, every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) = \sigma_p(A_d)$. Furthermore if $F(z) \not\equiv 0$, then by Lemma 19, we have that every transcendental solution $f(z)$ of (13) satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d)$.

(ii) We assume that f is a solution of (13). By the elementary theory of differential equations, all the solutions of (13) are entire functions and have the form

$$f = f^* + C_1 f_1 + C_2 f_2 + \cdots + C_k f_k, \tag{57}$$

where $C_1, \ldots, C_k$ are complex constants, $\{f_1, \ldots, f_k\}$ is a solution base of (12), and f^* is a solution of (13) and has the form

$$f^* = D_1 f_1 + D_2 f_2 + \cdots + D_k f_k, \tag{58}$$

where $D_1, \ldots, D_k$ are certain entire functions satisfying

$$D_j' = F \cdot G_j(f_1, \ldots, f_k) \cdot W(f_1, \ldots, f_k)^{-1} \quad (j = 1, \ldots, k), \tag{59}$$

where $G_j(f_1, \ldots, f_k)$ are differential polynomials in $f_1, \ldots, f_k$ and their derivative with constant coefficients, and $W(f_1, \ldots, f_k)$ is the Wronskian of $f_1, \ldots, f_k$. By Theorem A, we have $\sigma_{p+1}(f_j) \leq \sigma_p(A_d)$ $(j = 1, 2, \ldots, k)$; then by (57)–(59), we get

$$\begin{aligned} \sigma_{p+1}(f) &\leq \max\left\{\sigma_{p+1}(f_j), \sigma_{p+1}(F), j = 1, \ldots, k\right\} \\ &\leq \sigma_p(A_d). \end{aligned} \tag{60}$$

Since $\sigma_p(F) > \sigma_p(A_d)$, it is easy to see that $\sigma_p(f) \geq \sigma_p(F)$ by (13).

(iii) Suppose that f is a solution of (13), it is easy to see that $\sigma_{p+1}(f) \geq \sigma_{p+1}(F)$ by (13). On the other hand, since $\sigma_{p+1}(F) > \sigma_p(A_d)$ and by (57)–(59), we have

$$\begin{aligned} \sigma_{p+1}(f) &\leq \max\left\{\sigma_{p+1}(f_j), \sigma_{p+1}(F), j = 1, \ldots, k\right\} \\ &\leq \sigma_{p+1}(F). \end{aligned} \tag{61}$$

Therefore, all solutions of (13) satisfy $\sigma_{p+1}(f) = \sigma_{p+1}(F)$.

By the same proof in Theorem 4.2 in [8, page 401], we can obtain that all solutions of (13) satisfying $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(F)$ with at most one exceptional solution f_0 satisfying $\lambda_{p+1}(f_0) < \sigma_{p+1}(F)$. □

Proof of Theorem 10. Suppose that $f(z)$ is a transcendental solution of (13), by the same proof in Theorem 8, we have $\sigma_{p+1}(f) = \sigma_p(A_d) = \sigma$. Thus, it remains to show that $\mu_{p+1}(f) = \mu_p(A_d) = \sigma$. We choose α_2, β_2 to satisfy

$$\max\left\{\sigma_p(A_j), j \neq d, \sigma_p(F)\right\} < \alpha_2 < \beta_2 < \sigma. \tag{62}$$

Since the sequence of exponents $\{\lambda_n\}$ of A_d satisfies (15) and $\mu_p(A_d) = \sigma$, then by Lemma 15, there exists a set E_1 having finite logarithmic measure such that for all sufficiently large $r \notin E_1$, we have

$$\begin{aligned} &\left|A_d(z)\right| \geq \exp_p\left\{r^{\beta_2}\right\}, \\ &\left|A_j(z)\right| \leq \exp_p\left\{r^{\alpha_2}\right\}, \quad j \neq d. \end{aligned} \tag{63}$$

Hence from (51), (52), (54), and (63), for all z satisfying $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$, we have

$$\exp_p\left\{r^{\beta_2}\right\} \leq 2B(k+1) r^d \exp_p\left\{r^{\alpha_2}\right\} \left[T(2r, f)\right]^{2k}. \tag{64}$$

Since β_2 is arbitrarily close to σ, by (64) and Lemma 21 (ii), we have

$$\mu_{p+1}(f) = \underline{\lim_{r\to\infty}} \frac{\log_{p+1} T(r,f)}{\log r} \geq \sigma. \tag{65}$$

On the other hand, by Lemma 26, we have $\mu_{p+1}(f) \leq \mu_p(A_d) = \sigma$; therefore, every transcendental solution of (13) satisfies $\mu_{p+1}(f) = \sigma$. □

Proof of Theorem 11. (i) By Lemma 20, we know that every solution of (13) satisfies $\sigma_{p+1}(f) \leq \sigma_p(A_d)$. In the following, we show that every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) \geq \sigma_p(A_d)$. Suppose that $f(z)$ is a transcendental solution of (13). For each sufficiently large circle $|z| = r$, we take a point $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r, f)$. By Lemma 28, there exist a constant $\delta_r > 0$ and a set E_1 such that for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(d)}(z)}\right| \leq 2r^d. \tag{66}$$

By Lemma 13 (ii), there exist a set $H_1 \subset [0, 2\pi)$ having linear measure zero and a constant $B > 0$ such that for sufficiently large r and for all z satisfying $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left[T(2r, f)\right]^{2k} \quad (0 \leq i < j \leq k). \tag{67}$$

Setting $\max\{\sigma_p(A_j), \sigma_p(F), j \neq d\} = b < \sigma_p(A_d)$, for all z satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$ and for any given ε $(0 < 2\varepsilon < \sigma_p(A_d) - b)$, we have

$$\begin{aligned} &\left|A_j(z)\right| \leq \exp\left\{r^{b+\varepsilon}\right\} \quad (j \neq d), \\ &\left|\frac{F(z)}{f(z)}\right| \leq |F(z)| \leq \exp\left\{r^{b+\varepsilon}\right\}. \end{aligned} \tag{68}$$

Since $T(r, A_d) \sim \log M(r, A_d)$ as $r \to \infty$ $(r \notin E_1)$, by Lemma 30, for any given $\varepsilon > 0$, there exists a set $E_6 \subset (0, \infty)$ with $\overline{\log\text{dens}}\, E_6 > 0$ and a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that for all z satisfying $|z| = r \in E_6$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_2$, we have

$$\left|A_d(z)\right| > \exp_p\left\{r^{\sigma_p(A_d)-\varepsilon}\right\}. \tag{69}$$

Substituting (66)–(69) into (51), for all z satisfying $|z| = r \in E_6 \setminus E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\begin{aligned} &\exp_p\left\{r^{\sigma_p(A_d)-\varepsilon}\right\} \\ &\quad \le (k+1)\, B\left[T(2r, f)\right]^{2k} \cdot 2r^d \cdot \exp_p\left\{r^{b+\varepsilon}\right\}. \end{aligned} \tag{70}$$

From (70), we have $\sigma_{p+1}(f) \ge \sigma_p(A_d)$. Therefore, every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) = \sigma_p(A_d)$. Furthermore, if $F(z) \not\equiv 0$, then every transcendental solution $f(z)$ of (13) satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d)$.

(ii)–(iv) By the same proof in Theorems 8 and 10, we can obtain the conclusions (ii)–(iv). □

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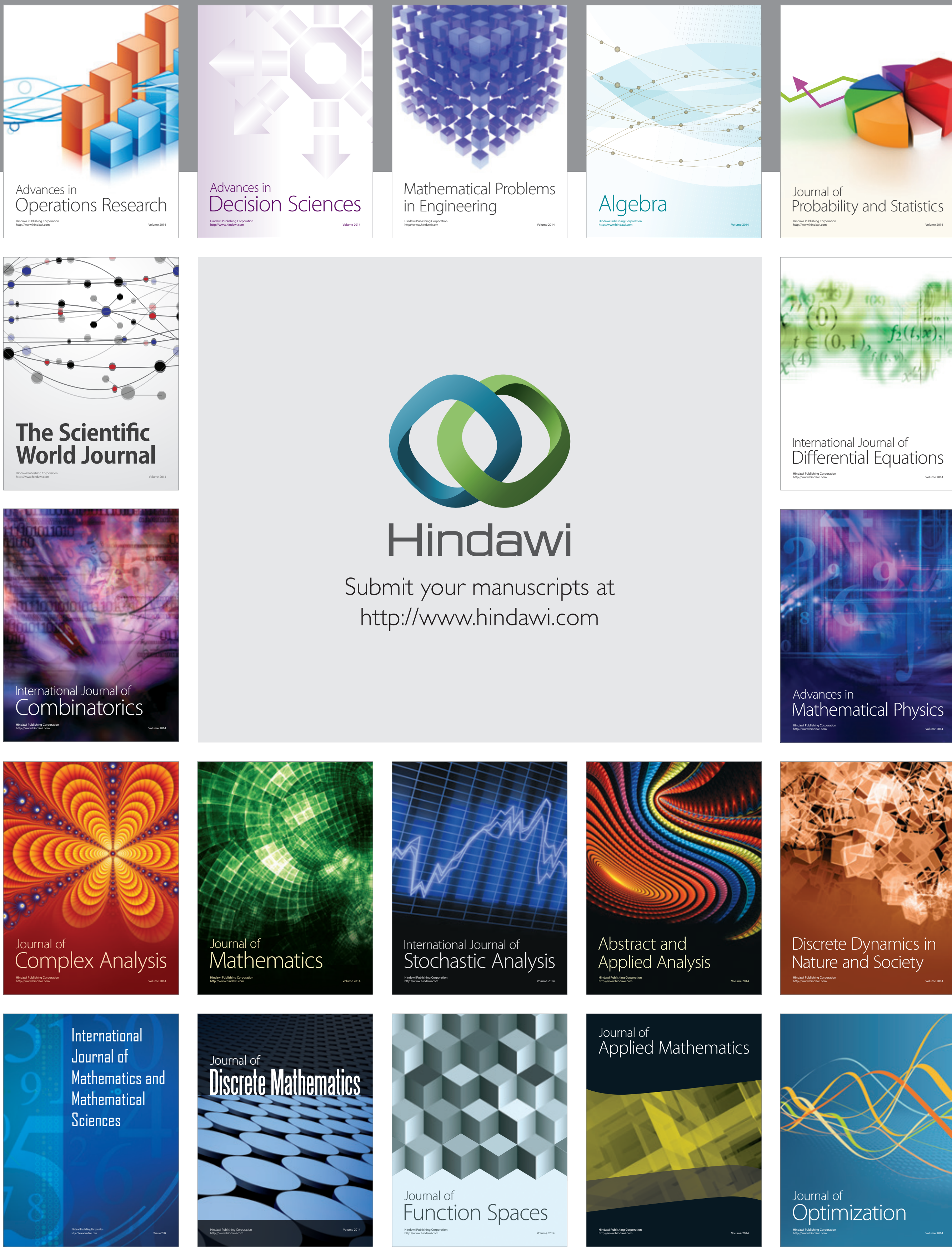

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