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Research Article

An Existence Result for Nonlocal Impulsive Second-Order Cauchy Problems with Finite Delay

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We deal with the existence of mild solutions of a class of nonlocal impulsive second-order functional differential equations with finite delay in a real Banach space X. An existence result on the mild solution is obtained by using the theory of the measures of noncompactness. An example is presented.

1. Introduction

The Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades (see, e.g., [1–5]).

The literature concerning second- and higher-order ordinary functional differential equations is very extensive. We only mention the works [1, 6–15], which are directly related to this work.

On the other hand, the impulsive conditions have advantages over traditional initial value problems because they can be used to model phenomena that cannot be modeled by traditional initial value problems, such as the dynamics of populations subject to abrupt changes (harvesting, diseases, etc.) (see [16–27] and references therein). For this reason, the theory of impulsive differential equations has become an important area of investigation in recent years. Partial differential equations of first and second order with impulses have been studied by Rogovchenko [26], Liu [25], Cardinali and Rubbioni [19], Liang et al. [24], Henríquez and Vásquez [1], Hernández et al., [21–23], Arthi and Balachandran [17], and so forth.

Moreover, we consider the nonlocal condition $x(0) = g(x)+x_0$, where g is a mapping from some space of functions so that it constitutes a nonlocal condition (see [24, 28–30] and the references therein), where it is demonstrated that nonlocal conditions have better effects in applications than traditional initial value problems.

In this paper, we pay our attention to the investigation of the existence of mild solutions to the following impulsive second-order functional differential equations with finite delay in a real Banach space X:

$$\frac{d^2}{dt^2}x(t) = Ax(t) + f(t, x_t, x(t)), \quad t \in (0,T],\ t \neq t_k,\ k = 1,2,\ldots,p, \tag{1}$$

$$x(t) = g(x)(t) + \phi(t), \quad t \in [-r, 0], \tag{2}$$

$$x'(0) = \xi \in X, \tag{3}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1,2,\ldots,p, \tag{4}$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $\{C(t)\}_{t\in\mathbf{R}}$ on X. f, g are given functions to be specified later. $\phi \in C([-r,0],X)$, where $C([a,b],X)$ denotes the space of all continuous functions from $[a,b]$ to X.

The impulsive moments $\{t_k\}$ are given such that $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $I_k : X \to X$ $(k = 1,2,\ldots,p)$ are appropriate functions, $\Delta x(t_k)$ represents the jump of a function x at t_k, which is defined by $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ are, respectively, the right and the left limits of x at t_k.

For any continuous function x defined on the interval $[-r,T]$ and any $t \in [0,T]$, we denote by x_t the element of $C([-r,0],X)$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

In this paper, motivated by above works, we study (1)–(4) in X and obtain the existence theorem based on theory on measures of noncompactness without the assumptions that the nonlinearity f satisfies a Lipschitz type condition and the cosine family of bounded linear operators $\{C(t)\}_{t\in\mathbf{R}}$ generated by A is compact.

2. Preliminaries

Throughout this paper, we set $J = [0, T]$, a compact interval in $\mathbf{R}$. We denote by X a Banach space with norm $\|\cdot\|$, by $L(X)$ the Banach space of all linear and bounded operators on X. We abbreviate $\|u\|_{L^1(J,\mathbf{R}^+)}$ with $\|u\|_{L^1}$, for any $u \in L^1(J, \mathbf{R}^+)$.

Let

$$\begin{aligned} PC(J,X) := \{ & x : J \longrightarrow X;\ x(t) \text{ is continuous at } t \neq t_k, \\ & \text{left continuous at } t = t_k, \text{ and} \\ & \text{the right limit } x(t_k^+) \text{ exists for } k = 1, 2, \ldots, p\}. \end{aligned} \tag{5}$$

It is easy to check that $PC(J, X)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in J} \|x(t)\|, \quad \text{for any } x \in PC(J, X). \tag{6}$$

We let $J_0 = (t_0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_p = (t_p, t_{p+1}]$.

For $\mathfrak{B} \subseteq PC(J, X)$, we denote by $\mathfrak{B}|_{\bar{J}_i}$ the set

$$\begin{aligned} \mathfrak{B}|_{\bar{J}_i} = \{ & x \in C([t_i, t_{i+1}], X);\ x(t_i) = v(t_i^+), x(t) = v(t), \\ & t \in J_i, v \in \mathfrak{B}\} \end{aligned} \tag{7}$$

$i = 0, 1, 2, \ldots, p$.

A family $\{C(t)\}_{t\in\mathbf{R}}$ in $L(X)$ is called a cosine function on X if

(i) $C(0) = I$ is the identity operator in X;
(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbf{R}$;
(iii) The map $t \rightarrow C(t)x$ is strongly continuous for each $x \in X$.

The associated sine function is the family $\{S(t)\}_{t\in\mathbf{R}}$ of operators defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad \text{for } x \in X,\ t \in \mathbf{R}. \tag{8}$$

One can define the infinitesimal generator A of $C(\cdot)$ by

$$\begin{aligned} D(A) &= \left\{x \in X;\ \lim_{t\to 0} 2t^{-2}(C(t)x - x) \in X\right\} \\ Ax &= \lim_{t\to 0} 2t^{-2}(C(t)x - x), \quad x \in D(A). \end{aligned} \tag{9}$$

In this paper, we assume there exist positive constants M and N such that

$$\|C(t)\| \le M, \qquad \|S(t)\| \le N \qquad \text{for every } t \in J. \tag{10}$$

The following properties are well known [6, 7, 11, 12]:

$$\begin{aligned} &\text{(i) } C(t)x \in D(A),\ C(t)Ax = AC(t)x \\ &\qquad\qquad \text{for } x \in D(A), t \in \mathbf{R}; \\ &\text{(ii) } S(t)x \in D(A),\ S(t)Ax = AS(t)x \\ &\qquad\qquad \text{for } x \in D(A), t \in \mathbf{R}; \\ &\text{(iii) } \int_0^t S(s)x\,ds \in D(A), \\ &\qquad A\int_0^t S(s)x\,ds = C(t)x - x \quad \text{for } x \in X,\ t \in \mathbf{R}; \\ &\text{(iv) } C(t)x - x = \int_0^t S(s)Ax\,ds \quad \text{for } x \in D(A),\ t \in \mathbf{R}. \end{aligned} \tag{11}$$

For more details on strongly continuous cosine and sine families, we refer the reader to [6, 7, 11, 12].

Next, we recall that the Hausdorff measure of noncompactness $\chi(\cdot)$ on each bounded subset Ω of Banach space Y is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0;\ \Omega \text{ has a finite } \varepsilon\text{-net in } X\}. \tag{12}$$

This measure of noncompactness satisfies some basic properties as follows.

Lemma 1 (see [31]). *Let Y be a real Banach space, and let $B, C \subseteq Y$ be bounded. Then*

(1) *$\chi(B) = 0$ if and only if B is precompact;*
(2) *$\chi(B) = \chi(\overline{B}) = \chi(convB)$, where $\overline{B}$ and $convB$ mean the closure and convex hull of B, respectively;*
(3) *$\chi(B) \le \chi(C)$ if $B \subseteq C$;*
(4) *$\chi(B \cup C) \le \max\{\chi(B), \chi(C)\}$;*
(5) *$\chi(B + C) \le \chi(B) + \chi(C)$, where $B + C = \{x + y;\ x \in B,\ y \in C\}$;*
(6) *$\chi(\alpha B) = |\alpha|\chi(B)$, for any $\alpha \in \mathbf{R}$;*
(7) *let Z be a Banach space and $Q : D(Q) \subseteq Y \rightarrow Z$ Lipschitz continuous with constant ν. Then $\chi(QB) \le \nu \cdot \chi(B)$ for all $B \subseteq D(Q)$ being bounded.*

Proposition 2 (see [32], Page 125). *Let Ω be a bounded set for a real Banach space X. Then, for every $\varepsilon > 0$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in Ω such that*

$$\chi(\Omega) \le 2\chi\left(\{x_n\}_{n=1}^{\infty}\right) + \varepsilon. \tag{13}$$

In the sequel, we make use of the following formulation of Theorem 4.2.2 of [33] obtained by using Theorem 2 of [34].

Proposition 3. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^1(J, X)$ such that there exist $v, q \in L_+^1([0, T])$ with the properties:*

(i) $\sup_{n\in\mathbf{N}}\|f_n(t)\| \le v(t)$, *a.e. $t \in J$;*
(ii) $\chi(\{f_n\}_{n=1}^{\infty}) \le q(t)$, *a.e. $t \in J$.*

Then, for every $t \in J$, we have

$$\chi\left(\left\{\int_0^t S(t-s) f_n(s)\,ds\right\}_{n=1}^{\infty}\right) \leq 2N \int_0^t q(s)\,ds, \tag{14}$$

where N is from (10).

A continuous map $Q : W \subseteq Y \to Y$ is said to be a χ-contraction if there exists a positive constant $\nu < 1$ such that $\chi(QC) \leq \nu \cdot \chi(C)$ for any bounded closed subset $C \subseteq W$.

Theorem 4 (see [31] (Darbo-Sadovskii)). *If $U \subseteq Y$ is bounded closed and convex, the continuous map $\mathcal{F} : U \to U$ is a χ-contraction, then the map $\mathcal{F}$ has at least one fixed point in U.*

Definition 5. A function $x : [-r, T] \to X$ is called a mild solution of system (1)–(4) if $x_0 = g(x) + \phi$, $x|_J \in PC(J, X)$ and

$$\begin{aligned} x(t) = {} & C(t)(\phi(0) + g(x)(0)) + S(t)\xi \\ & + \int_0^t S(t-s) f(s, x_s, x(s))\,ds \\ & + \sum_{0<t_k<t} C(t-t_k) I_k(x(t_k)), \quad t \in J. \end{aligned} \tag{15}$$

Remark 6. A mild solution of (1)–(4) satisfies (2) and (4). However, a mild solution may be not differentiable at zero.

3. Existence Result and Proof

In this section, we study the existence of mild solutions for the system (1)–(4).

Let $\mathfrak{F}(T)$ stand for the space

$$\begin{aligned} \mathfrak{F}(T) = \{ & x : [-r, T] \to X; x|_J \in PC(J, X), \\ & x_0 \in C([-r, 0], X)\} \end{aligned} \tag{16}$$

endowed with norm

$$\|x\|_{\mathfrak{F}(T)} = \sup_{t\in[-r,0]} \|x(t)\| + \sup_{t\in J} \|x(t)\|. \tag{17}$$

We will require the following assumptions.

(H1) (i) $f : J \times C([-r, 0], X) \times X \to X$ satisfies $f(\cdot, v, w) : J \to X$ is measurable for all $(v, w) \in C([-r, 0], X) \times X$ and $f(t, \cdot, \cdot) : C([-r, 0], X) \times X \to X$ is continuous for a.e. $t \in J$, and there exists a function $\mu(\cdot) \in L^1(J, \mathbf{R}^+)$ such that

$$\|f(t, v, w)\| \leq \mu(t)(1 + \|w\|) \tag{18}$$

for almost all $t \in J$;

(ii) there exists a function $\eta \in L^1(J, \mathbf{R}^+)$ such that for any bounded sets $D_1 \subset C([-r, 0], X)$, $D_2 \subset X$

$$\chi(f(t, D_1, D_2)) \leq \eta(t)\left(\sup_{\theta\in[-r,0]} \chi(D_1(\theta)) + \chi(D_2)\right), \quad \text{a.e. } t \in J. \tag{19}$$

(H2) $I_k : X \to X$ are compact operators and there exist positive constants M_1, M_2 such that

$$\|I_k(x)\| \leq M_1\|x\| + M_2, \quad \text{for any } x \in X,\ k = 1, 2, \ldots, p. \tag{20}$$

(H3) $g : C([-r, 0], X) \to X$ is a compact operator and there exists a constant $N_1 > 0$ such that

$$\|g(x)\|_{[-r,0]} \leq N_1 \quad \text{for all } x \in C([-r, 0], X), \tag{21}$$

where $\|g(x)\|_{[-r,0]} = \sup_{t\in[-r,0]} \|g(x)(t)\|$.

(H4) There exists $M^* \in (0, 1)$ such that $8N \int_0^T \eta(s)ds < M^*$.

Theorem 7. *Assume that (H1)–(H4) are satisfied, then there exists a mild solution of* (1)–(4) *on* $[-r, T]$ *provided that* $pMM_1 < 1$.

Proof. Define the operator $\Lambda : \mathfrak{F}(T) \to \mathfrak{F}(T)$ in the following way:

$$(\Lambda x)(t) = \begin{cases} g(x)(t) + \phi(t), & t \in [-r, 0], \\ C(t)(g(x)(0) + \phi(0)) + S(t)\xi & \\ + \int_0^t S(t-s) f(s, x_s, x(s))\,ds & \\ + \sum_{0<t_k<t} C(t-t_k) I_k(x(t_k)), & t \in J. \end{cases} \tag{22}$$

It is clear that the operator Λ is well defined, and the fixed point of Λ is the mild solution of problems (1)–(4).

The operator Λ can be written in the form $\Lambda = \Lambda_1 + \Lambda_2$, where the operators Λ_1, Λ_2 are defined as follows:

$$\begin{aligned} (\Lambda_1 x)(t) &= \begin{cases} g(x)(t) + \phi(t), & t \in [-r, 0], \\ C(t)(g(x)(0) + \phi(0)) & \\ +S(t)\xi, & t \in J, \end{cases} \\ (\Lambda_2 x)(t) &= \begin{cases} 0, & t \in [-r, 0], \\ \int_0^t S(t-s) f(s, x_s, x(s))\,ds & \\ + \sum_{0<t_k<t} C(t-t_k) I_k(x(t_k)), & t \in J. \end{cases} \end{aligned} \tag{23}$$

Obviously, under the assumptions of g, Λ_1 is continuous. For $t \in J$, we can prove that Λ_2 is continuous.

Indeed, let $\{x^n\}_{n\in\mathbf{N}}$ be a sequence such that $x^n \to x$ in $\mathfrak{F}(T)$ as $n \to \infty$. Since f satisfies (H1)(i), for almost every $t \in J$, we get

$$f(t, x_t^n, x^n(t)) \longrightarrow f(t, x_t, x(t)), \quad \text{as } n \longrightarrow \infty. \tag{24}$$

Noting that $x^n \rightarrow x$ in $\mathfrak{F}(T)$, we can see that there exists $\varepsilon > 0$ such that $\|x^n - x\|_{\mathfrak{F}(T)} \leq \varepsilon$ for n sufficiently large. Therefore, we have

$$\begin{aligned} &\left\| f\left(t, x_t^n, x^n(t)\right) - f\left(t, x_t, x(t)\right) \right\| \\ &\quad \leq \mu(t)\left(1 + \left\|x^n(t)\right\|\right) + \mu(t)\left(1 + \|x(t)\|\right) \\ &\quad \leq 2\mu(t) + \mu(t)\left\|x^n(t) - x(t)\right\| + 2\mu(t)\|x(t)\| \\ &\quad \leq 2\mu(t) + \mu(t)\varepsilon + 2\mu(t)\|x\|_{\mathfrak{F}(T)}. \end{aligned} \tag{25}$$

It follows from the Lebesgue's dominated convergence theorem that

$$\begin{aligned} &\int_0^t \left\| S(t-s)\left[f\left(s, x_s^n, x^n(s)\right) - f\left(s, x_s, x(s)\right)\right]\right\| ds \\ &\quad \leq N \int_0^t \left\| f\left(s, x_s^n, x^n(s)\right) - f\left(s, x_s, x(s)\right)\right\| ds \\ &\quad \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{26}$$

Moreover, noting that (H2), we obtain that

$$\lim_{n\to\infty} \left\|\Lambda_2 x^n - \Lambda_2 x\right\|_{\mathfrak{F}(T)} = 0. \tag{27}$$

This shows that Λ_2 is continuous. Therefore, Λ is continuous.

Let us introduce in the space $\mathfrak{F}(T)$ the equivalent norm defined as

$$\|x\|_* = \sup_{t\in[-r,0]} \|x(t)\| + \sup_{t\in J}\left(e^{-Lt}\|x(t)\|\right), \tag{28}$$

where $L > 0$ is a constant chosen so that

$$N \sup_{t\in J} \int_0^t \mu(s) e^{-L(t-s)} ds < 1. \tag{29}$$

Noting that for any $\psi \in L^1(J, X)$, we have

$$\lim_{L\to+\infty} \sup_{t\in J} \int_0^t e^{-L(t-s)} \psi(s)\, ds = 0, \tag{30}$$

so, we can take the appropriate L to satisfy (29).

Consider the set

$$B_\rho = \left\{x \in \mathfrak{F}(T); \|x\|_* \leq \rho\right\}, \tag{31}$$

where ρ is a constant chosen so that

$$\rho \geq \frac{N_1 + \|\phi\|_{[-r,0]} + \ell + pMM_2}{1 - pMM_1} > 0, \tag{32}$$

where $\ell := M(N_1 + \|\phi(0)\|) + N(\|\xi\| + \|\mu\|_{L^1})$ and $\|\phi\|_{[-r,0]} = \sup_{t\in[-r,0]}\|\phi(t)\|$.

Now, if $t \in [-r, 0]$, $x \in B_\rho$, then

$$\|(\Lambda x)(t)\| = \left\|g(x)(t) + \phi(t)\right\| \leq N_1 + \|\phi\|_{[-r,0]}. \tag{33}$$

For $t \in J$, $x \in B_\rho$, we have

$$\begin{aligned} \|(\Lambda x)(t)\| \leq{} & \left\|C(t)\left(g(x)(0) + \phi(0)\right)\right\| + \left\|S(t)\xi\right\| \\ & + \int_0^t \left\|S(t-s) f\left(s, x_s, x(s)\right)\right\| ds \\ & + \sum_{0<t_k<t} \left\|C\left(t-t_k\right) I_k\left(x\left(t_k\right)\right)\right\| \\ \leq{} & M\left(N_1 + \|\phi(0)\| + \sum_{0<t_k<t} \left\|I_k\left(x\left(t_k\right)\right)\right\|\right) \\ & + N\left(\|\xi\| + \int_0^t \mu(s)\left(1 + e^{Ls}e^{-Ls}\|x(s)\|\right) ds\right) \\ ={} & \ell + M \sum_{0<t_k<t} \left\|I_k\left(x\left(t_k\right)\right)\right\| \\ & + N\int_0^t \mu(s) e^{Ls} e^{-Ls} \|x(s)\|\, ds, \end{aligned} \tag{34}$$

then

$$\begin{aligned} e^{-Lt}\|(\Lambda x)(t)\| \leq{} & e^{-Lt}\left[\ell + M \sum_{0<t_k<t} \left\|I_k\left(x\left(t_k\right)\right)\right\| \right. \\ & \left. + N \int_0^t \mu(s) e^{Ls} e^{-Ls}\|x(s)\|\, ds\right] \\ \leq{} & \ell + \rho pMM_1 + pMM_2 \\ & + N\int_0^t \mu(s) e^{-L(t-s)} ds \cdot \|x\|_*, \end{aligned} \tag{35}$$

therefore,

$$\begin{aligned} &\sup_{t\in J}\left(e^{-Lt}\|(\Lambda x)(t)\|\right) \\ &\quad \leq \ell + pMM_2 \\ &\quad + \left[pMM_1 + \sup_{t\in J}\left(N\int_0^t \mu(s) e^{-L(t-s)} ds\right)\right]\rho. \end{aligned} \tag{36}$$

It results that

$$\begin{aligned} \|\Lambda x\|_* ={} & \sup_{t\in[-r,0]} \|(\Lambda x)(t)\| + \sup_{t\in J}\left(e^{-Lt}\|(\Lambda x)(t)\|\right) \\ \leq{} & N_1 + \|\phi\|_{[-r,0]} + \ell + pMM_2 \\ & + \left(pMM_1 + N \sup_{t\in J} \int_0^t \mu(s) e^{-L(t-s)} ds\right)\rho. \end{aligned} \tag{37}$$

Let $L \rightarrow +\infty$, we obtain

$$\|\Lambda x\|_* \leq N_1 + \|\phi\|_{[-r,0]} + \ell + pMM_2 + pMM_1\rho \leq \rho. \tag{38}$$

Hence for some positive number ρ, $\Lambda B_\rho \subset B_\rho$.

Using the strong continuity of $\{C(t)\}_{t\in\mathbf{R}}$ and the compactness condition on the operators I_k, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|(C(t+h) - C(t)) I_k(x)\| \le \varepsilon, \quad x \in B_\rho, \\ t \in J, \ k = 1, 2, \dots, p, \tag{39}$$

when $|h| < \delta$. If $t \in [t_k, t_{k+1}]$ and $h < \delta$ such that $t + h \in [t_k, t_{k+1}]$, then

$$\begin{aligned} &\left\| \sum_{0<t_k<t} (C(t+h-t_k) - C(t-t_k)) I_k(x(t_k)) \right\| \\ &\quad \le \sum_{k=1}^{p} \|(C(t+h-t_k) - C(t-t_k)) I_k(x(t_k))\| \le p\varepsilon. \end{aligned} \tag{40}$$

For $x \in B_\rho$, by the hypothesis (H1)(i) and (40), we get

$$\begin{aligned} &\|(\Lambda_2 x)(t+h) - (\Lambda_2 x)(t)\| \\ &\quad \le \left\| \int_0^{t+h} S(t+h-s) f(s, x_s, x(s)) \, ds \right. \\ &\qquad \left. - \int_0^t S(t-s) f(s, x_s, x(s)) \, ds \right\| + p\varepsilon \\ &\quad \le p\varepsilon + \int_0^t \|(S(t+h-s) - S(t-s)) f(s, x_s, x(s))\| \, ds \\ &\qquad + \int_t^{t+h} \|S(t+h-s) f(s, x_s, x(s))\| \, ds \\ &\quad \le p\varepsilon + \left[Mh \int_0^t \mu(s) \, ds + N \int_t^{t+h} \mu(s) \, ds \right] \cdot (1+\rho). \end{aligned} \tag{41}$$

As $h \to 0$ and $\varepsilon \to 0$, the right-hand side of the inequality above tends to zero independent of x, so Λ_2 maps bounded sets into equicontinuous sets.

For bounded set $B \subset PC(J, X)$, we consider the map

$$\chi_{pc}(B) = \max_{i=0,1,\dots,p} \chi_i \left(B|_{\overline{J}_i} \right), \tag{42}$$

where χ_i is the Hausdorff measure of noncompactness on the Banach space $C(J_i, X)$ and $B|_{\overline{J}_i}$ is defined in (7).

Furthermore, we define the Hausdorff measure of noncompactness $\chi_{\mathfrak{F}}$ on $\mathfrak{F}(T)$ as follows:

$$\chi_{\mathfrak{F}}(\mathbb{Y}) := \chi_{pc}\left(\mathbb{Y}|_{PC(J,X)}\right) + \sup_{t\in[-r,0]} \chi(\mathbb{Y}(t)), \quad \mathbb{Y} \subset \mathfrak{F}(T). \tag{43}$$

For every bounded subset $\widetilde{\Omega} \subset PC(J, X)$, by applying Proposition 2, for any $\varepsilon > 0$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \widetilde{\Omega}$ such that

$$\chi_{pc}\left(\Lambda_2 \widetilde{\Omega}\right) \le 2\chi_{pc}(\Lambda_2 \{x_n\}) + \varepsilon, \tag{44}$$

noting that the definition of χ_{pc}, we have

$$\chi_{pc}\left(\Lambda_2 \widetilde{\Omega}\right) \le 2 \max_{i=0,1,\dots,p} \chi_i \left(\Lambda_2 \{x_n\}|_{\overline{J}_i}\right) + \varepsilon. \tag{45}$$

Then, noting the equicontinuity of $\Lambda_2|_{\overline{J}_i}$, $i = 0, 1, \dots, p$, we can apply Lemmas 2.1 and 2.2 of [35] to obtain

$$\chi_i\left(\Lambda_2 \{x_n\}|_{\overline{J}_i}\right) = \sup_{t\in\overline{J}_i} \chi(\Lambda_2 \{x_n\}(t)). \tag{46}$$

Then from (45) and (46), we have

$$\begin{aligned} \chi_{pc}\left(\Lambda_2 \widetilde{\Omega}\right) &\le 2 \max_{i=0,1,\dots,p} \left(\sup_{t\in\overline{J}_i} \chi(\Lambda_2 \{x_n\}(t)) \right) \\ &\quad + \varepsilon = 2 \sup_{t\in J} \chi(\Lambda_2 \{x_n\}(t)) + \varepsilon. \end{aligned} \tag{47}$$

For every bounded subset $\Omega \subset \mathfrak{F}(T)$, we have

$$\begin{aligned} \chi_{\mathfrak{F}}(\Lambda_2 \Omega) &= \chi_{pc}\left((\Lambda_2 \Omega)|_{PC(J,X)}\right) \\ &\quad + \sup_{t\in[-r,0]} \chi((\Lambda_2 \Omega)(t)) = \chi_{pc}\left((\Lambda_2 \Omega)|_{PC(J,X)}\right), \end{aligned} \tag{48}$$

moreover, by applying Proposition 2, for any $\varepsilon > 0$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset \Omega$ such that

$$\begin{aligned} \chi_{\mathfrak{F}}(\Lambda_2 \Omega) &\le 2\chi_{\mathfrak{F}}(\Lambda_2 \{y_n\}) + \varepsilon \\ &= 2\chi_{pc}\left((\Lambda_2 \{y_n\})|_{PC(J,X)}\right) + \varepsilon. \end{aligned} \tag{49}$$

Combining with (48) and (49), we have

$$\begin{aligned} \chi_{\mathfrak{F}}(\Lambda_2 \Omega) &= \chi_{pc}\left((\Lambda_2 \Omega)|_{PC(J,X)}\right) \\ &\le 2\chi_{pc}\left((\Lambda_2 \{y_n\})|_{PC(J,X)}\right) + \varepsilon. \end{aligned} \tag{50}$$

Using the induction of (45)–(47) above, we can see

$$\begin{aligned} \chi_{\mathfrak{F}}(\Lambda_2 \Omega) &= \chi_{pc}\left((\Lambda_2 \Omega)|_{PC(J,X)}\right) \\ &\le 2 \sup_{t\in J} \chi\left(\Lambda_2 \{y_n\}(t)|_{t\in J}\right) + \varepsilon. \end{aligned} \tag{51}$$

Thus, from (51), (H2) and Proposition 3 and (3) in Lemma 1, we can see

$$\begin{aligned}\chi_{\mathfrak{F}}(\Lambda_2\Omega) &\le 2\sup_{t\in J}\chi(\Lambda_2\{y_n\}(t)|_{t\in J})+\varepsilon\\ &= 2\sup_{t\in J}\left[\chi\left(\left\{\int_0^t S(t-s)f(s,\tilde{y}_{ns},\tilde{y}_n(s))\,ds\right.\right.\right.\\ &\qquad\left.\left.\left.+\sum_{0<t_k<t}C(t-t_k)I_k(\tilde{y}_n(t_k))\right\}\right)\right]+\varepsilon\\ &\le 2\sup_{t\in J}\left[2N\int_0^t\eta(s)\left(\sup_{\theta\in[-r,0]}\chi(\{y_n(s+\theta)\})\right.\right.\\ &\qquad\left.\left.+\chi(\{\tilde{y}_n(s)\})\right)ds\right]+\varepsilon,\end{aligned}\tag{52}$$

where $\tilde{y}_n(t) := y_n(t)|_{t\in J}$.

Noting that

$$\begin{aligned}\sup_{\theta\in[-r,0]}\chi(\{y_n(s+\theta)\}) &\le \sup_{\theta\in[-r,0]}\chi(\{y_n(\theta)\})+\sup_{s\in J}\chi(\{\tilde{y}_n(s)\})\\ &\le \sup_{\theta\in[-r,0]}\chi(\Omega(\theta))+\sup_{s\in J}\chi(\Omega(s))\\ &\le \sup_{\theta\in[-r,0]}\chi(\Omega(\theta))+\chi_{pc}(\Omega)=\chi_{\mathfrak{F}}(\Omega),\end{aligned}\tag{53}$$

$$\begin{aligned}\chi(\{\tilde{y}_n(s)\}) &\le \chi(\Omega(s))\\ &\le \chi_{pc}(\Omega).\end{aligned}\tag{54}$$

Thus, by (52), we see

$$\begin{aligned}\chi_{\mathfrak{F}}(\Lambda_2\Omega) &\le 2\sup_{t\in J}\left[2N\int_0^t\eta(s)\left(\sup_{\theta\in[-r,0]}\chi(\{y_n(s+\theta)\})\right.\right.\\ &\qquad\left.\left.+\chi(\{\tilde{y}_n(s)\})\right)ds\right]+\varepsilon\\ &\le 2\sup_{t\in J}\left(4N\int_0^t\eta(s)\,ds\right)\cdot\chi_{\mathfrak{F}}(\Omega)+\varepsilon\\ &= 8N\int_0^T\eta(s)\,ds\cdot\chi_{\mathfrak{F}}(\Omega)+\varepsilon.\end{aligned}\tag{55}$$

Since ε is arbitrary, we can obtain

$$\chi_{\mathfrak{F}}(\Lambda_2\Omega) \le 8N\int_0^T\eta(s)\,ds\cdot\chi_{\mathfrak{F}}(\Omega).\tag{56}$$

Combining with (H3), we have

$$\begin{aligned}\chi_{\mathfrak{F}}(\Lambda\Omega) &\le \chi_{\mathfrak{F}}(\Lambda_1\Omega)+\chi_{\mathfrak{F}}(\Lambda_2\Omega)\\ &\le 8N\int_0^T\eta(s)\,ds\cdot\chi_{\mathfrak{F}}(\Omega) < M^*\chi_{\mathfrak{F}}(\Omega),\end{aligned}\tag{57}$$

hence Λ is a $\chi_{\mathfrak{F}}$-contraction on $\mathfrak{F}(T)$. According to Theorem 4, the operator Λ has at least one fixed point x in B_ρ. This completes the proof. □

Next, we establish a condition that guarantee that a mild solution satisfies (3).

Proposition 8. *Assume that the hypotheses of Theorem 7 are fulfilled and that $\phi(0)+g(x)(0)\in D(A)$. If $x(\cdot)$ is a mild solution of* (1)–(4), *then condition* (3) *holds.*

Proof. Clearly, $(1/t)\int_0^t S(t-s)f(s,x_s,x(s))ds \to 0$ as $t\to 0$. Moreover, noting that $\phi(0)+g(x)(0)\in D(A)$ and (11), we have $C(\cdot)(\phi(0)+g(x)(0))$ is of class C^1. Therefore, we can see that

$$\begin{aligned}&\lim_{t\to 0^+}\frac{x(t)-x(0)}{t}\\ &\quad=\lim_{t\to 0^+}\frac{1}{t}\left[(C(t)-I)(\phi(0)+g(x)(0))+S(t)\xi\right.\\ &\qquad\left.+\int_0^t S(t-s)f(s,x_s,x(s))\,ds\right]=\xi,\end{aligned}\tag{58}$$

which shows the assertion. □

4. Application

In this section, we consider an application of the theory developed in Section 3 to the study of an impulsive partial differential equation with unbounded delay.

Example 9. $X = L^2([0,\pi])$, $A : D(A)\subseteq X\to X$ is the map defined by $A\vartheta=\vartheta''$ with domain $D(A)=\{\vartheta\in X:\vartheta''\in X, \vartheta(0)=\vartheta(\pi)=0\}$.

We consider the following integrodifferential model:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}v(t,\xi) &= \frac{\partial^2}{\partial \xi^2}v(t,\xi)+\sin|v(t,\xi)|\\ &\quad+t^2\int_{t-r}^t\gamma(\theta-t)\cdot\cos\left(\frac{|v(\theta,\xi)|}{t}\right)d\theta,\\ &\qquad v(t,\pi)=v(t,0)=0,\\ v(\theta,\xi) &= v_0(\theta,\xi)+\int_0^\pi c(\xi,s)\sin(1+v(\theta,s))\,ds,\\ &\qquad -r\le\theta\le 0,\quad \frac{\partial}{\partial t}v(0,\xi)=\omega(\xi),\\ \Delta v(t_k,\xi) &= \int_0^\pi\rho_k(\xi,y)\,dy\cdot\cos^2(v(t_k,\xi)),\qquad 1\le k\le p,\end{aligned}\tag{59}$$

where $t \in [0,T]$, $r > 0$, $\xi \in [0,\pi]$, $0 < t_1 < t_2 < \cdots < t_p < T$, $\omega \in X$ and $v_t(\theta,\xi) = v(t+\theta,\xi)$. $\gamma : [-r,0] \to \mathbf{R}$ and $c(\xi,s)$, $\rho_k(\xi,z) \in L^2([0,\pi]\times[0,\pi],\mathbf{R})$ satisfy the following assumptions.

(1) The function $\gamma : [-r,0] \to \mathbf{R}$ is a continuous function and $\int_{-r}^{0} |\gamma(\theta)| d\theta < \infty$.

(2) The function $c(\xi,s)$, ξ, $s \in [0,\pi]$ is measurable and there exists a constant N_1 such that $(\pi \int_0^\pi \int_0^\pi c^2(\xi,s) ds d\xi)^{1/2} \le N_1$.

(3) For every $k = 1,2,\ldots,p$, the function $\rho_k(\xi,z), z \in [0,\pi]$, is measurable and there exists a constant $\overline{N}$ such that

$$\left(\int_0^\pi \left(\int_0^\pi \rho_k(\xi,z)dz\right)^2 d\xi\right)^{1/2} \le \overline{N}. \tag{60}$$

To treat the above problem, we define

$$\begin{gathered} D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi]), \\ Au = u''. \end{gathered} \tag{61}$$

A is the infinitesimal generator of a strongly continuous cosine function $\{C(t)\}_{t\in\mathbf{R}}$ on X. Moreover, A has a discrete spectrum, the eigenvalues are $-n^2, n \in \mathbf{N}$, with the corresponding normalized eigenvectors $\omega_n(x) = \sqrt{(2/\pi)}\sin(nx)$; the set $\{\omega_n; n \in \mathbf{N}\}$ is an orthonormal basis of X and the following properties hold.

(a) If $\omega \in D(A)$, then $A\omega = -\sum_{n=1}^{\infty} n^2 \langle \omega, \omega_n\rangle \omega_n$.

(b) For each $\omega \in X$, $C(t)\omega = \sum_{n=1}^{\infty} \cos(nt)\langle \omega,\omega_n\rangle\omega_n$ and $S(t)\omega = \sum_{n=1}^{\infty}(\sin(nt)/n)\langle\omega,\omega_n\rangle\omega_n$. Consequently, $\|C(t)\| = \|S(t)\| \le 1$ for $t \in \mathbf{R}$, and $\{S(t)\}$ is compact for every $t \in \mathbf{R}$.

For $\xi \in [0,\pi]$ and $\varphi \in C([-r,0],X)$, we set

$$\begin{aligned} & x(t)(\xi) = v(t,\xi), \\ & \phi(\theta)(\xi) = v_0(\theta,\xi), \quad \theta \in [-r,0], \\ & g(\varphi(\theta))(\xi) = \int_0^\pi c(\xi,s)\sin(1+\varphi(\theta)(s))\,ds, \\ & f(t,\varphi,x(t))(\xi) = \sin|x(t)(\xi)| \\ & \qquad\qquad + t^2 \int_{-r}^0 \gamma(\theta)\cdot\cos\left(\frac{|\varphi(\theta)(\xi)|}{t}\right)d\theta, \\ & I_k(x(t_k))(\xi) = \int_0^\pi \rho_k(\xi,y)\,dy \cdot \cos^2(x(t_k)(\xi)). \end{aligned} \tag{62}$$

Then the above equation (59) can be reformulated as the abstract (1)–(4).

For $t \in [0,T]$, we can see

$$\begin{aligned} \|f(t,\varphi,x(t))\| &\le \|x(t)\| + t^2 \int_{-r}^0 |\gamma(\theta)|\,d\theta \\ &\le \mu(t)(1+\|x(t)\|), \end{aligned} \tag{63}$$

where $\mu(t) := \max\{1, t^2 \int_{-r}^0 |\gamma(\theta)| d\theta\}$.

For any $x_1, x_2 \in X$, φ, $\widetilde{\varphi} \in C([-r,0],X)$,

$$\begin{aligned} &\|f(t,\varphi,x_1(t))(\xi) - f(t,\widetilde{\varphi},x_2(t))(\xi)\| \\ &\quad \le \|x_1(t) - x_2(t)\| + t\int_{-r}^0 |\gamma(\theta)|\,\|\varphi(\theta)(\xi) - \widetilde{\varphi}(\theta)(\xi)\|\,d\theta. \end{aligned} \tag{64}$$

Therefore, for any bounded sets $D_1 \subset C([-r,0],X)$, $D_2 \subset X$, we have

$$\begin{aligned} \chi(f(t,D_1,D_2)) &\le \chi(D_2) + t\int_{-r}^0 |\gamma(\theta)|\,\chi(D_1(\theta))\,d\theta \\ &\le \chi(D_2) + t \sup_{-r\le\theta\le0}\chi(D_1(\theta))\int_{-r}^0|\gamma(\theta)|\,d\theta \\ &\le \eta(t)\left(\sup_{-r\le\theta\le0}\chi(D_1(\theta)) + \chi(D_2)\right), \\ &\quad \text{a.e. } t \in [0,T], \end{aligned} \tag{65}$$

where $\eta(t) := \max\{1, t\int_{-r}^0 |\gamma(\theta)| d\theta\}$.

For $x \in X$,

$$\|I_k(x)\| \le \overline{N}(1+\|x\|), \quad k = 1,2,\ldots,p. \tag{66}$$

Suppose further that there exists a constant $\widetilde{M}^* \in (0,1)$ such that $8\int_0^T \eta(s)ds < \widetilde{M}^*$ and $p\overline{N} < 1$, then (59) has at least a mild solution by Theorem 7.

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