Research Article

The Use of Sumudu Transform for Solving Certain Nonlinear Fractional Heat-Like Equations

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We make use of the properties of the Sumudu transform to solve nonlinear fractional partial differential equations describing heat-like equation with variable coefficients. The method, namely, homotopy perturbation Sumudu transform method, is the combination of the Sumudu transform and the HPM using He's polynomials. This method is very powerful, and professional techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering.

1. Introduction

In the literature one can find a wide class of methods dealing with the problem of approximate solutions to problems described by nonlinear fractional differential equations, for instance, asymptotic methods and perturbation methods [1]. The perturbation methods have some limitations; for instance, the approximate solution engages series of small parameters which causes difficulty since most nonlinear problems have no small parameters at all [1]. Even though a suitable choice of small parameters occasionally lead to ideal solution, in most cases unsuitable choices leads to serious effects in the solutions [1]. Therefore, an analytical method which does not require a small parameter in the equation modeling of the phenomenon is welcome [2–4]. To deal with the pitfall presented by these perturbation methods for solving nonlinear equations, a literature review in some new asymptotic methods for the search for the solitary solutions of nonlinear differential equations, nonlinear differential-difference equations, and nonlinear fractional differential equations is presented in [5]. The homotopy perturbation method (HPM) was first initiated by He [6]. The HPM was also studied by many authors to present approximate and exact solution of linear and nonlinear equations arising in various scientific and technological fields [7–13]. The Adomian decomposition method (ADM) [14–19] and variational iteration method (VIM) [2–4] have also been applied to study the various physical problems. The homotopy decomposition method (HDM) was recently proposed by [20, 21] to solve the groundwater flow equation and the modified fractional KDV equation [20, 21]. The homotopy decomposition method is actually the combination of the perturbation method and Adomian decomposition method. Singh et al. [22] have made used of studying the solutions of linear and nonlinear partial differential equations by using the homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform, HPM, and He's polynomials.

2. Sumudu Transform

The Sumudu transform is an integral transform similar to the Laplace transform, introduced in the early 1990s by Watugala [23] to solve differential equations and control engineering problems.

First we will summon up the following useful definitions and theorems for this integral transform operator. Note that these theorems and definitions will be used in the rest of the paper.

2.1. Definitions and Theorems

Definition 1. The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by

$$S\left(f\left(t\right)\right) = F_s\left(u\right) = \int_0^\infty \frac{1}{u} \exp\left[-\frac{t}{u}\right] f\left(t\right) dt. \tag{1}$$

Definition 2. The double Sumudu transform of a function $f(x,t)$, defined for all real numbers $(x \geq 0, t \geq 0)$, is defined by

$$F\left(u, v\right) = S_2\left[f\left(x, t\right), \left(u, v\right)\right]$$
$$= \frac{1}{vu} \iint_0^\infty \exp\left[-\left(\frac{t}{v} + \frac{x}{u}\right)\right] f\left(x, t\right) dx\, dt. \tag{2}$$

In the same line of ideas, the double Sumudu transform of second partial derivative with respect to x is of form [24]

$$S_2\left[\frac{\partial^2 f\left(x, t\right)}{\partial x^2}; \left(u, v\right)\right]$$
$$= \frac{1}{u^2} F\left(u, v\right) - \frac{1}{u^2} F\left(0, v\right) - \frac{1}{u} \frac{\partial F\left(0, v\right)}{\partial x}. \tag{3}$$

Similarly, the double Sumudu transform of second partial derivative with respect to t is of form [24]

$$S_2\left[\frac{\partial^2 f\left(x, t\right)}{\partial t^2}; \left(u, v\right)\right]$$
$$= \frac{1}{v^2} F\left(u, v\right) - \frac{1}{v^2} F\left(u, 0\right) - \frac{1}{u} \frac{\partial F\left(u, 0\right)}{\partial t}. \tag{4}$$

Theorem 3. *Let $G(u)$ be the Sumudu transform of $f(t)$ such that*

(i) *$G(1/s)/s$ is a meromorphic function, with singularities having $\mathrm{Re}[s] \leq \gamma$ and*

(ii) *there exist a circular region Γ with radius R and positive constants M and K with $|G(1/s)/s| < MR^{-K}$; then the function $f(t)$ is given by*

$$S^{-1}\left(G\left(s\right)\right) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp\left[st\right] G\left(\frac{1}{s}\right) \frac{ds}{s}$$
$$= \sum residual\left[\exp\left[st\right] \frac{G\left(1/s\right)}{s}\right]. \tag{5}$$

For the proof see [23].

2.2. Properties of Sumudu Transform [25–28]

(i) The transform of a Heaviside unit step function is a Heaviside unit step function in the transformed domain [26, 27].

(ii) The transform of a Heaviside unit ramp function is a Heaviside unit ramp function in the transformed domain [26, 27].

(iii) The transform of a monomial t^n is the scaled monomial $S(t^n) = n! u^n$ [26, 27].

(iv) If $f(t)$ is a monotonically increasing function, so is $F(u)$, and the converse is true for decreasing functions [26, 27].

(v) The Sumudu transform can be defined for functions which are discontinuous at the origin. In that case the two branches of the function should be transformed separately. If $f(t)$ is C^n continuous at the origin, so is the transformation $F(u)$ [26, 27].

(vi) The limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero provided both limits exist [26, 27].

(vii) The limit of $f(t)$ as t tends to infinity is equal to the limit of $F(u)$ as u tends to infinity provided both limits exist [26, 27].

(viii) Scaling of the function by a factor $c > 0$ to form the function $f(ct)$ gives a transform $F(cu)$ which is the result of scaling by the same factor [26, 27].

2.3. Basic Definition of Fractional Calculus

Definition 4. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 5. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^x \left(x - t\right)^{\alpha - 1} f\left(t\right) dt, \quad \alpha > 0, \ x > 0,$$
$$J^0 f\left(x\right) = f\left(x\right). \tag{6}$$

Properties of the operator can be found in [30–33] one mentions only the following.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$

$$J^\alpha J^\beta f\left(x\right) = J^{\alpha + \beta} f\left(x\right),$$
$$J^\alpha J^\beta f\left(x\right) = J^\beta J^\alpha f\left(x\right),$$
$$J^\alpha x^\gamma = \frac{\Gamma\left(\gamma + 1\right)}{\Gamma\left(\alpha + \gamma + 1\right)} x^{\alpha + \gamma}. \tag{7}$$

Lemma 6. *If $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, and $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x),$$

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{8}$$

Definition 7 (partial derivatives of fractional order). Assume now that $f(\mathbf{x})$ is a function of n variables x_i $i = 1, \ldots, n$ also of class C on $D \in \mathbb{R}_n$

$$a\partial_{\underline{x}}^\alpha f = \frac{1}{\Gamma(m - \alpha)} \int_a^{x_i} (x_i - t)^{m-\alpha-1} \partial_{x_i}^m f(x_j)\Big|_{x_j = t} dt, \tag{9}$$

where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

Definition 8. The Sumudu transform of the Caputo fractional derivative is defined as follows [28]:

$$S\left[D_t^\alpha f(t)\right] = u^{-\alpha} S\left[f(t)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^+) \tag{10}$$

$$(m - 1 < \alpha \leq m).$$

3. Solution by HPSTM

3.1. Basic Idea of HPSTM. We illustrate the basic idea of this method, by considering a general fractional nonlinear non-homogeneous partial differential equation with the initial condition of the form of general form

$$D_t^\alpha U(x,t) = L(U(x,t)) + N(U(x,t)) + f(x,t), \quad \alpha > 0 \tag{11}$$

subject to the initial condition

$$D_0^k U(x,0) = g_k, \quad (k = 0, \ldots, n-1),$$
$$D_0^n U(x,0) = 0, \quad n = [\alpha], \tag{12}$$

where, D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator, and L represents a linear fractional differential operator.

Applying the Sumudu Transform on both sides of (11), we obtain

$$S\left[D_t^\alpha U(x,t)\right] = S\left[L(U(x,t))\right] + S\left[N(U(x,t))\right] + S\left[f(x,t)\right]. \tag{13}$$

Using the property of the Sumudu transform, we have

$$S\left[U(x,t)\right] = u^\alpha S\left[L(U(x,t))\right] + u^\alpha S\left[N(U(x,t))\right] + u^\alpha S\left[f(x,t)\right] + g(x,t). \tag{14}$$

Now applying the Sumudu inverse on both sides of (24) we obtain

$$U(x,t)$$
$$= S^{-1}\left[u^\alpha S\left[L(U(x,t))\right] + u^\alpha S\left[N(U(x,t))\right]\right] + G(x,t), \tag{15}$$

where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the initial conditions [1].

Now we apply the HPM

$$U(x,t) = \sum_{n=0}^\infty p^n U_n(x,t). \tag{16}$$

The nonlinear term can be decomposed into

$$NU(x,t) = \sum_{n=0}^\infty p^n \mathscr{H}_n(U) \tag{17}$$

using the He's polynomial $\mathscr{H}_n(U)$ [17, 18] given as

$$\mathscr{H}_n(U_0, \ldots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^\infty p^j U_j(x,t)\right)\right], \tag{18}$$

$$n = 0, 1, 2, \ldots.$$

Substituting (16) and (17)

$$\sum_{n=0}^\infty p^n U_n(x,t)$$
$$= G(x,t) + p\left[S^{-1}\left[u^\alpha S\left[L\left(\sum_{n=0}^\infty p^n U_n(x,t)\right)\right]\right.\right.$$
$$\left.\left. + u^\alpha S\left[N\left(\sum_{n=0}^\infty p^n U_n(x,t)\right)\right]\right]\right] \tag{19}$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials [1]. Comparing the coefficients of like powers of p, the following approximations are obtained:

$$p^0 : U_0(x,t) = G(x,t),$$
$$p^1 : U_1(x,t) = S^{-1}\left[u^\alpha S\left[L(U_0(x,t)) + H_0(U)\right]\right],$$
$$p^2 : U_2(x,t) = S^{-1}\left[u^\alpha S\left[L(U_1(x,t)) + H_1(U)\right]\right],$$
$$p^3 : U_3(x,t) = S^{-1}\left[u^\alpha S\left[L(U_2(x,t)) + H_2(U)\right]\right],$$
$$p^n : U_n(x,t) = S^{-1}\left[u^\alpha S\left[L(U_{n-1}(x,t)) + H_{n-1}(U)\right]\right]. \tag{20}$$

Finally, we approximate the analytical solution $U(x,t)$ by truncated series [1]

$$U(x,t) = \lim_{N \to \infty} \sum_{n=0}^{N} U_n(x,t). \tag{21}$$

The above series solutions generally converge very rapidly [1, 34–37].

4. Application

In this section we apply this method for solving fractional differential equation in form of (11) together with (12).

Example 9. Consider the following three-dimensional fractional heat-like equation:

$$\partial_t^\alpha u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right),$$

$$0 < x, y, z < 1, \quad 0 < \alpha \le 1 \tag{22}$$

subject to the initial condition

$$u(x,y,z,0) = 0. \tag{23}$$

Following carefully the steps involved in the HDM, we arrive at the following equation:

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)$$

$$= \frac{p}{\Gamma(\alpha)} S^{-1}$$

$$\times \left(u^{-\alpha} \times \left[\int_0^t (t-\tau)^{\alpha-1} \right. \right.$$

$$\times \left(x^4 y^4 z^4 + \frac{1}{36} \right.$$

$$\times \left(x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{xx} \right.$$

$$+ y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{yy}$$

$$\left. \left. \left. \left. + z^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{zz} \right) \right) d\tau \right] \right). \tag{24}$$

Now comparing the terms of the same power of p yields

$$p^0 : u_0(x,y,z,t),$$

$$p^1 : u_1(x,y,z,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x^4 y^4 z^4 \, d\tau,$$

$$\vdots$$

$$p^n : u_n(x,y,z,t)$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}$$

$$\times \left(\frac{1}{36} \left(x^2 (u_{n-1})_{xx} \right. \right.$$

$$\left. \left. + y^2 (u_{n-1})_{yy} + z^2 (u_{n-1})_{zz} \right) \right) d\tau,$$

$$u_n(x,y,z,0) = 0, \quad n \ge 2. \tag{25}$$

Thus the following components are obtained as results of the above integrals:

$$u_0(x,y,z,t) = 0,$$

$$u_1(x,y,z,t) = \frac{t^\alpha x^4 y^4 z^4}{\Gamma(\alpha+1)},$$

$$u_2(x,y,z,t) = \frac{t^{2\alpha} x^4 y^4 z^4}{\Gamma(2\alpha+1)},$$

$$u_3(x,y,z,t) = \frac{t^{3\alpha} x^4 y^4 z^4}{\Gamma(3\alpha+1)}, \tag{26}$$

$$\vdots$$

$$u_n(x,y,z,t) = \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}.$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x,y,z,t) = \sum_{n=1}^{N} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}. \tag{27}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x,y,z,t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)} - x^4 y^4 z^4 \tag{28}$$

$$= x^4 y^4 z^4 \left(E_\alpha(t^\alpha) - 1 \right),$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x, y, z, t) = x^4 y^4 z^4 (\exp(t) - 1).\tag{29}$$

This is the exact solution for this case.

Example 10. We consider the three-dimensional fractional wave-like equation

$$\partial_t^\alpha u(x, y, z, t)$$
$$= x^2 + y^2 + z^2 + \frac{1}{2}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right),\tag{30}$$
$$0 < x, y, z < 1, \quad 1 < \alpha \leq 2$$

subject to the initial condition

$$u(x, y, z, 0) = 0, \qquad u_t(x, y, z, 0) = x^2 + y^2 - z^2.\tag{31}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$u_0(x, y, z, t) = \left(x^2 + y^2 - z^2\right)t,$$
$$u_1(x, y, z, t) = \frac{t^\alpha}{\Gamma(1 + \alpha)}\left(x^2 + y^2 - z^2\right),$$
$$u_2(x, y, z, t) = \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}\left(x^2 + y^2 + z^2\right),$$
$$u_3(x, y, z, t) = \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)}\left(x^2 + y^2 - z^2\right),\tag{32}$$
$$\vdots$$
$$u_n(x, y, z, t) = \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}\left(x^2 + y^2 + (-1)^n z^2\right).$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x, y, z, t) = \sum_{n=1}^{N} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}\left(x^2 + y^2 + (-1)^n z^2\right).\tag{33}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x, y, z, t) = \sum_{n=1}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}\left(x^2 + y^2 + (-1)^n z^2\right).\tag{34}$$

In the case of $\alpha = 2$ we obtain

$$u(x, y, z, t) = \left(x^2 + y^2\right)\exp(t)$$
$$+ z^2 \exp(-t) - \left(x^2 + y^2 + z^2\right).\tag{35}$$

This is the exact solution for this case.

Example 11. We consider the one-dimensional fractional wave-like equation

$$\partial_t^\alpha u(x, t) = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1, \ 1 < \alpha \leq 2, \ t > 0\tag{36}$$

with the initial conditions as

$$u(x, 0) = x^2.\tag{37}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$u_0(x, t) = x^2,$$
$$u_1(x, t) = \frac{t^\alpha x^2}{\Gamma(\alpha + 1)},$$
$$u_2(x, t) = \frac{t^{2\alpha} x^2}{\Gamma(2\alpha + 1)},$$
$$u_3(x, t) = \frac{t^{3\alpha} x^2}{\Gamma(3\alpha + 1)},\tag{38}$$
$$\vdots$$
$$u_n(x, t) = \frac{t^{n\alpha} x^2}{\Gamma(n\alpha + 1)}.$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x, t) = \sum_{n=1}^{N} \frac{t^{n\alpha} x^2}{\Gamma(n\alpha + 1)}.\tag{39}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^2}{\Gamma(n\alpha + 1)} = x^2 E_\alpha(t^\alpha),\tag{40}$$

where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x, t) = x^2 \exp(t).\tag{41}$$

This is the exact solution for this case.

Example 12. In this example we consider the two-dimensional fractional heat-like equation (Figures 1 and 2)

$$\partial_t^\alpha u(x, t) = u_{xx} + u_{yy},$$
$$0 < x, \quad y < 2\pi, \quad t > 0, \quad 0 < \alpha \leq 1.\tag{42}$$

subject to the initial condition

$$u(x, y, 0) = \sin(x)\sin(y). \tag{43}$$

Following carefully the steps involved in the HPSTM, we arrive at the following series solutions:

$$u_0(x, y, t) = \sin(x)\sin(y),$$

$$u_1(x, y, t) = -2\frac{t^\alpha \sin(x)\sin(y)}{\Gamma(\alpha + 1)},$$

$$u_2(x, y, t) = 4\frac{t^{2\alpha} \sin(x)\sin(y)}{\Gamma(2\alpha + 1)},$$

$$u_3(x, y, t) = -8\frac{t^{3\alpha} \sin(x)\sin(y)}{\Gamma(3\alpha + 1)}, \tag{44}$$

$$\vdots$$

$$u_n(x, y, z, t) = (-2)^n\frac{t^{n\alpha} \sin(x)\sin(y)}{\Gamma(n\alpha + 1)}.$$

Therefore the approximate solution of equation for the first n is given as

$$u_N(x, y, t) = \sum_{n=1}^{N}(-2)^n\frac{t^{n\alpha} \sin(x)\sin(y)}{\Gamma(n\alpha + 1)}. \tag{45}$$

Now when $N \to \infty$, we obtained the following solution:

$$u(x, y, t) = \sum_{n=0}^{\infty}\frac{(-2)^n t^{n\alpha} \sin(x)\sin(y)}{\Gamma(n\alpha + 1)}. \tag{46}$$

Note that in the case $\alpha = 1$

$$u(x, y, z, t) = \sin(x)\sin(y)\exp(-2t). \tag{47}$$

This is the exact solution for this case.

Example 13. Consider the following time-fractional derivative in x, y-plane as

$$D_t^\alpha u(x, y, t) = \frac{1}{2}\nabla^2 u(x, y, t), \quad 1 < \alpha \le 2, \ x, y \in \mathbb{R}, \ t > 0 \tag{48}$$

subject to the initial conditions

$$u(x, y, 0) = \sin(x + y), \qquad u_t(x, y, 0) = -\cos(x + y). \tag{49}$$

Applying the steps involved in HPSTM as presented in Section 3.1 to (49) we obtain

$$p^0: \ u_0(x, y, t) = \sin(x + y) - \cos(x + y)t,$$

$$p^1: u_1(x, t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2\left[(u_0)_{xx} + (u_0)_{yy}\right]\right)\right]$$

$$= -\sin(x + y)\frac{t^2}{2} + \cos(x + y)\frac{t^3}{3!},$$

$$p^2: u_2(x, t) = S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2\left[(u_1)_{xx} + (u_1)_{yy}\right]\right)\right]$$

$$= \sin(x + y)\left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^{4-\alpha}}{\Gamma(5 - \alpha)}\right]$$

$$+ \cos(x + y)\left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6 - \alpha)}\right],$$

$$p^3: u_3(x, t)$$

$$= S^{-1}\left[u^\alpha S\left(\frac{1}{2}x^2\left[(u_2)_{xx} + (u_2)_{yy}\right]\right)\right]$$

$$= \sin(x + y)\left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{2t^{4-\alpha}}{\Gamma(5 - \alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7 - \alpha)}\right.$$

$$\left.- \frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6 - 2\alpha)(5 - 2\alpha)\Gamma(3 - \alpha)\Gamma(2.5 - \alpha)}\right]$$

$$+ \cos(x + y)$$

$$\times\left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7 - \alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8 - 2\alpha)}\right]. \tag{50}$$

Therefore the series solution is given as

$$u(x, y, t)$$

$$= \sin(x + y)$$

$$\times\left[1 - \frac{3t^2}{2!} + \frac{t^4}{8} + \frac{t^6}{6!} + \frac{3t^{4-\alpha}}{\Gamma(5 - \alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7 - \alpha)}\right.$$

$$\left.- \frac{4^{\alpha-2}\sqrt{\pi}t^{6-2\alpha}}{(6 - 2\alpha)(5 - 2\alpha)\Gamma(3 - \alpha)\Gamma(2.5 - \alpha)}\right]$$

$$+ \cos(x + y)$$

$$\times\left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{3t^{7-\alpha}}{\Gamma(7 - \alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8 - 2\alpha)}\right] + \cdots. \tag{51}$$

It is important to point out that, if $\alpha = 2$, the above solution takes the form

$$u_{N=4}(x, y, t) = \sin(x + y)\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right]$$
$$- \cos(x + y)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right]$$

(52)

which is the first four terms of the series expansion of the exact solution $u(x, y, t) = \sin(x + y - t)$.

Example 14. Consider the following two-dimensional heat-like equation:

$$D_{tt}^{\alpha}u = \frac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right),$$

(53)

$$0 < x, \quad y < 1, \quad 1 < \alpha \leq 2, \quad t > 0$$

subject to the initial conditions

$$u(x, y, 0) = x^2, \qquad u_t(x, y, 0) = y^2.$$

(54)

The exact solution is given as

$$u(x, y, t) = x^2 \cosh(t) + y^2 \sinh(t).$$

(55)

Applying the Sumudu transform on both sides of (53), we obtain the following:

$$S[u(x, y, t)] = x^2 t^2 + y^2 + u^{\alpha}\left[\frac{1}{12}S\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]$$

(56)

Applying the inverse Sumudu transform on both sides of (56), we obtain the following:

$$u(x, y, t) = x^2 t^2 + y^2 + S^{-1}$$
$$\times \left[u^{\alpha}\left[\frac{1}{12}S\left(x^2 u_{xx} + y^2 u_{yy}\right)\right]\right].$$

(57)

Now applying the homotopy perturbation technique on the above equation we obtain the following:

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t)$$
$$= x^2 t^2 + y^2 + S^{-1}$$

$$\times \left[u^{\alpha}\left[\frac{1}{12}S\left(x^2\left[\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right]_{xx}\right.\right.\right.$$
$$\left.\left.\left.+ y^2\left[\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right]_{yy}\right)\right]\right].$$

(58)

By comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x, y, t) = x^2 t^2 + y^2,$$

$$p^1 : u_1(x, y, t)$$
$$= S^{-1}\left[u^{\alpha}\left[\frac{1}{12}S\left(x^2(u_0)_{xx} + y^2(u_0)_{yy}\right)\right]\right]$$
$$= x^2 t^2 + y^2 + \frac{1}{6}y^4 t^3 + \frac{1}{2}x^4 t^2,$$

$$p^2 : u_2(x, y, t)$$
$$= S^{-1}\left[u^{\alpha}\left[\frac{1}{12}S\left(x^2(u_1)_{xx} + y^2(u_1)_{yy}\right)\right]\right]$$

(59)

$$= \frac{1}{3}y^4 t^3 + \frac{1}{2}x^4 t^2 + \frac{(yt)^4}{24} + \frac{(yt)^5}{120}$$

$$+ \frac{x^4 t^{5-\alpha}}{\Gamma(3-\alpha)(4-\alpha)} + \frac{y^4 t^{5-\alpha}}{\Gamma(5-\alpha)(5-\alpha)}$$

$$- \frac{x^4 t^{4-\alpha}}{\Gamma(3-\alpha)(3-\alpha)} - \frac{y^4 t^{5-\alpha}}{\Gamma(4-\alpha)(4-\alpha)}$$

$$\vdots$$

Example 15. Consider the following one-dimensional fractional heat-like equation:

$$D_{tt}^{\alpha}u = x^2\frac{\partial[u_x u_{xx}]}{\partial x} - x^2(u_{xx})^2 - u,$$

(60)

$$0 \leq x \leq 1, \quad 0 \leq t, \quad 1 < \alpha \leq 2$$

Subject to the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = x^2.$$

(61)

Stimulation of $u(x, -1, t)$ of approximate solution

(a)

Stimulation of $u(x, -4, t)$ of approximate solution

(b)

Stimulation of $u(x, -8, t)$ of approximate solution

(c)

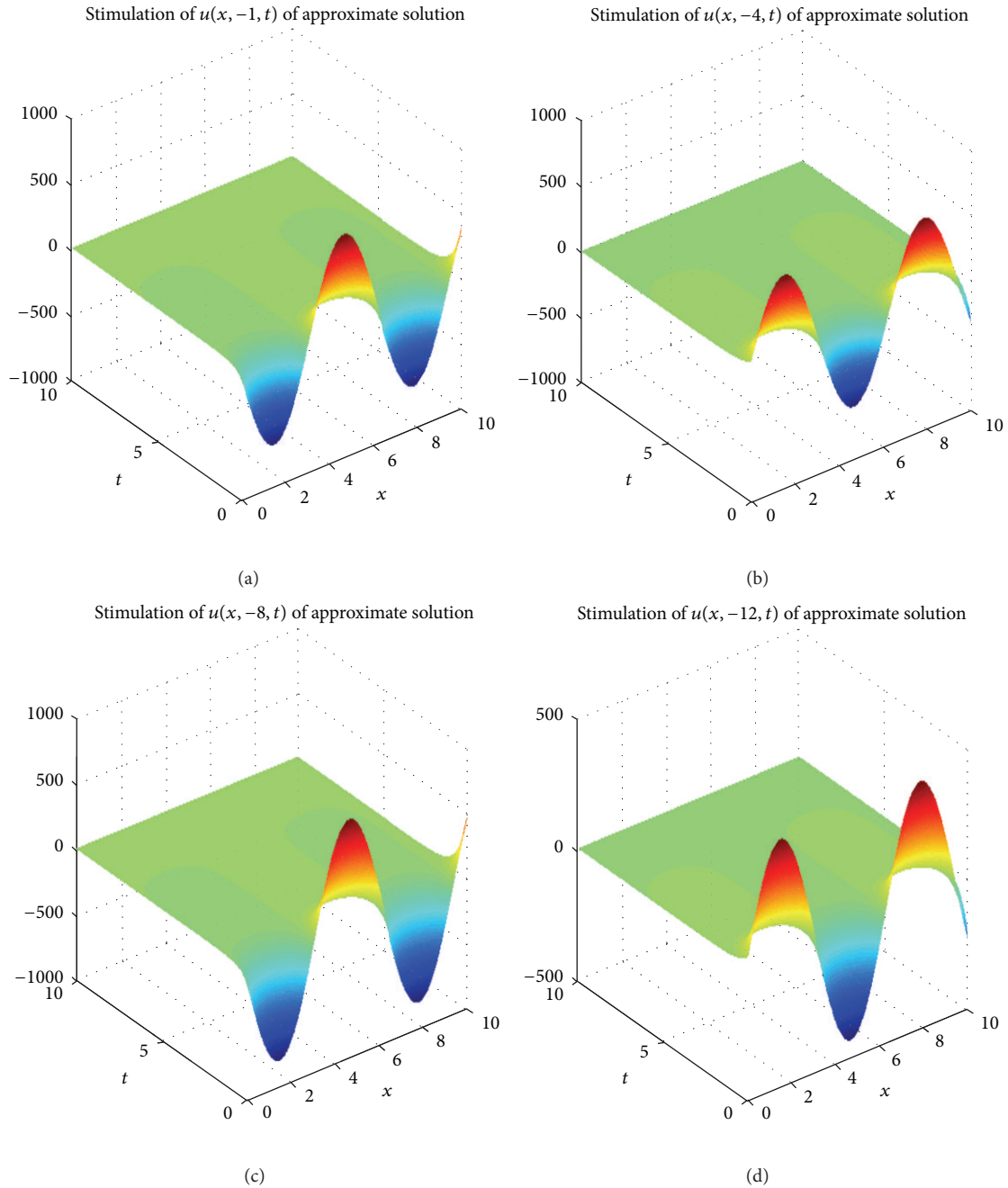Stimulation of $u(x, -12, t)$ of approximate solution

(d)

FIGURE 1: Numerical simulations of the approximate solution of (42) for a fixed y.

The exact solution is given as

$$u(x, t) = x^2 \sin[t]. \tag{62}$$

Applying the Sumudu transform on both sides of (60), we obtain the following:

$$S[u(x, t)] = x^2 + u^\alpha \left[x^2 \frac{\partial [u_x u_{xx}]}{\partial x} - x^2 (u_{xx})^2 - u \right]. \tag{63}$$

Applying the inverse Sumudu transform on both sides of (63), we obtain the following:

$$u(x, y, t) = x^2 + S^{-1}$$
$$\times \left[u^\alpha \left[S \left(x^2 \frac{\partial [u_x u_{xx}]}{\partial x} - x^2 (u_{xx})^2 - u \right) \right] \right]. \tag{64}$$

Stimulation of $u(x, y, 5)$ of approximate solution

(a)

Stimulation of $u(x, y, 10)$ of approximate solution

(b)

Stimulation of $u(x, y, 15)$ of approximate solution

(c)

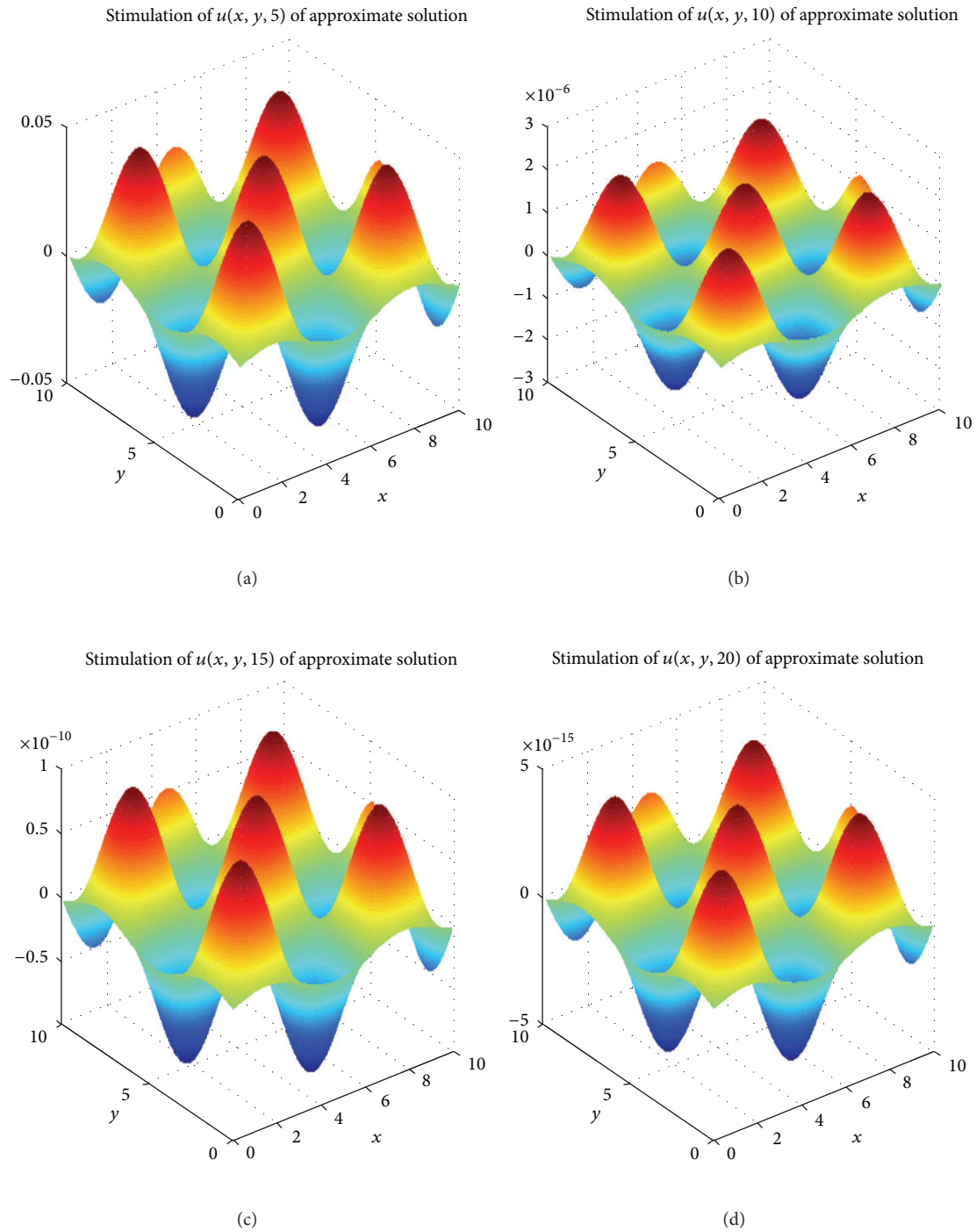Stimulation of $u(x, y, 20)$ of approximate solution

(d)

FIGURE 2: Numerical simulations of the approximate solution of (42) for a fixed t.

Now applying the homotopy perturbation technique on the above equation we obtain the following:

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t)$$

$$= x^2 + S^{-1}$$

$$\times \left[u^\alpha \left[S \left(x^2 \sum_{n=0}^{\infty} p^n H_n - x^2 \sum_{n=0}^{\infty} p^n H_n^1 \right. \right. \right.$$

$$\left. \left. \left. - \sum_{n=0}^{\infty} p^n u_n(x, t) \right) \right] \right].$$

$$(65)$$

TABLE 1: Numerical values for approximate solutions of (49) via ADM, VIM, and HPSTM.

t	x	y	ADM [29]	VIM [29]	HPSTM	Exact
0.25	0.5	0.5	0.68163209	0.68163219	0.681639	0.681639
	0.5	1.0	0.94898215	0.94898245	0.948985	0.948985
	1.0	0.5	0.94898215	0.94898245	0.948985	0.948985
	1.0	1.0	0.98398623	0.98398643	0.983986	0.983986
0.5	0.5	0.5	0.47942925	0.47942985	0.479425	0.479426
	0.5	1.0	0.84147331	0.84147361	0.841471	0.841471
	1.0	0.5	0.84147331	0.84147361	0.841471	0.841471
	1.0	1.0	0.99749205	0.99749235	0.997495	0.997495
0.75	0.5	0.5	0.2474231	0.2474232	0.247402	0.247404
	1.0	1.0	0.68163452	0.68163456	0.681636	0.681639
	0.5	0.5	0.68163453	0.68163456	0.681636	0.681639
	1.0	1.0	0.94898533	0.94898532	0.948982	0.948985
1.0	0.5	0.5	−0.000001905	−0.000001925	−0.0000019	−0.00000018
	1.0	1.0	0.4794205	0.4794215	0.479401	0.479426
	0.5	0.5	0.4794205	0.4794215	0.479401	0.479426
	1.0	1.0	0.8414352	0.8414582	0.841448	0.841448

By comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x, y, t) = x^2,$$

$$p^1 : u_1(x, y, t)$$

$$= S^{-1}\left[u^\alpha\left[S\left[x^2 H_0 - x^2 H_0^1 - u_0\right]\right]\right] = -x^2 t,$$

$$p^2 : u_2(x, y, t)$$

$$= S^{-1}\left[u^\alpha\left[u^\alpha\left[S\left[x^2 H_1 - x^2 H_1^1 - u_1\right]\right]\right]\right]$$

$$= x^2\left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)}\right],$$

$$p^3 : u_3(x, y, t)$$

$$= S^{-1}\left[u^\alpha\left[\left[S\left[x^2 H_2 - x^2 H_2^1 - u_2\right]\right]\right]\right]$$

$$= x^2\left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)}\right],$$

$$u(x, t)$$

$$= x^2\left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!}\right.$$

$$\left. + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} + \cdots + \cdots\right].$$

$$(66)$$

Now if we replace $\alpha = 2$, we recover the following series approximation:

$$u(x, t) = x^2\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + \cdots\right] = x^2\sin[t]$$

$$(67)$$

which is the exact solution of this case.

5. Conclusion

The aim of this work was to make use of the properties of the so-called Sumudu transform to solve nonlinear fractional heat-like equations. The basic idea of the method combines Sumudu transform and the HPM using He's polynomials. In addition the method is friendly user, and it does not require anything like Adomian polynomial. From the numerical comparison in Table 1, we can see that, these three methods are very powerful, and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HPSTM has an advantage over the ADM and VIM which is that it solves the nonlinear problems without anything like the Lagrangian multiplier as in the case of VIM. We do not need to calculate anything like Adomian polynomial as in the case of ADM. In addition the calculations involved in HPSTM are very simple and straightforward.

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