Research Article
A Study on N_θ-Quasi-Cauchy Sequences

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Recently, the concept of N_θ-ward continuity was introduced and studied. In this paper, we prove that the uniform limit of N_θ-ward continuous functions is N_θ-ward continuous, and the set of all N_θ-ward continuous functions is a closed subset of the set of all continuous functions. We also obtain that a real function f defined on an interval E is uniformly continuous if and only if $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, and biological science.

A real function f is continuous if and only if it preserves convergent sequences. A subset E of $\mathbf{R}$, the set of real numbers, is compact if any sequence of points in E has a convergent subsequence whose limit is in E. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, many kinds of continuities and compactness were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity, slowly oscillating compactness [1], quasi-slowly oscillating continuity, quasi-slowly oscillating compactness [2], Δ-quasi-slowly oscillating continuity, Δ-quasi-slowly oscillating compactness [3–5], ward continuity, ward compactness [6, 7], δ-ward continuity, δ-ward compactness [8], statistical ward continuity, and lacunary statistical ward continuity [9, 10].

In [11], the notion of N_θ convergence was introduced, and studied by Freedman et al. Using the main idea for

continuity and compactness given above the concepts of N_θ-ward compactness of a subset E of $\mathbf{R}$ and N_θ-ward continuity of a real function are introduced and investigated recently in [12].

The purpose of this paper is to continue the investigation given in [12] and obtain further interesting results on N_θ-ward continuity.

2. Preliminaries

Boldface letters $\boldsymbol{\alpha}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... will be used for sequences $\boldsymbol{\alpha} = (\alpha_k)$, $\boldsymbol{\alpha} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in the set of real numbers $\mathbf{R}$ for the sake of abbreviation. Sums of the form $\sum_{k_{r-1}+1}^{k_r} |\alpha_k|$ frequently occur and will often be written for convenience as $\sum_{k \in I_r} |\alpha_k|$.

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (α_n) of points in $\mathbf{R}$ is quasi-Cauchy if $(\Delta\alpha_n)$ is a null sequence where $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$. These sequences were named as quasi-Cauchy by Burton and Coleman [13, page 328], while they were called as forward convergent to 0 sequences in [7, page 226].

A sequence (α_k) of points in **R** is called N_θ-convergent to an element L of **R** if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |\alpha_k - L| = 0, \tag{1}$$

where $I_r = (k_{r-1}, k_r]$ and $\theta = (k_r)$ is a lacunary sequence, that is, an increasing sequence of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} is abbreviated by q_r. A sequence (α_n) of points in **R** is called N_θ-quasi-Cauchy if $(\Delta \alpha_n)$ is N_θ-convergent to 0. A function defined on a subset A of **R** is called N_θ-ward continuous if it preserves N_θ-quasi-Cauchy sequences, that is, $(f(\alpha_k))$ is an N_θ-quasi-Cauchy sequence whenever (α_k) is. $|\Delta N_\theta^0|$ will denote the set of N_θ-quasi-Cauchy sequences of points in **R**. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for N_θ-quasi-Cauchy sequences. A counterexample for the case is the sequence $(a_n) = (\sqrt{n})$ with the subsequence $(a_{n^2}) = (n)$.

A sequence (α_n) of points in **R** is slowly oscillating [14, Definition 2, page 947] if

$$\lim_{\lambda \to 1^+} \overline{\lim_n} \max_{n+1 \le k \le [\lambda n]} |\alpha_k - \alpha_n| = 0, \tag{2}$$

where $[\lambda n]$ denotes the integer part of λn (see also [14]).

An ideal I is a family of subsets of positive integers **N** which is closed under taking finite unions and subsets of its elements. A sequence (α_n) of real numbers is said to be ideal convergent to a real number L, if for each $\varepsilon > 0$ the set $\{n : |\alpha_n - L| \ge \varepsilon\}$ belongs to I. Ideal ward compactness of a subset of **R** and ideal ward continuity of a real function were recently introduced by Çakalli and Hazarika in [15].

3. Results

Any quasi-Cauchy sequence is N_θ-quasi-Cauchy, so any slowly oscillating sequence is N_θ-quasi-Cauchy, and so any Cauchy sequence is. A sequence $\boldsymbol{\alpha} = (\alpha_k)$ is called Cesaro summable to a real number L if $\lim_{n \to \infty}(1/n) \sum_{i=1}^n \alpha_i = L$. This is denoted by $C_1 - \lim \alpha_k = L$, and the set of all Cesaro sequences is denoted by σ_1. We call a sequence $\boldsymbol{\alpha} = (\alpha_k)$ Cesaro quasi-Cauchy if $C_1 - \lim \Delta \alpha_k = 0$. The set of all Cesaro quasi-Cauchy sequences is denoted by $\Delta \sigma_1^0$. A sequence $\boldsymbol{\alpha} = (\alpha_k)$ is called strongly Cesaro summable to a real number L if $\lim_{n \to \infty}(1/n) \sum_{i=1}^n |\alpha_i - L| = 0$. This is denoted by $|C_1| - \lim \alpha_k = L$. The set of all strongly Cesaro summable sequences is denoted by $|\sigma_1|$. We call a sequence $\boldsymbol{\alpha} = (\alpha_k)$ strongly Cesaro quasi-Cauchy if $|C_1| - \lim \Delta \alpha_k = 0$. The set of all strongly Cesaro quasi-Cauchy sequences is denoted by $|\Delta \sigma_1^0|$. The following inclusions are satisfied: $|\sigma_1| \subset \sigma_1$ and $|\Delta \sigma_1^0| \subset \Delta \sigma_1^0$.

Using a similar idea to that of [11], one can easily find out the following inclusion properties between the set of strongly Cesaro quasi-Cauchy sequences and the set of N_θ-quasi-Cauchy sequences (see also [16]).

(i) $|\Delta N_\theta^0| \subset |\Delta \sigma_1^0|$ if and only if $\lim \sup q_r < \infty$ for any lacunary sequence θ.

(ii) $|\Delta \sigma_1^0| \subset |\Delta N_\theta^0|$ if and only if $\lim \inf q_r > 1$ for any lacunary sequence θ.

Combining these facts, for any lacunary sequence θ, we have the following:

(iii) $|\Delta N_\theta^0| = |\Delta \sigma_1^0|$ if and only if $1 < \lim \inf q_r \le \lim \sup q_r < \infty$;

(iv) $|\Delta N_\theta^0| = |\Delta \sigma_1^0|$ if and only if $|\sigma_1| = N_\theta^0$.

In the sequel, we will always assume that $\lim \inf_r q_r > 1$.

We observe that N_θ-summability is a kind of strong A-summability where $A = (a_{rk})$ is a regular matrix generated by the lacunary sequence $\theta = (k_r)$ as follows:

$$a_{rk} = \frac{1}{h_r} \quad \text{if } k \in I_r, \ a_{rk} = 0 \text{ otherwise.} \tag{3}$$

On the other hand, we see that N_θ-ward continuity cannot be given as a strong A-continuity by any kind of regular summability matrix (related to continuity for strong matrix methods see [17]).

As far as ideal continuity is considered, we note that any N_θ-ward continuous function is ideal continuous; furthermore any N_θ continuous function is ideal continuous for an admissible ideal.

Theorem 1. *If a function f is uniformly continuous on a subset E of **R**, then, $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.*

Proof. Let E be a subset of **R**, and let f be a uniformly continuous function on E. Take any quasi-Cauchy sequence (α_k) of points in E, and let ε be any positive real number. By uniform continuity of f, there exists a $\delta > 0$ such that $|f(\alpha) - f(\beta)| < \varepsilon$ whenever $|\alpha - \beta| < \delta$ and $\alpha, \beta \in E$. Since (α_k) is a quasi-Cauchy sequence, there exists a positive integer k_0 such that $|\alpha_{k+1} - \alpha_k| < \delta$ for $k \ge k_0$. Hence,

$$\frac{1}{h_r} \sum_{k \in I_r} |f(\alpha_{k+1}) - f(\alpha_k)| < \frac{1}{h_r}(k_r - k_{r-1})\varepsilon = \varepsilon, \tag{4}$$

for $r \ge k_0$. Thus, $(f(\alpha_k))$ is an N_θ-quasi-Cauchy sequence. This completes the proof of the theorem. $\square$

We have much more below for a real function f defined on an interval that f is uniformly continuous if and only if $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E. First, we give the following lemma.

Lemma 2. *If (ξ_n, η_n) is a sequence of ordered pairs of points in an interval such that $\lim_{n \to \infty}|\xi_n - \eta_n| = 0$, then, there exists an N_θ-quasi-Cauchy sequence (α_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (\alpha_{j-1}, \alpha_j)$.*

Proof. Although the following proof is similar to that of [13], we give it for completeness. For each positive integer k, we can

fix $z_0^k, z_1^k, \ldots, z_{n_k}^k$ in E with $z_0^k = \eta_k$, $z_{n_k}^k = \xi_{k+1}$, and $|z_i^k - z_{i-1}^k| < 1/k$ for $1 \leq i \leq n_k$. Now write

$$\left(\xi_1, \eta_1, z_1^1, \ldots, z_{n_1-1}^1, \xi_2, \eta_2, z_1^2, \ldots, z_{n_2-1}^2, \xi_3, \eta_3, \ldots, \right. \tag{5}$$
$$\left. \xi_k, \eta_k, z_1^k, \ldots, z_{n_{k-1}}^k, \xi_{k+1}, \eta_{k+1}, \ldots \right).$$

Then denoting this sequence by (α_n), we obtain that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (\alpha_{j-1}, \alpha_j)$. The sequence constructed is a quasi-Cauchy sequence, and it is an N_θ-quasi-Cauchy sequence, since any quasi-Cauchy sequence is an N_θ-quasi-Cauchy sequence. This completes the proof of the lemma. □

Theorem 3. *If a function f defined on an interval E is N_θ-ward continuous, then, it is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on E. Then, there is an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exist $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For every integer $n \geq 1$ fix $\xi_n, \eta_n \in E$ with $|\xi_n - \eta_n| < 1/n$ and $|f(\xi_n) - f(\eta_n)| \geq \varepsilon_0$. By the lemma, there exists an N_θ-quasi-Cauchy sequence (α_i) such that for any integer $i \geq 1$ there exists a j with $\xi_i = \alpha_j$ and $\eta_i = \alpha_{j+1}$. This implies that $|f(\alpha_{j+1}) - f(\alpha_j)| \geq \varepsilon_0$; hence, $(f(\alpha_i))$ is not N_θ-quasi-Cauchy. Thus, f does not preserve N_θ-quasi-Cauchy sequences. This completes the proof of the theorem. □

Observing that the sequence, constructed in the proof of the preceding theorem, is also a quasi-Cauchy sequence, we obtain that a real function f defined on an interval E is uniformly continuous if $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E. Combining this with Theorem 1, we have that a real function f defined on an interval E is uniformly continuous if and only if $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.

Corollary 4. *If a function defined on an interval is N_θ-ward continuous, then, it is ward continuous.*

Proof. The proof follows from Theorem 3 and [7, Theorem 6] so it is omitted. □

Corollary 5. *If a function defined on an interval is N_θ-ward continuous, then, it is slowly oscillating continuous.*

Proof. The proof follows from Theorem 3 and [7, Theorem 5] so it is omitted. □

It is a well-known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of N_θ-ward continuity; that is, uniform limit of a sequence of N_θ-ward continuous functions is N_θ-ward continuous.

Theorem 6. *If (f_n) is a sequence of N_θ-ward continuous functions on a subset E of $\mathbf{R}$ and (f_n) is uniformly convergent to a function f, then, f is N_θ-ward continuous on E.*

Proof. Let (α_k) be any N_θ-quasi-Cauchy sequence of points in E, and let ε be any positive real number. By uniform convergence of (f_n), there exists an $n_1 \in \mathbf{N}$ such that $|f(\alpha) - f_k(\alpha)| < \varepsilon/3$ for $n \geq n_1$ and every $\alpha \in E$. Hence,

$$\frac{1}{h_r} \sum_{k \in I_r} |f(\alpha) - f_k(\alpha)| < \frac{1}{h_r} (k_r - k_{r-1}) \frac{\varepsilon}{3} = \frac{\varepsilon}{3}, \tag{6}$$

for $r \geq n_1$ and every $\alpha \in E$. As f_{n_1} is N_θ-ward continuous on E, there exists an $n_2 \in \mathbf{N}$ such that for $r \geq n_2$

$$\frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_{k+1}) - f_{n_1}(\alpha_k) \right| < \frac{\varepsilon}{3}. \tag{7}$$

Now write $n_0 = \max\{n_1, n_2\}$. Thus for $r \geq n_0$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} |f(\alpha_{k+1}) - f(\alpha_k)|$$
$$\leq \frac{1}{h_r} \sum_{k \in I_r} \left| f(\alpha_{k+1}) - f_{n_1}(\alpha_{k+1}) \right|$$
$$+ \frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_{k+1}) - f_{n_1}(\alpha_k) \right| \tag{8}$$
$$+ \frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_k) - f(\alpha_k) \right|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |f(\alpha_{k+1}) - f(\alpha_k)| = 0. \tag{9}$$

Thus, f preserves N_θ-quasi-Cauchy sequences. This completes the proof of the theorem. □

Theorem 7. *The set of all N_θ-ward continuous functions on a subset E of $\mathbf{R}$ is a closed subset of the set of all continuous functions on E, that is, $\overline{\Delta N_\theta WC(E)} = \Delta N_\theta WC(E)$, where $\Delta N_\theta WC(E)$ is the set of all N_θ-ward continuous functions on E and $\overline{\Delta N_\theta WC(E)}$ denotes the set of all cluster points of $\Delta N_\theta WC(E)$.*

Proof. Let f be any element in the closure of $\Delta N_\theta WC(E)$. Then, there exists a sequence (f_n) of points in $\Delta N_\theta WC(E)$ such that $\lim_{k \to \infty} f_k = f$. To show that f is N_θ-ward continuous, take any N_θ-quasi-Cauchy sequence (α_k) of points in E. Let $\varepsilon > 0$. Since (f_n) converges to f, there exists an $n_1 \in \mathbf{N}$ such that $|f(\alpha_k) - f_{n_1}(\alpha_k)| < \varepsilon/3$ for all $k \in \mathbf{N}$. Hence,

$$\frac{1}{h_r} \sum_{k \in I_r} \left| f(\alpha_k) - f_{n_1}(\alpha_k) \right| < \frac{1}{h_r} (k_r - k_{r-1}) \frac{\varepsilon}{3} = \frac{\varepsilon}{3}, \tag{10}$$

for $r \geq n_1$. As f_{n_1} is N_θ-ward continuous on E, there exists a positive integer $n_2 \in \mathbf{N}$ such that $r \geq n_2$ implies that

$$\frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_{k+1}) - f_{n_1}(\alpha_k) \right| < \frac{\varepsilon}{3}. \tag{11}$$

Now write $n_0 = \max\{n_1, n_2\}$. Thus for $r \geq n_0$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} \left| f(\alpha_{k+1}) - f(\alpha_k) \right|$$

$$\leq \frac{1}{h_r} \sum_{k \in I_r} \left| f(\alpha_{k+1}) - f_{n_1}(\alpha_{k+1}) \right|$$

$$+ \frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_{k+1}) - f_{n_1}(\alpha_k) \right| \qquad (12)$$

$$+ \frac{1}{h_r} \sum_{k \in I_r} \left| f_{n_1}(\alpha_k) - f(\alpha_k) \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon;$$

Hence,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| f(\alpha_{k+1}) - f(\alpha_k) \right| = 0. \qquad (13)$$

Thus, f preserves N_θ-quasi-Cauchy sequences. This completes the proof of the theorem. □

Corollary 8. *The set of all N_θ-ward continuous functions on a subset E of $\mathbf{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from the preceding theorem. □

4. Conclusion

In this paper, new results concerning N_θ-ward continuity are obtained namely; a real function f defined on an interval E is uniformly continuous if and only if $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E, the uniform limit of N_θ-ward continuous functions is N_θ-ward continuous, and the set of all N_θ-ward continuous functions is a closed subset of the set of all continuous functions. We also prove that if a function f is uniformly continuous on a subset E of $\mathbf{R}$, then, $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in E.

As a further study one can find out if Theorem 3 is valid when the set E is replaced by a G-sequentially connected subset of $\mathbf{R}$ for a regular sequential method G [18]. For another further study, we suggest to investigate the present work for the fuzzy case. However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [19] for the definitions in the fuzzy setting). One can introduce and give an investigation of N_θ-quasi-Cauchy sequences in cone normed spaces (see [20] for basic concepts in cone normed spaces).

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