$N-$LAPLACIAN EQUATIONS IN $\mathbb{R}^N$ WITH CRITICAL GROWTH

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ABSTRACT. We study the existence of nontrivial solutions to the following problem:

$$\begin{cases} u \in W^{1,N}(\mathbb{R}^N), \ u \geq 0 \text{ and} \\ -div(| \nabla u |^{N-2} \nabla u) + a(x) \mid u \mid^{N-2} u = f(x,u) \text{ in } \mathbb{R}^N \ (N \geq 2), \end{cases}$$

where a is a continuous function which is coercive, i.e., $a(x) \to \infty$ as $\mid x \mid \to \infty$ and the nonlinearity f behaves like $\exp\left(\alpha \mid u \mid^{N/(N-1)}\right)$ when $\mid u \mid \to \infty$.

1. INTRODUCTION

In this paper, we apply a mountain pass type argument to prove the existence of nontrivial weak solutions to the following class of semilinear elliptic problems in $\mathbb{R}^N$ $(N \geq 2)$, involving critical growth:

$$\begin{cases} u \in W^{1,N}(\mathbb{R}^N), \qquad u \geq 0 \qquad \text{and} \\ -div(| \nabla u |^{N-2} \nabla u) + a(x) \mid u \mid^{N-2} u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1}$$

where $a : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying $a(x) \geq a_0, \quad \forall x \in \mathbb{R}^N$, and such that $a(x) \to \infty$ as $\mid x \mid \to \infty$. It is assumed that the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is also continuous and $f(x,0) \equiv 0$. Thus, $u \equiv 0$ is a solution of (1) and has critical growth, i.e., f behaves like $\exp\left(\alpha \mid u \mid^{N/(N-1)}\right)$ when $\mid u \mid \to \infty$.

For $N = 2$, problems of this type, that is, involving the Laplacian operator and critical growth in the whole $\mathbb{R}^2$, have been considered by Cao in [10] and by Cao and Zhengjie in [11], under the decisive hypothesis that the function

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a be a constant. For that purpose, they used the concentration-compactness principle of P. L. Lions.

Recently, Rabinowitz in [29], among other results, obtained a nontrivial solution to the problem $-\Delta u + a(x)u = f(x,u)$ in $\mathbb{R}^N$, under the assumption that a is coercive and that the potential $F(x,u) = \int_0^u f(x,s)ds$ is superquadratic and $f(x,u)$ has subcritical growth, that is, $\mid f(x,u) \mid \leq b_1 + b_2 \mid u \mid^s$, where $s \in (1, (N+2)/(N-2))$. This result was extended by Costa [15] to a class of potentials $F(x,u)$ which are nonquadratic at infinity. Miyagaki, in [24], has treated this problem for $N \geq 3$, involving critical Sobolev exponent, namely for $f(x,u) = \lambda \mid u \mid^{q-1} u + \mid u \mid^{p-1} u$, where $1 < q < p \leq (N-2)/(N+2)$ and $\lambda > 0$. In [3], this result was generalized by Alves to the $p-$Laplacian operator.

In this paper, we complement the results mentioned above by establishing sufficient conditions for the existence of nontrivial solutions to (1). To treat variationally this class of problems, with f behaving like $\exp\left(\alpha \mid u \mid^{\frac{N}{N-1}}\right)$ when $\mid u \mid \to \infty$, we introduce a Trudinger-Moser type inequality. On the other hand, to overcome the lack of compactness that has arisen from the critical growth and the unboundedness of the domain, we use some recent ideas from [16, 19] together with a compact imbedding result essentially given by the coerciveness of a (cf. [15]).

We would also like to mention that problems involving the Laplacian operator with critical growth range in bounded domains of $\mathbb{R}^2$ have been investigated, among others, by [2, 5, 6, 9, 16, 17, 22, 23, 30]. We refer to [1, 19, 26] for semilinear problems with critical growth for the $N-$Laplacian in bounded domains of $\mathbb{R}^N$.

Now we shall describe the conditions on the functions a and f. Namely, for a we suppose that:

(a_1) there exists a positive real number a_0 such that $a(x) \geq a_0$, $\forall x \in \mathbb{R}^N$,

(a_2) $a(x) \to \infty$ as $\mid x \mid \to \infty$.

On the other hand, motivated by a Trudinger-Moser type inequality (cf. Lemma 1 below), we assume the following growth condition on the nonlinearity $f(x,u)$,

(f_1) the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous and for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$,

$$\mid f(x,u) \mid \leq b_1 \mid u \mid^{N-1} + b_2 \left[\exp\left(\alpha_0 \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha_0, u)\right],$$

for some constants α_0, b_1, $b_2 > 0$, where

$$S_{N-2}(\alpha_0, u) = \sum_{k=0}^{N-2} \frac{\alpha_0^k}{k!} \mid u \mid^{\frac{N}{N-1}k}.$$

Moreover, f is assumed to satisfy the following conditions:

(f_2) there is a constant $\mu > N$ such that, for all $x \in \mathbb{R}^N$ and $u > 0$,

$$0 \leq \mu F(x,u) \equiv \mu \int_0^u f(x,t)dt \leq u f(x,u),$$

(f_3) there are constants R_0, $M_0 > 0$ such that, for all $x \in \mathbb{R}^N$ and $u \geq R_0$,

$$0 < F(x,u) \leq M_0 f(x,u);$$

(f_4) $\lim\limits_{u \to +\infty} u f(x,u) \exp\left(-\alpha_0 \mid u \mid^{\frac{N}{N-1}}\right) \geq \beta_0 > 0$ uniformly on compact subsets of $\mathbb{R}^N$.

As usual, $W^{1,N}(\mathbb{R}^N)$ denotes the Sobolev space of functions in $L^N(\mathbb{R}^N)$ such that their weak derivatives are also in $L^N(\mathbb{R}^N)$ with the norm

$$\| u \|_{W^{1,N}}^N \doteq \int_{\mathbb{R}^N} (\mid \nabla u \mid^N + \mid u \mid^N) dx,$$

and we consider the subspace $E \subset W^{1,N}(\mathbb{R}^N)$ given by

$$E = \{u \in W^{1,N}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} a(x) \mid u \mid^N dx < \infty\}$$

endowed with the norm

$$\| u \|_E^N \doteq \int_{\mathbb{R}^N} (\mid \nabla u \mid^N + a(x) \mid u \mid^N) dx.$$

Since $a(x) \geq a_0 > 0$, we clearly see that the Banach space E is a continuously embedded in $W^{1,N}\left(\mathbb{R}^N\right)$ and, moreover,

$$\lambda_1(N) = \inf_{0 \neq u \in E} \frac{\| u \|_E^N}{\| u \|_{L^N}^N} \geq a_0 > 0. \tag{2}$$

The main result of this paper is the following

Theorem 1. *Suppose $(a_1)-(a_2)$ and $(f_1)-(f_4)$ are satisfied. Furthermore, assume that*

(f_5) $$\limsup_{u \to 0^+} \frac{NF(x,u)}{\mid u \mid^N} < \lambda_1(N) \quad uniformly\ in\ x \in \mathbb{R}^N.$$

Then the problem (1) has a nontrivial weak solution $u \in E$.

Remark 1. *The assumption (a_2) implies that the Banach space E is compactly immersed in L^q if $N \leq q < \infty$. We observe that this compact embedding result is used here only to prove that the Palais-Smale sequence obtained by mountain pass type argument converges to a weak nontrivial solution. Therefore, the same device can be applied when we have some assumption which implies a compact embedding result as the cited above. For instance, when the function a is a radially symmetric function, that is, $a(x) = a(y)$ if $\mid x \mid = \mid y \mid$ (cf.* [4, 18]*).*

2. A Trudinger-Moser inequality

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. The Trudinger-Moser inequality (cf. [25, 31]) asserts that

$$\exp(\alpha \mid u \mid^{\frac{N}{N-1}}) \in L^1(\Omega), \quad \forall u \in W_0^{1,N}(\Omega), \quad \forall \alpha > 0$$

and that there exists a constant $C(N)$ which depends on N only, such that

$$\sup_{\|u\|_{W_0^{1,N}} \leq 1} \int_\Omega \exp(\alpha \mid u \mid^{\frac{N}{N-1}}) \leq C(N) \mid \Omega \mid, \quad if \quad \alpha \leq \alpha_N,$$

where $\mid \Omega \mid = \int_\Omega dx$, $\alpha_N = N w_{N-1}^{\frac{1}{N-1}}$ and w_{N-1} is the $(N-1)-$dimensional measure of the $(N-1)-$sphere.

Inspired of this inequality and based on the related results [7, 10, 11, 12, 13, 14], we get the following result.

Lemma 1. *If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}\left(\mathbb{R}^N\right)$, then*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] < \infty. \tag{3}$$

Moreover, if $\| \nabla u \|_{L^N}^N \leq 1$, $\| u \|_{L^N} \leq M < \infty$ and $\alpha < \alpha_N = N w_{N-1}^{\frac{1}{N-1}}$, then there exists a constant $C = C(N, M, \alpha)$, which depends only on N, M and α, such that

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right] \leq C(N, M, \alpha). \tag{4}$$

Proof. We may assume $u \geq 0$, since we can replace u by $\mid u \mid$ without causing any increase in the integral of the gradient. Since we shall use Schwarz symmetrization method, we recall briefly some of theirs basic properties (cf. [20, 27]). Let $1 \leq p \leq \infty$ and $u \in L^p(\mathbb{R}^N)$ such that $u \geq 0$. Thus, there is a unique nonnegative function $u^* \in L^p(\mathbb{R}^N)$, called the Schwarz symmetrization of u, such that it depends only on $\mid x \mid$, u^* is a decreasing function of $\mid x \mid$; for all $\lambda > 0$

$$\mid \{x : u^*(x) \geq \lambda\} \mid = \mid \{x : u(x) \geq \lambda\} \mid$$

and there exists $R_\lambda > 0$ such that $\{x : u^* \geq \lambda\}$ is the ball $B(0, R_\lambda)$ of radius R_λ centered at origin. Moreover, suppose that $G : [0, +\infty) \to [0, +\infty)$ is a continuous and increasing function such that $G(0) = 0$. Then, we have

$$\int_{\mathbb{R}^N} G(u^*(x))\, dx = \int_{\mathbb{R}^N} G(u(x))\, dx.$$

Further, if $u \in W_0^{1,p}\left(\mathbb{R}^N\right)$ then $u^* \in W_0^{1,p}\left(\mathbb{R}^N\right)$ and

$$\int_{\mathbb{R}^N} \mid \nabla u^* \mid^p (x)\, dx \leq \int_{\mathbb{R}^N} \mid \nabla u \mid^p (x)\, dx.$$

Thus, we can write

$$\int_{\mathbb{R}^N}\left[\exp\left(\alpha \mid u \mid^{\frac{N}{N-1}}\right) -S_{N-2}\left(\alpha,u\right)\right] = \int_{\mathbb{R}^N}\left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\left(\alpha,u^*\right)\right],$$

and, for a real number $r > 1$ to be determined, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\left(\alpha,u^*\right)\right]\\ &= \int_{|x|<r}\left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\left(\alpha,u^*\right)\right]\\ &\quad + \int_{|x|\geq r}\left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\left(\alpha,u^*\right)\right]\\ &\leq \int_{|x|<r}\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) + \int_{|x|\geq r}\left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\left(\alpha,u^*\right)\right].\end{aligned}$$

Let us recall two elementary inequalities. Using the fact that the function $h : (0,+\infty) \to \mathbb{R}$ given by $h\left(t\right) = [(t+1)^{\frac{N}{N-1}} - t^{\frac{N}{N-1}} - 1]/t^{\frac{1}{N-1}}$ is bounded, we have a positive constant $A = A(N)$ such that

$$(u+v)^{\frac{N}{N-1}} \leq u^{\frac{N}{N-1}} + Au^{\frac{1}{N-1}}v + v^{\frac{N}{N-1}}, \quad \forall u,v \geq 0. \tag{5}$$

If γ and γ' are positive real numbers such that $\gamma+\gamma' = 1$, then for all $\varepsilon > 0$, we have

$$u^{\gamma}v^{\gamma'} \leq \varepsilon u + \varepsilon^{-\frac{\gamma}{\gamma'}}v, \quad \forall u,v \geq 0, \tag{6}$$

because $g : [0,+\infty) \to \mathbb{R}$, given by $g\left(t\right) = t^{\gamma} - \varepsilon t$, is bounded.

Let $v\left(x\right) = u^*\left(x\right) - u^*\left(rx_0\right)$ where x_0 is some fixed unit vector in $\mathbb{R}^N$. Notice that $v \in W_0^{1,N}\left(B\left(0,r\right)\right)$. Here, $B(0,r)$ denotes the ball of radius r centered at the origin of $\mathbb{R}^N$. Now, from (5) and (6), we have, respectively,

$$\mid u^* \mid^{\frac{N}{N-1}} = \mid v + u^*\left(rx_0\right) \mid^{\frac{N}{N-1}} \leq v^{\frac{N}{N-1}} + Av^{\frac{1}{N-1}}u^*\left(rx_0\right) + u^*\left(rx_0\right)^{\frac{N}{N-1}},$$

$$\begin{aligned}v^{\frac{1}{N-1}}u^*\left(rx_0\right) &= \left(v^{\frac{N}{N-1}}\right)^{\frac{1}{N}}\left(u^*\left(rx_0\right)^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}}\\ &\leq \frac{\varepsilon}{A}v^{\frac{N}{N-1}} + \left(\frac{\varepsilon}{A}\right)^{\frac{1}{1-N}}u^*\left(rx_0\right)^{\frac{N}{N-1}},\end{aligned}$$

and hence,

$$\mid u^* \mid^{\frac{N}{N-1}} \leq (1+\varepsilon)\, v^{\frac{N}{N-1}} + K(\varepsilon,N)u^*\left(rx_0\right)^{\frac{N}{N-1}},$$

where $K(\varepsilon,N) = A^{\frac{N}{N-1}}\varepsilon^{\frac{1}{1-N}} + 1$. Therefore,

$$\begin{aligned}&\textstyle\int_{|x|\leq r}\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) \leq\\ &\exp\left(K(\varepsilon,N)u^*\left(rx_0\right)^{\frac{N}{N-1}}\right)\textstyle\int_{|x|\leq r}\exp\left(\alpha \mid (1+\varepsilon)\, v \mid^{\frac{N}{N-1}}\right),\end{aligned}$$

which, in view of Trudinger-Moser inequality, implies,

$$\int_{|x|\leq r} \exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) < \infty, \quad \forall u \in W^{1,N}\left(\mathbb{R}^N\right), \quad \forall \alpha > 0. \tag{7}$$

Furthermore, taking $\varepsilon > 0$ such that $(1+\varepsilon)\,\alpha < \alpha_N$, we obtain

$$\begin{aligned}
&\int_{|x|\leq r} \exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) \\
&\qquad \leq C\,(N)\, \frac{w_{N-1} r^N}{N} \exp\left(K(\varepsilon,N) u^*\,(r x_0)^{\frac{N}{N-1}}\right) \\
&\qquad \leq C\,(N)\, \frac{w_{N-1} r^N}{N} \exp\left(\left(\frac{N M^N}{w_{N-1}}\right)^{\frac{1}{N-1}} \frac{K(\varepsilon,N)}{r^{\frac{N}{N-1}}}\right),
\end{aligned} \tag{8}$$

for all $u \in W^{1,N}\left(\mathbb{R}^N\right)$ such that $\| \nabla u \|_{L^N}^N \leq 1$ and $\| u \|_{L^N} \leq M$, where in the last inequality we have used Radial Lemma A.IV in [8]:

$$| u^*\,(x) | \leq | x |^{-1} \left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}} \| u^* \|_{L^N(\mathbb{R}^N)}, \quad \forall x \neq 0.$$

On the other hand, we have

$$\begin{aligned}
&\int_{|x|\geq r} \left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\,(u^*)\right] \\
&= \frac{\alpha^{N-1}}{(N-1)!} \int_{|x|\geq r} | u^* |^N + \sum_{k=N}^{\infty} \frac{\alpha^k}{k!} \int_{|x|\geq r} | u^* |^{\frac{N}{N-1}k}.
\end{aligned} \tag{9}$$

Now, notice that the estimate

$$\begin{aligned}
\int_{|x|\geq r} \frac{1}{| x |^{\frac{N}{N-1}k}} dx &= w_{N-1} \int_r^{\infty} \frac{t^{N-1}}{t^{\frac{N}{N-1}k}} dt \\
&= \left(\frac{w_{N-1}}{\frac{N}{N-1}k - N}\right) r^{N-\frac{N}{N-1}k} \leq \frac{w_{N-1} r^N}{r^{\frac{N}{N-1}k}}, \ \forall k \geq N,
\end{aligned}$$

together with Radial Lemma, lead to

$$\begin{aligned}
&\sum_{k=N}^{\infty} \frac{\alpha^k}{k!} \int_{|x|\geq r} | u^* |^{\frac{N}{N-1}k} \\
&\qquad \leq w_{N-1} r^N \sum_{k=N}^{\infty} \frac{\alpha^k}{k!} \left[\left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}} \left(\frac{\| u^* \|_{L^N(\mathbb{R}^N)}}{r}\right)\right]^{\frac{N}{N-1}k}.
\end{aligned} \tag{10}$$

Finally, (7), (9) and (10) imply the existence of the integral in (3). Furthermore, in the case that $\alpha < \alpha_N$ and $\| u \|_{L^N} \leq M$, if we choose $r = M\left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}}$, we have

$$\int_{|x|\geq r} \left[\exp\left(\alpha \mid u^* \mid^{\frac{N}{N-1}}\right) - S_{N-2}\,(u)\right] \leq N M^N \exp\left(\alpha_N\right),$$

which, in combination with (8), implies (4). ∎

3. The variational formulation

First, we observe that since we are interested in obtaining nonnegative solutions, it is convenient to define

$$f(x,u)=0, \quad \forall (x,u)\in \mathbb{R}^N\times(-\infty,0].$$

From (f_1), we obtain for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$,

$$| F(x,u) | \le b_3\left[\exp\left(\alpha_1 \mid u \mid^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha_1,u)\right], \tag{11}$$

for some constants α_1, $b_3>0$. Thus, by lemma 1, we have $F(x,u)\in L^1(\mathbb{R}^N)$ for all $u\in W^{1,N}\left(\mathbb{R}^N\right)$. Therefore, the functional $I:E\to\mathbb{R}$ given by

$$I(u)=\frac{1}{N}\parallel u\parallel_E^N-\int\limits_{\mathbb{R}^N}F(x,u)dx$$

is well defined. Furthermore, using standard arguments (cf. Theorem A.VI in [8]) as well as the fact that for any given strong convergent sequence (u_n) in $W^{1,N}\left(\mathbb{R}^N\right)$ there is a subsequence (u_{n_k}) and there exists $h\in W^{1,N}\left(\mathbb{R}^N\right)$ such that $| u_{n_k}(x) | \le h(x)$ almost everywhere in $\mathbb{R}^N$, we see that I is a C^1 functional on E with

$$I'(u)v=\int\limits_{\mathbb{R}^N}(| \nabla u |^{N-2}\nabla u\nabla v+a(x) \mid u \mid^{N-2} uv)dx-\int\limits_{\mathbb{R}^N}f(x,u)vdx,\ \forall v\in E.$$

Consequently, critical points of the functional I are precisely the weak solutions of problem (1). Here, like in [29, 24, 19], we are going to use a Mountain-Pass Theorem without a compactness condition such as the one of Palais-Smale type. This version of Mountain -Pass Theorem is a consequence of Ekeland's variational principle (cf. [21]). In the next two lemmas we check that the functional I satisfies the geometric conditions of the Mountain-Pass Theorem (cf. [28]).

Lemma 2. *Assume that* (a_1), (f_1), (f_2) *and* (f_3) *are satisfied. Then for any* $u\in W^{1,N}\left(\mathbb{R}^N\right)-\{0\}$ *with compact support and* $u\ge 0$, *we have* $I(tu)\to-\infty$ *as* $t\to\infty$.

Proof. Let $u\in W^{1,N}\left(\mathbb{R}^N\right)-\{0\}$ with compact support and $u\ge0$. By (f_2) and (f_3) there are positive constants c, d such that

$$F(x,s)\ge cs^\mu-d, \quad \forall x\in supp\,(u),\forall s\in[0,+\infty).$$

Thus,

$$I(tu)\le\frac{t^N}{N}\parallel u\parallel_E^N -ct^\mu\int\limits_{\mathbb{R}^N}u^\mu+d \mid supp\,(u) \mid,$$

which implies that $I(tu)\to-\infty$ as $t\to\infty$, since $\mu>N$. ∎

Lemma 3. *Suppose that* (a_1), (f_1) *and* (f_5) *hold. Then there exist* α, $\rho>0$ *such that*

$$I(u)\ge\alpha \quad if \quad \parallel u\parallel_E=\rho.$$

Proof. From (f_5), there exist ε, $\delta > 0$ in such a way that $\mid u \mid \leq \delta$ implies

$$F(x,u) \leq \frac{(\lambda_1(N) - \varepsilon)}{N} \mid u \mid^N$$

for all $x \in \mathbb{R}^N$. On the other hand, for $q > N$, by (f_1), there are positive constants $\beta,\ C = C(q, \delta)$ such that $\mid u \mid \geq \delta$ implies

$$F(x,u) \leq C \mid u \mid^q \left[\exp\left(\beta \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)\right]$$

for all $x \in \mathbb{R}^N$. These two estimates yield,

$$F(x,u) \leq \frac{(\lambda_1(N) - \varepsilon)}{N} \mid u \mid^N + C \mid u \mid^q \left[\exp\left(\beta \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)\right],$$

for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$. In what follows we make use of the inequality

$$\int_{\mathbb{R}^N} \mid u \mid^q \left[\exp\left(\beta \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)\right] \leq C(\beta, N) \parallel u \parallel_E^q, \tag{12}$$

to be proved later, assuming that $\parallel u \parallel_E \leq M$ holds, where M is sufficiently small. Under the assumption we have just done, by means of (2) and the continuous imbedding $E \hookrightarrow L^N(\mathbb{R}^N)$, we achieve

$$\begin{aligned} I(u) &\geq \frac{1}{N} \parallel u \parallel_E^N - \frac{(\lambda_1(N) - \varepsilon)}{N} \parallel u \parallel_{L^N}^N - C \parallel u \parallel_E^q \\ &\geq \frac{1}{N}\left(1 - \frac{(\lambda_1(N) - \varepsilon)}{\lambda_1(N)}\right) \parallel u \parallel_E^N - C \parallel u \parallel_E^q . \end{aligned}$$

Thus, since $\varepsilon > 0$ and $q > N$, we may choose α, $\rho > 0$ such that $I(u) \geq \alpha \quad if \quad \parallel u \parallel_E = \rho$.

Now, let us obtain inequality (12). As it has been done in the proof of lemma 1, we use shall the method of symmetrization. Letting $R(\beta, u) = \exp\left(\beta \mid u \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)$, we have

$$\int_{\mathbb{R}^N} R(\beta, u) \mid u \mid^q dx = \int_{\mathbb{R}^N} R(\beta, u^*) \mid u^* \mid^q dx$$

and

$$\int_{\mathbb{R}^N} R(\beta, u^*) \mid u^* \mid^q dx = \int_{|x| \leq \sigma} R(\beta, u^*) \mid u^* \mid^q dx + \int_{|x| \geq \sigma} R(\beta, u^*) \mid u^* \mid^q dx,$$

where σ is a number to be determined later. Using the Hölder inequality, we obtain

$$\begin{aligned} \int_{|x| \leq \sigma} R(\beta, u^*) \mid u^* \mid^q dx &\leq \int_{|x| \leq \sigma} [\exp(\beta \mid u^* \mid^{\frac{N}{N-1}})] \mid u^* \mid^q dx \\ &\leq \left(\int_{|x| \leq \sigma} \exp(\beta r \mid u^* \mid^{\frac{N}{N-1}})\right)^{\frac{1}{r}} \left(\int_{|x| \leq \sigma} \mid u^* \mid^{qs}\right)^{\frac{1}{s}}, \end{aligned}$$

where $1/r + 1/s = 1$. Now, proceeding as in the proof of lemma 1, we obtain

$$\int_{|x| \leq \sigma} \exp(\beta r \mid u^* \mid^{\frac{N}{N-1}}) dx \leq C(\beta, N)$$

if $\| u \|_E \leq M$, where M is such that $\beta r M^{\frac{N}{N-1}} < \alpha_N$. Thus, using the continuous imbedding $E \hookrightarrow L^{qs}(\mathbb{R}^N)$, we have

$$\int_{|x|\leq\sigma} R(\beta, u^*) \mid u^* \mid^q dx \leq C(\beta, N) \| u \|_E^q . \tag{13}$$

On the other hand, the Radial Lemma leads to

$$\begin{aligned}
&\int_{|x|\geq\sigma} \mid u^* \mid^{\frac{N}{N-1}k} \mid u^* \mid^q dx \\
&\leq \left(\left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}} \| u^* \|_{L^N(\mathbb{R}^N)}\right)^{\frac{N}{N-1}k} \int_{|x|\geq\sigma} \frac{\mid u^* \mid^q}{\mid x \mid^{\frac{N}{N-1}k}} dx \\
&\leq \left(\left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}} \| u^* \|_{L^N(\mathbb{R}^N)}\right)^{\frac{N}{N-1}k} \left(\int_{|x|\geq\sigma} \frac{1}{\mid x \mid^{\frac{N}{N-1}kr}}\right)^{\frac{1}{r}} \left(\int_{|x|\geq\sigma} \mid u^* \mid^{qs}\right)^{\frac{1}{s}} \\
&\leq w_{N-1}\sigma^N \left(\frac{\left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}} \| u^* \|_{L^N(\mathbb{R}^N)}}{\sigma^r}\right)^{\frac{N}{N-1}k} \| u \|_{L^{sq}(\mathbb{R}^N)}^q \\
&\leq C(N, M) \| u \|_E^q,
\end{aligned}$$

for all $k \geq N$, if we choose $\sigma^r = M_0 \left(\frac{N}{w_{N-1}}\right)^{\frac{1}{N}}$ where $\| u \|_{L^N(\mathbb{R}^N)} \leq M_0 = \lambda_1(N)^{1/N} M$. We also have that

$$\begin{aligned}
\int_{|x|\geq\sigma} \mid u^* \mid^N \mid u^* \mid^q dx &\leq \left(\int_{|x|\geq\sigma} \mid u^* \mid^{Nr} dx\right)^{\frac{1}{r}} \left(\int_{|x|\geq\sigma} \mid u^* \mid^{qs} dx\right)^{\frac{1}{s}} \\
&\leq \| u^* \|_{L^{Nr}(\mathbb{R}^N)}^N \| u^* \|_{L^{qs}(\mathbb{R}^N)}^q \\
&\leq C\,(N, M) \| u^* \|_E^q,
\end{aligned}$$

if $\| u^* \|_E^q \leq M$, via the continuous imbedding $E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^{Nr}(\mathbb{R}^N)$. Therefore,

$$\int_{|x|\geq R} R_N(\beta, u^*) \mid u^* \mid^q dx \leq C(N, M) \exp(\beta) \| u \|_E^q . \tag{14}$$

Finally, the combination of estimates (13) and (14) leads to (12). ∎

In order to get a more precise information about the minimax level obtained by the Mountain Pass Theorem, let us consider the following sequence of nonnegative functions

$$\widetilde{\mathfrak{M}}_n(x, r) = w_{N-1}^{-\frac{1}{N}} \begin{cases} (\log n)^{\frac{N-1}{N}} & if \quad \mid x \mid \leq r/n \\ \log(\dfrac{r}{\mid x \mid})/(\log n)^{\frac{1}{N}} & if \quad r/n \leq \mid x \mid \leq r \\ 0 & if \quad \mid x \mid \geq r. \end{cases}$$

Notice that: $\widetilde{\mathfrak{M}}_n(\cdot, r) \in W^{1,N}(\mathbb{R}^N)$, the support of $\widetilde{\mathfrak{M}}_n(x, r)$, is the ball $B[0, r]$ of radius r centered at zero, $\int_{\mathbb{R}^N} \mid \nabla\widetilde{\mathfrak{M}}_n(x, r) \mid^N dx = 1$ and

$\int_{\mathbb{R}^N} \mid \widetilde{\mathfrak{M}}_n(x,r) \mid^N dx = O(1/\log n) \quad as \quad n \to \infty$. Moreover, let $\mathfrak{M}_n(x,r) = \widetilde{\mathfrak{M}}_n(x,r)/ \parallel \widetilde{\mathfrak{M}}_n \parallel_E$. Thus, it is not difficult to see that

$$\mathfrak{M}_n^{\frac{N}{N-1}}(x,r) = w_{N-1}^{-\frac{1}{N-1}} \log n + d_n, \ \forall \mid x \mid \leq r/n, \tag{15}$$

where d_n is a bounded sequence of nonnegative numbers.

Lemma 4. *Suppose that* (a_1) *and* $(f_1) - (f_5)$ *hold true. Then there exists* $\mathfrak{M}_n(\cdot, r)$ *such that*

$$\max\{I(t\mathfrak{M}_n) : t \geq 0\} < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Proof. Let r be a fixed positive real number such that

$$\beta_0 > \frac{1}{r^N}\left(\frac{N}{\alpha_0}\right)^{N-1}. \tag{16}$$

Suppose, by contradiction, that for all n we have

$$\max\{I(t\mathfrak{M}_n) : t \geq 0\} \geq \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1},$$

where $\mathfrak{M}_n(x) = \mathfrak{M}_n(x,r)$. In view of Lemma 2, given n there exists $t_n > 0$ such that

$$I\left(t_n\mathfrak{M}_n\right) = \max\{I\left(t\mathfrak{M}_n\right) : t \geq 0\}.$$

So,

$$I\left(t_n\mathfrak{M}_n\right) = \frac{t_n^N}{N} - \int_{\mathbb{R}^N} F\left(x, t_n\mathfrak{M}_n\right) dx \geq \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}, \tag{17}$$

and using the fact that $F\left(x,u\right) \geq 0$, we obtain

$$t_n^N \geq (\frac{\alpha_N}{\alpha_0})^{N-1}. \tag{18}$$

Since at $t = t_n$, we have $\dfrac{d}{dt}I\left(t\mathfrak{M}_n\right) = 0$, it follows that

$$t_n^N = \int_{\mathbb{R}^N} t_n\mathfrak{M}_n f\left(x, t_n\mathfrak{M}_n\right) dx = \int_{|x|\leq r} t_n\mathfrak{M}_n f\left(x, t_n\mathfrak{M}_n\right) dx. \tag{19}$$

Now, using hypothesis (f_5), given $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for all $u \geq R_\varepsilon$ and for all $\mid x \mid \leq r$,

$$uf(x,u) \geq (\beta_0 - \varepsilon)\exp(\alpha_0 \mid u \mid^{\frac{N}{N-1}}). \tag{20}$$

From (19) and (20), for large n, we obtain,

$$\begin{aligned} t_n^N &\geq (\beta_0 - \varepsilon)\int_{|x|\leq \frac{r}{n}} \exp(\alpha_0 \mid t_n\mathfrak{M}_n \mid^{\frac{N}{N-1}})dx \\ &= (\beta_0 - \varepsilon)\frac{w_{N-1}}{N}\left(\frac{r}{n}\right)^N \exp(\alpha_0 t_n{}^{\frac{N}{N-1}} w_{N-1}^{-\frac{1}{N-1}} \log n + \alpha_0 t_n{}^{\frac{N}{N-1}} d_n). \end{aligned}$$

Thus,

$$1 \geq (\beta_0 - \varepsilon)\frac{w_{N-1}}{N} r^N \exp[\frac{\alpha_0 N \log n}{\alpha_N} t_n{}^{\frac{N}{N-1}} + \alpha_0 t_n{}^{\frac{N}{N-1}} d_n - N\log t_n - N\log n].$$

Therefore, the sequence t_n is bounded, since otherwise, up to subsequences, we would have

$$\lim_{n\to\infty} \frac{\alpha_0 N \log n}{\alpha_N} t_n{}^{\frac{N}{N-1}} + \alpha_0 t_n{}^{\frac{N}{N-1}} d_n - N\log t_n - N\log n = +\infty,$$

which leads to a contradiction. Moreover, by (18) and

$$t_n^N \geq (\beta_0 - \varepsilon)\frac{w_{N-1}}{N} r^N \exp[(\frac{\alpha_0 t_n{}^{\frac{N}{N-1}}}{\alpha_N} - 1)N\log n + \alpha_0 t_n{}^{\frac{N}{N-1}} d_n]$$

it follows that

$$t_n^N \to (\frac{\alpha_N}{\alpha_0})^{N-1}, \quad \text{as} \quad n\to\infty. \tag{21}$$

Now, in order to estimate (19) more precisely, we consider the sets

$$A_n = \{x \in B[0,r] : t_n \mathfrak{M}_n \geq R_\varepsilon\} \quad \text{and} \quad B_n = B[0,r] - A_n.$$

From (19) and (20) we arrive at

$$\begin{aligned} t_n^N \geq (\beta_0 - \varepsilon)\int_{|x|\leq r} \exp(\alpha_0 \mid t_n \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx + \int_{B_n} t_n \mathfrak{M}_n f\,(x, t_n\mathfrak{M}_n)\,dx \\ - (\beta_0 - \varepsilon)\int_{B_n} \exp(\alpha_0 \mid t_n \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx. \end{aligned}$$

Notice that $\mathfrak{M}_n(x) \to 0$ and the characteristic functions $\chi_{B_n} \to 1$ for almost every x such that $\mid x \mid\leq r$. Therefore, the Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned} \int_{B_n} t_n \mathfrak{M}_n f\,(x, t_n \mathfrak{M}_n)\,dx \quad &\to \quad 0 \text{ and} \\ \int_{B_n} \exp(\alpha_0 \quad \mid \quad t_n \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx &\to \frac{w_{N-1}}{N} r^N \text{ as } n \to \infty. \end{aligned}$$

Note also that, by (18), $t_n^N \geq (\frac{\alpha_N}{\alpha_0})^{N-1}$,

$$\begin{aligned} &\int_{|x|\leq r} \exp(\alpha_0 \mid t_n \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx \geq \int_{|x|\leq r} \exp(\alpha_N \mid \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx \\ &= \int_{|x|\leq \frac{r}{n}} \exp(\alpha_N \mid \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx + \int_{\frac{r}{n}\leq|x|\leq r} \exp(\alpha_N \mid \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx, \end{aligned}$$

$$\begin{aligned} \int_{|x|\leq \frac{r}{n}} \exp(\alpha_N \mid \mathfrak{M}_n \mid^{\frac{N}{N-1}})dx &= \int_{|x|\leq \frac{r}{n}} \exp[\alpha_N \omega_{N-1}^{-\frac{1}{N-1}} \log n + d_n \alpha_N] \\ &= \frac{\omega_{N-1}}{N}\frac{r^N}{n^N} \exp[N\log n] \exp[d_n \alpha_N] \\ &= \frac{\omega_{N-1} r^N}{N} \exp[d_n \alpha_N] \geq \frac{\omega_{N-1} r^N}{N}, \end{aligned}$$

since $d_n \geq 0$. Using the change of variable $\tau = \log \frac{r}{s}/(\zeta_n \log n)$, with $\zeta_n = \| \widetilde{\mathfrak{M}}_n \|_E$, we have, by straightforward computation,

$$\int_{r/n\leq|x|\leq r} \exp(\alpha_N \mid M_n \mid^{\frac{N}{N-1}})dx$$
$$= w_{N-1}r^N\zeta_n \log n \int_0^{\zeta_n^{-1}} \exp[N \log n(\tau^{\frac{N}{N-1}} - \zeta_n\tau)]d\tau \to w_{N-1}r^N as\ n \to \infty$$

Finally, passing to limits, using (21) and the latter fact we obtain

$$\begin{aligned}(\frac{\alpha_N}{\alpha_0})^{N-1} &\geq (\beta_0 - \varepsilon)\frac{\omega_{N-1}r^N}{N}\{\exp[d_0\alpha_N] - 1\} \\ &\quad +(\beta_0 - \varepsilon)w_{N-1}r^N,\end{aligned}$$

where $d_0 = \liminf_{n\to\infty} d_n$ is a nonnegative number. Thus,

$$(\frac{\alpha_N}{\alpha_0})^{N-1} \geq (\beta_0 - \varepsilon)w_{N-1}r^N,$$

which implies

$$\beta_0 \leq \frac{1}{r^N}\left(\frac{N}{\alpha_0}\right)^{N-1},$$

contradicting (16). ∎

4. Proof of Theorem 1.1

In view of lemmas 2 and 3 we can apply the Mountain-Pass Theorem to obtain a sequence $(u_n) \subset E$ such that $I(u_n) \to c > 0$ and $I'(u_n) \to 0$, that is,

$$\frac{1}{N} \| u_n \|_E^N - \int_{\mathbb{R}^N} F(x,u_n)dx \to c, \quad as \quad n \to \infty, \tag{22}$$

$$\begin{aligned}&\mid \int_{\mathbb{R}^N} \left[\mid \nabla u_n \mid^{N-2} \nabla u_n \nabla v - a(x) \mid u_n \mid^{N-2} u_n v\right] - \int_{\mathbb{R}^N} f(x,u_n)v \mid \\ &\leq \varepsilon_n \| v \|_E,\end{aligned} \tag{23}$$

for all $v \in E$, where $\varepsilon_n \to 0$ as $n \to \infty$. Furthermore, by lemma 4, the level c is less than $\frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1})$. From now on, we shall be working in order to prove that (u_n) converges to a weak nontrivial solution u of problem (1). From (22), (23) and (f_2),

$$\begin{aligned}C + \varepsilon_n \| u_n \|_E &\geq (\frac{\mu}{N} - 1) \| u_n \|_E - \int_{\mathbb{R}^N}(\mu F(x,u_n) - f(x,u_n)u_n)dx \\ &\geq (\frac{\mu}{N} - 1) \| u_n \|_E^N,\end{aligned}$$

which implies that

$$\| u_n \|_E \leq C, \quad \int_{\mathbb{R}^N} f(x,u_n)u_n dx \leq C, \quad \int_{\mathbb{R}^N} F(x,u_n)dx \leq C.$$

Now, using the same argument as in Proposition 2.1 of [15], in view of Sobolev's Theorem together with conditions $(a_1) - (a_2)$, the Banach space E is compactly immersed in L^q if $N \leq q < \infty$, (cf. [3]). Therefore, up to subsequences, we have $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ in $L^q(\mathbb{R}^N)$, $\forall q \geq N$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. Moreover, arguing as in lemma 4 of [19], we get

$$\begin{cases} f(x,u_n) \to f(x,u) \text{ in } L^1(B(0,R)), \\ \mid \nabla u_n \mid^{N-2} \nabla u_n \to \mid \nabla u \mid^{N-2} \nabla u \\ \text{weakly in } \left(L^{N/(N-1)}(B(0,R))\right)^N, \end{cases} \tag{24}$$

for all $R > 0$. Therefore, by (23), passing to the limit,

$$\int_{\mathbb{R}^N} \left(\mid \nabla u \mid^{N-2} \nabla u \nabla \varphi - a(x) \mid u \mid^{N-2} u\varphi\right) dx = \int_{\mathbb{R}^N} f(x,u)\varphi dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, that is, u is a weak solution of (1). Let us show that u is nontrivial. Assume, by contradiction, that $u \equiv 0$. Using the Generalized Lebesgue Dominated Convergence Theorem (cf. [20]), by (f_3) and the first result in (24), we conclude that $F(x,u_n) \to 0$ in $L^1(B(0,R))$, for all $R > 0$. Thus, using (11), in view of Radial Lemma, we obtain $F(x,u_n) \to 0$ in $L^1(\mathbb{R}^N)$. This together with (22) imply

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mid \nabla u_n \mid^N = Nc, \tag{25}$$

and hence given $\epsilon > 0$, we have $\| \nabla u_n \|_{L^N}^N \leq Nc + \epsilon$, for large n. Using $c < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$ and choosing $q > 1$ sufficiently close to 1 and ϵ sufficiently small, we obtain $q\alpha_0 \| \nabla u_n \|_{L^N}^{\frac{N}{N-1}} < \alpha_N$. Hence, by the same kind of argument as it has been done in the proof of lemma 1, we conclude that

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha \mid u_n \mid^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u_n)\right] \leq C, \ \forall n,$$

which, in combination with the Hölder inequality and (f_1), implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mid f(x,u_n) \mid^q dx = 0.$$

Therefore, from (23) with $v = u_n$, we achieve

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mid \nabla u_n \mid^N = 0,$$

which contradicts (25) since $c > 0$. Thus, u is nontrivial and the proof of our main result is complete. ∎

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