LONG-TIME ASYMPTOTICS OF SOLUTIONS OF THE SECOND INITIAL-BOUNDARY VALUE PROBLEM FOR THE DAMPED BOUSSINESQ EQUATION

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Abstract. For the damped Boussinesq equation $u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}$, $x \in (0,\pi)$, $t > 0$; $\alpha, b = const > 0$, $\beta = const \in R^1$, the second initial–boundary value problem is considered with small initial data. Its classical solution is constructed in the form of a series in small parameter present in the initial conditions and the uniqueness of solutions is proved. The long-time asymptotics is obtained in the explicit form and the question of the blow up of the solution in a certain case is examined. The possibility of passing to the limit $b \to +0$ in the constructed solution is investigated.

1. Introduction

One of the equations describing the propagation of long waves on the surface of shallow water is the Boussinesq one which first appeared in the paper [6]. It takes into account the effects of dispersion and nonlinearity and can be written as

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.1}$$

where $u(x,t)$ is an elevation of the free surface of fluid, subscripts denote partial derivatives, and $\alpha, \beta = const \in R^1$ depend on the depth of fluid and the characteristic speed of long waves. Recently Milewsky and Keller [13] have deduced an isotropic pseudodifferential equation governing the evolution of the free surface of liquid with a constant depth. The equation (1.1) can be derived from it in the appropriate limit. Other versions of the Boussinesq equation can be obtained in a way similar to the one proposed in [13]. In

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fact, they are all perturbations of the linear wave equation that take into account the effects of small nonlinearity and dispersion.

Although (1.1) was proposed earlier as a model equation describing the propagation of small amplitude, nonlinear waves on shallow water, the mathematical theory for it is not as complete as for the Korteweg-de Vries-type equations ([1, 5]). The latter ones are of the first order in time and govern waves travelling only in one direction while (1.1) describes both left- and right-running solutions.

The equation (1.1) and its generalizations have been studied in the papers [7-12, 15, 16, 21] (see also the references there). Zakharov [21] has constructed the Lax pair for the inverse scattering transform. Further development of this theory has been done in [7], where the authors have showed the way of constructing global in time solutions and those that blow up in finite time. Galkin, Pelinovsky, and Stepanyants [8] have obtained rational solutions of the one-dimensional Boussinesq equation for the cases of the zero and nonzero boundary conditions at the infinity in space.

A generalization of (1.1), namely

$$u_{tt} = -u_{xxxx} + u_{xx} + (f(u))_{xx} \tag{1.2}$$

has been considered in [4,16]. It has been proposed in [4] that certain, solitary-wave solutions of (1.2) are nonlinearly stable for a range of their wave speeds. The authors obtained some sufficient conditions for the initial data to evolve into a global solution of the equation. In [16] local and global well-posedness has been proved by means of transforming the Cauchy problem for (1.2) into the system of nonlinear Schrödinger equations. Further improvement of these results has been done in [11], where some refined time estimates of the solution have been obtained.

Abstract Cauchy problems for the generalization of (1.2) in Banach spaces have been examined in [9, 10], where some sufficient conditions for the blow up of solutions have been deduced.

The equations (1.1) and (1.2) take into account the effects of dispersion and nonlinearity, but in real processes viscosity also plays an important role [3-5]. Therefore it is interesting to consider the equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.3}$$

where the term with the mixed derivative on the left-hand side is responsible for strong internal damping. Here α, $b = const > 0$, $\beta = const \in R^1$. Note that (1.1) with $\alpha > 0$, or the "good" Boussinesq equation, describes nonlinear beam oscillations (see [9, 11]). In the present paper we shall consider its damped version (1.3) on a finite interval with homogeneous second boundary conditions and inhomogeneous small initial conditions. It corresponds to nonlinear damped oscillations of a piece of a beam with free ends.

The equation of the (1.3)-type with weak damping $k_1 u_t$ and a linear feedback term $k_2(u - [u])$ on a periodic domain has been examined in [12] from the point of view of establishing the global well-posedness. In [3] Biler has studied some abstract Cauchy problems for the operator analog of (1.3) with a different nonlinearity (the latter one contained some powers of L_2-norm of the solution). The "oscillation condition" used in [3] for the case of constant coefficients (1.3) takes the form $\alpha > b^2$. This assumption excluding the overdamping phenomenon will also be used in the present paper. Biler has obtained sufficient conditions for both power and exponential decay in time of the solution in question. Pego and Weinstein [15] have investigated the behavior of solitary waves for the damped equation (1.3) with a nonlinearity $(f(u))_{xx}$. Having applied the spectral theory, they proved the existence of the real eigenvalues responsible for a non-oscillatory instability and explained the mechanism of the transition to this instability in terms of the motion of the poles of the resolvent.

However, none of the authors mentioned above has obtained long-time asymptotics of the solutions in question in the explicit form (although some time estimates have been deduced). The present paper is a continuation of the investigations [17-20], where the classical solutions of various problems for (1.3) have been constructed and their long-time asymptotics have been found explicitly.

One of the methods of studying Cauchy problems for nonlinear evolution equations is the inverse scattering transform (see [1]). However, this technique does not work for a wide class of dissipative equations which are not completely integrable. Another approach has been proposed by Naumkin and Shishmarev [14], who have considered nonlocal evolution equations of the first order in time and with small initial data. By means of using both the spectral and perturbation theories they have succeeded in constructing the exact solutions of the Cauchy problems in question and have calculated their long-time asymptotics. In [17-20] this method has been developed further and has been adapted for the equations of the second and third order in time governing wave propagation. Global in time classical solutions have been constructed in the form of a series in small parameter present in the initial conditions. Then the major terms of the long-time asymptotics have been calculated in the explicit form. These asymptotic representations are convenient for the qualitative analysis of the processes in question.

Apart from solving the Cauchy problems for the equations of the second and third order in time (see [17, 19]) and studying spatially periodic solutions of (1.3) (see [18]), the first initial-boundary value problem for (1.3) has been examined in [20]. The long-time asymptotics obtained there clearly showed the presence of the time and space oscillations exponentially decaying in time because of damping. The second initial-boundary value problem studied below has some relevance to the periodic problem [18] (some periodicity is incorporated in the boundary conditions). In the present paper the technique is much improved, and several new questions are considered, namely: the possibility of passing to the limit $b \to +0$ in the constructed solution (limiting absorption principle), the uniqueness of solutions, and the blow up of the classical solution in finite time in the case $\beta\widehat{\psi}_0 < 0$, where $\widehat{\psi}_0$ is the zero Fourier coefficient of the second initial function. The change of boundary conditions leads to this effect when $\beta\widehat{\psi}_0 < 0$. In the main case $\beta\widehat{\psi}_0 > 0$

two terms of the long-time asymptotic expansion are found. The major term increases linearly in time which is typical for the solution of the linear problem as well. The second term contains the Airy functions of a negative argument corresponding to the damped oscillations. The presence of the exponential multiplier emphacizes the damping effect.

2. Statement of the problem and main results

Consider the following initial-boundary value problem:

$$
\begin{aligned}
&u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx,}\, x \in (0,\pi),\, t > 0,\\
&u(x,0) = \varepsilon^2\varphi(x),\, u_t(x,0) = \varepsilon^2\psi(x),\, x \in (0,\pi),\\
&u_x(0,t) = u_x(\pi,t) = 0,\, t > 0,\\
&u_{xxx}(0,t) = u_{xxx}(\pi,t) = 0,\, t > 0,
\end{aligned}
\tag{2.1}
$$

where $\alpha,\ \beta,\ \varepsilon\ = const > 0,$ and $\beta = const \in R^1$.

Definition 1. The function $u(x,t)$ defined on $(0,\pi)\times(0,+\infty)$ is said to be the classical solution of the problem (2.1) if it is continuous together with its derivatives included in the equation, satisfies the equation, and continuously adjoins the initial and boundary conditions.

Definition 2. The function $f(x)$ belongs to the class $\widehat{C}^{2n}(0,\pi),\ n \geq 1,$ if $f'(0) = f'(\pi) = f'''(0) = f'''(\pi) = ... = f^{(2n-1)}(0) = f^{(2n-1)}(\pi) = 0$ and $f^{(2n)}(x) \in L_2(0,\pi)$.

In the sequel we shall denote the norm of the space of functions belonging to $L_2(-\pi,\pi)$ for each fixed $t > 0$ by

$$||u(t)|| = (\int_{-\pi}^{\pi} |u(x,t)|^2 dx)^{1/2}.$$

We shall also use the notation

$$\widehat{f}_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx.$$

Theorem 1. *If $\alpha > b^2$, $\beta\widehat{\psi}_0 \geq 0$, $\varphi(x) \in \widehat{C}^6(0,\pi)$ and $\psi\,(x) \in \widehat{C}^4(0,\pi)$, then there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ there exists a unique classical solution of the problem (2.1) represented in the form*

$$u(x,t) = \sum_{N=0}^{\infty} \varepsilon^N u_\varepsilon^{(N)}(x,t), \tag{2.2}$$

where the functons $u_\varepsilon^{(N)}(x,t)$ will be defined in the proof (see (3.11), (3.14), and (4.6)). This series converges absolutely and uniformly with respect to $x \in [0,\pi]$, $t \in [0,+\infty]$, and $\varepsilon \in [0,\varepsilon_0]$.

The solution (2.2) has the following asymptotics as $t \to +\infty$:

a) if $\widehat{\psi}_0 = 0$, then

$$u(x,t) = \varepsilon^2\widehat{\varphi}_0 + e^{-bt}\{[A_\varepsilon\cos(\sigma_1 t) + B_\varepsilon\sin(\sigma_1 t)]\cos x + O(e^{-\nu bt})\}; \tag{2.3}$$

b) if $\beta\widehat{\psi}_0 > 0$, then

$$u(x,t) = \varepsilon^2(\widehat{\varphi}_0 + \widehat{\psi}_0 t) + e^{-bt}\{[D_\varepsilon Bi(-\xi_1(t)) + E_\varepsilon Ai(-\xi_1(t))]\cos x + O(e^{-\nu bt})\}, \tag{2.4}$$

where $0 < \nu < 1$, $k = \alpha - b^2 > 0$, $\sigma_1 = \sqrt{k + 1 + 2\varepsilon^2\beta\widehat{\varphi}_0}$; A_ε, B_ε, D_ε, E_ε are constant coefficients defined by (3.17), (3.19), (4.7), and (4.8); $Ai(-z)$ and $Bi(-z)$, $z > 0$, are the Airy functions of the first and second kind respectively, and $\xi_1(t) > 0$ is a linear function of t (see (4.3)). The estimates of the remainders in (2.3) and (2.4) are uniform with respect to $x \in (0,\pi)$ and $\varepsilon \in (0, \varepsilon_0]$.

Remark 1. We do not include the boundedness of $u(x,t)$ for all $t > 0$ into our definition of the classical solution. Although $u(x,t) \to +\infty$ as $t \to +\infty$ when $\beta\widehat{\psi}_0 > 0$ (see (2.4)), it has continuous derivatives and satisfies the equation and the initial and boundary conditions in (2.1) for all $t > 0$. It will be shown later that even a solution of the linear problem contains the term $\varepsilon^2(\widehat{\varphi}_0 + \widehat{\psi}_0 t)$ and thus tends to infinity as $t \to +\infty$.

Remark 2. The relation $\alpha > b^2$ corresponds to the existence of an infinite number of damped oscillations. It is the most interesting case both from the mathematical and physical points of view. In the overdamping case $0 < \alpha < b^2$ there exists only a limited number of damped oscillations and aperiodic processes play the main role.

Corollary 1. *If $\beta\widehat{\psi}_0 < 0$ and the rest of the assumptions of Theorem 1 holds, then for any $T > 0$ there is such $\varepsilon_0(T) > 0$ that for $0 < \varepsilon \le \varepsilon_0(T)$ there exists a unique classical solution of the problem (2.1) on the interval $(0,T]$ represented in the form (2.2). Here $\varepsilon_0(T) \to 0$ as $T \to +\infty$. For any fixed ε there exists such $\overline{T} < +\infty$ that the solution (2.2) blows up as $t \to \overline{T}$.*

Remark 3. Blowing up of the solution in finite time means that the series (2.2) diverges as $t \to \overline{T}$ and the function (2.2) ceases to be a classical solution of the problem (2.1) for $t \ge \overline{T}$.

Next we consider the problem of (2.1)-type for the classical Boussinesq equation (without dissipation) on a bounded time interval, namely

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx},\ x \in (0,\pi),\ t \in (0,T],\ T < +\infty,$$

with the same initial and boundary conditions as in (2.1). We shall call it (2.1*) and its solution $u^*(x,t)$. In the case $\beta\widehat{\psi}_0 < 0$ it is supposed that $T < \overline{T}$.

Theorem 2. *If $\alpha > b^2$, $\varphi(x) \in \widehat{C}^6(0,\pi)$, $\psi(x) \in \widehat{C}^4(0,\pi)$, then for $x \in (0,\pi)$, $t \in (0,T]$*

$$\lim_{b \to +0} u(x,t) = u^*(x,t).$$

Remark 4. The sign of $\beta\widehat{\psi}_0$ doesn't matter here since both problems are considered on a bounded time interval.

The rest of the paper is organized as follows. Sections 3, 4, and 5 are devoted to the proof of Theorem 1. In Section 3 the solution of (2.1) is constructed for the case $\widehat{\psi}_0 = 0$ and its asymptotics as $t \to +\infty$ is obtained. In Section 4 the same is done for $\beta\widehat{\psi}_0 > 0$. In Section 5 the uniqueness of solutions of (2.1) is proved for both cases. Corollary 1 is proved in Section 6, and Theorem 2 in Section 7. Some final remarks are given in Section 8.

3. Construction of a solution and long-time asymptotics for the case $\widehat{\psi}_0 = 0$ (proof of Theorem 1)

3.1. **Construction of a solution.** Assume that a classical solution of (2.1) exists. Making an even continuation of $u(x,t)$ to the segment $[-\pi, 0]$ in order to satisfy the boundary conditions we expand it into the complex Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} \widehat{u}_n(t) e^{inx}, \ \widehat{u}_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x,t) e^{-inx} dx \tag{3.1}$$

with the additional condition $\widehat{u}_{-n}(t) = \widehat{u}_n(t)$, $n \geq 1$. Evidently, it corresponds to the expansion

$$\begin{aligned} u(x,t) = \widehat{u}_0(t) + 2\sum_{n=1}^{\infty} \widehat{u}_n(t) \cos nx, \ \widehat{u}_n(t) \\ = \frac{1}{\pi} \int_0^{\pi} u(x,t) \cos nx dx, \ n = 0, 1, 2, \dots . \end{aligned} \tag{3.2}$$

Repeating the same procedure for the initial functions we can represent them as

$$\begin{aligned} \varphi(x) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}_n e^{inx}, \ \psi(x) = \sum_{n=-\infty}^{\infty} \widehat{\psi}_n e^{inx}, \\ \widehat{\varphi}_{-n} = \widehat{\varphi}_n, \ \widehat{\psi}_{-n} = \widehat{\psi}_n, \ n \geq 1, \end{aligned} \tag{3.3}$$

or correspondingly

$$\varphi(x) = \widehat{\varphi}_0 + 2\sum_{n=1}^{\infty} \widehat{\varphi}_n \cos nx, \ \widehat{\varphi}_n = \frac{1}{\pi} \int_0^{\pi} \varphi(x) \cos nx dx,$$

$$\psi(x) = \widehat{\psi}_0 + 2\sum_{n=1}^{\infty} \widehat{\psi}_n \cos nx, \ \widehat{\psi}_n = \frac{1}{\pi} \int_0^{\pi} \psi(x) \cos nx dx, \ n = 0, 1, 2, \dots .$$

Integrating by parts in the representations of $\widehat{\varphi}_n$ and $\widehat{\psi}_n$ and using the smoothness of $\varphi(x)$ and $\psi(x)$ stated in the hypothesis we deduce that

$$|\widehat{\varphi}_n| \leq n^{-6}, \ |\widehat{\psi}_n| \leq n^{-4}, \ n \geq 1. \tag{3.4}$$

where the constants in the right-hand sides can be made unit ones by the appropriate choice of ε. Substituting (3.1) and (3.3) into (2.1) we get

$$\begin{aligned} \widehat{u}_0''(t) = 0, \ t > 0, \\ \widehat{u}_0(0) = \varepsilon^2 \widehat{\varphi}_0, \ \widehat{u}_0'(0) = 0, \end{aligned} \tag{3.5}$$

$$\widehat{u}_n''(t) + 2bn^2\widehat{u}_n'(t) + (\alpha n^4 + n^2 + 2\beta n^2\widehat{u}_0(t))\widehat{u}_n(t) = -\beta n^2 Q(\widehat{u}(t)),\ t > 0,$$

$$\widehat{u}_n(t) = \varepsilon^2\widehat{\varphi}_n,\ \widehat{u}_n'(0) = \varepsilon^2\widehat{\psi}_n,\ n \in Z,\ n \neq 0, \tag{3.6}$$

where a prime denotes a derivative of a function of a single variable,

$$Q(\widehat{u}(t)) = \sum_{\substack{q=-\infty \\ q\neq 0,n}}^{\infty} \widehat{u}_{n-q}(t)\widehat{u}_q(t), \text{ and } \widehat{u}_{-n}(t) = \widehat{u}_n(t),\ n \geq 1.$$

This convolution term can be written as

$$Q(\widehat{u}(t)) = \epsilon_n \sum_{q=1}^{n-1} \widehat{u}_{n-q}(t)\widehat{u}_q(t) + 2\sum_{q=1}^{\infty} \widehat{u}_{n+q}(t)\widehat{u}_q(t),\ n \geq 1, \tag{3.7}$$

$$\text{where } \epsilon_n = 0 \text{ for } n = 1 \text{ and } \epsilon_n = 1 \text{ for } n \geq 2.$$

Note that we have transfered two terms containing $\widehat{u}_0(t)\widehat{u}_n(t)$ to the left-hand side of the equation in (3.6). As a result, all the Fourier coefficients of the index n are separated in the left-hand side of the equation ($\widehat{u}_0(t) = \varepsilon^2\widehat{\varphi}_0$ is a known function), and the convolution term $Q(\widehat{u}(t))$ depends on the indeces not equal to n.

Now we can consider (3.6) with $n \geq 1$ and $Q(\widehat{u}(t))$ defined by (3.7), and then reconstruct $u(x,t)$ by means of (3.2). Our goal is to obtain a refined long-time estimate of $\widehat{u}_1(t)$ which will contribute to the second term of the asymptotics, while $\widehat{u}_0(t)$ will form its major term. Then we shall estimate the remaining series $\sum_{n=2}^{\infty}\widehat{u}_n(t)\cos nx$.

Seeking the fundamental solutions of the homogeneous equation associated with (3.6) in the form $e^{-\lambda t}$ we deduce that

$$\lambda_{1,2}(n) = bn^2 \mp i\sigma_n,\ \sigma_n = n\sqrt{kn^2 + 1 + 2\varepsilon^2\beta\widehat{\varphi}_0},\ k = \alpha - b^2 > 0. \tag{3.8}$$

Setting $\Phi_n = \varepsilon\widehat{\varphi}_n$, $\Psi_n = \varepsilon\widehat{\psi}_n$ (it is convenient to keep a small parameter in these coefficients in order to simplify the future estimates) we integrate (3.6) in t and get

$$\begin{aligned} \widehat{u}_n(t) &= \varepsilon e^{-bn^2 t}\{[\cos(\sigma_n t) + bn^2\frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\} \\ &\quad - \frac{\beta n^2}{\sigma_n}\int_0^t \exp[-bn^2(t-\tau)]\sin[\sigma_n(t-\tau)]Q(\widehat{u}(\tau))d\tau,\ n \geq 1. \end{aligned} \tag{3.9}$$

Representing $\widehat{u}_n(t)$, $n \geq 1$, as a formal series in ε

$$\widehat{u}_n(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{v}_n^{(N)}(t), \tag{3.10}$$

substituting it into (3.9) and comparing the coefficients of equal powers of ε we obtain for $n \geq 1$, $t > 0$

$$\widehat{v}_n^{(0)}(t) = e^{-bn^2 t}\{[\cos(\sigma_n t) + bn^2 \frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n\},$$

$$\begin{aligned} \widehat{v}_n^{(N)}(t) & \\ &= -\frac{\beta n^2}{\sigma_n}\int_0^t \exp[-bn^2(t-\tau)]\sin[\sigma_n(t-\tau)]F_N(\widehat{v}(\tau))d\tau,\ N \geq 1, \end{aligned} \tag{3.11}$$

where

$$F_N(\widehat{v}(t)) = \epsilon_n \sum_{q=1}^{n-1}\sum_{j=1}^{N} \widehat{v}_{n-q}^{(j-1)}(t)\widehat{v}_q^{(N-j)}(t) + 2\sum_{q=1}^{\infty}\sum_{j=1}^{N} \widehat{v}_{n+q}^{(j-1)}(t)\widehat{v}_q^{(N-j)}(t).$$

Now we have to prove that the formally constructed function (3.2), (3.10), (3.11) really represents a classical solution of the problem (2.1). To this end we shall show that the series

$$\widehat{u}_0(t) + 2\sum_{n=1}^{\infty}\cos nx \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{v}_n^{(N)}(t)$$

converges absolutely and uniformly for sufficiently small ε together with its derivatives included in the equation.

First we shall prove the following inequalities for $n \geq 1$, $N \geq 0$, $t > 0$, and any constant $\gamma \in (0, 1/2)$:

$$|\widehat{v}_n^{(N)}(t)| \leq c^N (N+1)^{-2} n^{-6}\exp(-\gamma bt). \tag{3.12}$$

Here and in the sequel we denote by c generic positive constants not dependent on N, n, ε, x, and t. They may depend on the coefficients of the equation and the initial functions.

We use the induction on the number N. For $N = 0$ and $n \geq 1$ we have from (3.8) and (3.11) for sufficiently small ε and $\gamma \in (0, 1/2)$

$$|\widehat{v}_n^{(0)}(t)| \leq n^{-6}\exp(-bn^2 t) \leq n^{-6}\exp(-\gamma bt).$$

Assuming that (3.12) is valid for all $\widehat{v}_n^{(s)}(t)$ with $0 \leq s \leq N-1$ we shall prove that it holds for $s = N$. Since for all integer $n \geq 1$, $q \geq 1$, $n \neq q$ (see [14])

$$q^{-6}|n-q|^{-6} \leq 2^6[q^{-6} + |n-q|^{-6}],$$
$$j^{-2}(N+1-j)^{-2} \leq 2^2(N+1)^{-2}[j^{-2} + (N+1-j)^{-2}],$$

we have

$$|\widehat{v}_n^{(N)}(t)| \leq c|\beta|(N+1)^{-2}n^{-6}\sum_{q=1}^{\infty}(q^{-6} + |n-q|^{-6})$$
$$\times \sum_{j=1}^{N} c^{j-1}c^{N-j}[(N+1-j)^{-2} + j^{-2}]\, S_N(n,t),$$

$$S_N(n,t) = exp(-bn^2 t)\int_0^t \exp[b(n^2 - 2\gamma)\tau]d\tau \leq \frac{\exp(-2\gamma bt)}{b(n^2 - 2\gamma)} \text{ for } 0 < \gamma < 1/2.$$

Hence

$$S_N(n,t) \le \frac{\exp(-\gamma bt)}{b(1-2\gamma)},$$

and consequently (3.12) is proved by induction with some $c = c(b) \to +\infty$ as $b \to +0$. By means of analogous arguments it is easy to prove that for $n \ge 2$, $t > 0$, $N \ge 0$ and any constant æ $\in (0,2)$

$$|\widehat{v}_n^{(N)}(t)| \le c^N (N+1)^{-2} n^{-6} \exp(-æ bt). \tag{3.13}$$

Indeed, on the final step of the proof we shall use the inequality

$$S_N(n,t) = \exp(-bn^2 t) \int_0^t \exp[b(n^2 - 2æ)\tau] d\tau \le \frac{\exp(-æ bt)}{2b(2-æ)}.$$

Now we can recall (3.2), (3.10), and (3.11) and deduce the formula (2.2). Performing the interchange of summation in the series we have

$$u(x,t) = \widehat{u}_0(t) + 2\sum_{n=1}^{\infty} \cos(nx) \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{v}_n^{(N)}(t) = \sum_{N=0}^{\infty} \varepsilon^N u_\varepsilon^{(N)}(x,t), \tag{3.14}$$

where

$$u_\varepsilon^{(0)}(x,t) = \widehat{u}_0(t),\ u_\varepsilon^{(N)}(x,t) = 2\varepsilon \sum_{n=1}^{\infty} \widehat{v}_n^{(N)}(t) \cos(nx),\ N \ge 1.$$

This interchange is possible due to the absolute and uniform in $x \in [0,\pi]$, $t \ge 0$, $\varepsilon \in [0, \varepsilon_0]$ convergence of the series in question. The last statement in its own term follows from the estimates (3.12) with $\varepsilon \le \varepsilon_0 < c^{-1}$.

Now in order to prove that (3.14) represents a classical solution of (2.1) we need to obtain the following estimates of the time derivatives of $\widehat{u}_n(t)$ for $n \ge 1$, $t > 0$:

$$|\partial_t^k \widehat{u}_n(t)| \le c n^{2k-6} \exp(-\gamma bt),\ k = 1,2. \tag{3.15}$$

Differentiating (3.11) with respect to t we get for $k = 1,2$

$$\begin{aligned}\partial_t^k \widehat{v}_n^{(0)} &= \sum_{l=0}^{k} c_k^l (-1)^l (bn^2)^l \exp(-bn^2 t) \partial_t^{k-l} \\ &\quad \cdot \left\{ [\cos(\sigma_n t) + bn^2 \frac{\sin(\sigma_n t)}{\sigma_n}]\Phi_n + \frac{\sin(\sigma_n t)}{\sigma_n}\Psi_n \right\},\end{aligned}$$

$$\partial_t^k \widehat{v}_n^{(N)}(t) = -\frac{\beta n^2}{\sigma_n} \int_0^t H_k(n, t-\tau) F_N(\widehat{v}(\tau)) d\tau + R_k(n,t),\ N \ge 1,$$

where

$$H_k(n,t) = \sum_{l=0}^{k} c_k^l (-1)^l (bn^2)^l \exp(-bn^2 t)(\sigma_n)^{k-l} \sin[\sigma_n t + (k-l)\pi/2],$$

$F_N(\widehat{v}(t))$ is defined by (3.11), c_k^l are binomial coefficients, and $R_k(n,t)$ are the results of the differentiating the integrals with respect to the upper limit, namely

$$R_1(n,t) = 0,\ R_2(n,t) = -\beta n^2 F_N(\widehat{v}(t)).$$

Hence it follows that for $n \geq 1, N \geq 0,\ t > 0,\ k = 1, 2$, and $\gamma \in (0, 1/2)$

$$|\partial_t^k \widehat{v}_n^{(N)}(t)| \leq c^N (N+1)^{-2} n^{2k-6} \exp(-\gamma bt) \tag{3.16}$$

which implies (3.15). By means of (3.15) it can be verified straightforwardly that (3.14) represents a classical solution of (2.1).

3.2. **Long-time asymptotics.** Here we shall find a refined asymptotic estimate of $\widehat{u}_1(t)$ which will contribute to the second term of the expansion (2.3) and make a rough estimate of the remaining series. Using (3.9) with $n = 1$ and adding and subtracting integrals from t to ∞ we can represent $\widehat{v}_1^{(N)}(t)$ as folows:

$$\begin{aligned}
\widehat{v}_1^{(N)}(t) &= e^{-bt}[A_\varepsilon^{(0)} \cos(\sigma_1 t) + B_\varepsilon^{(0)} \sin(\sigma_1 t)],\\
\widehat{v}_1^{(N)}(t) &= e^{-bt}[(A_\varepsilon^{(N)} + R_A^{(N)}) \cos(\sigma_1 t) + (B_\varepsilon^{(N)} + R_B^{(N)}) \sin(\sigma_1 t)],\\
A_\varepsilon^{(0)} &= \varepsilon\widehat{\varphi}_1,\ B_\varepsilon^{(0)} = \frac{\varepsilon}{\sigma_1}(b\widehat{\varphi}_1 + \widehat{\psi}_1),\\
A_\varepsilon^{(N)} &= \frac{\beta}{\sigma_1}\int_0^\infty e^{b\tau} \sin(\sigma_1\tau) L_N(\tau) d\tau,\\
B_\varepsilon^{(N)} &= -\frac{\beta}{\sigma_1}\int_0^\infty e^{b\tau} \cos(\sigma_1\tau) L_N(\tau) d\tau,\\
R_A^{(N)} &= \frac{\beta}{\sigma_1}\int_t^\infty e^{b\tau} \sin(\sigma_1\tau) L_N(\tau) d\tau,\\
R_B^{(N)} &= -\frac{\beta}{\sigma_1}\int_t^\infty e^{b\tau} \cos(\sigma_1\tau) d\tau,\\
L_N(t) &= 2\sum_{q=1}^{\infty}\sum_{j=1}^{N} \widehat{v}_{1+q}^{(j-1)} \widehat{v}_q^{(N-j)}(t),\ N \geq 1,
\end{aligned} \tag{3.17}$$

and the functions $\widehat{v}_n^{(s)}(t)$, $s = 0, 1, ..., N-1$, for the calculation of the sum $L_N(t)$ are defined by (3.11).

Next we shall prove that for $N \geq 1,\ t > 0,\ 0 < \rho < 3/2$

$$|R_{A,B}^{(N)}(t)| \leq c^N e^{-\rho bt}, \tag{3.18}$$

Taking into account (3.12) and (3.13) and choosing γ and æ there so that $\gamma + æ > 1$ and æ $> \gamma$ we can write that

$$\begin{aligned}
|R_{A,B}^{(N)}(t)| &\leq 2\frac{|\beta|}{\sigma_1}\int_t^\infty e^{b\tau} \sum_{j=1}^{N} [|\widehat{v}_1^{(j-1)}(\tau)\widehat{v}_1^{(N-j)}(\tau)| + \sum_{q=2}^{\infty} |\widehat{v}_{1+q}^{(j-1)}(\tau)\widehat{v}_q^{(N-j)}(\tau)|] d\tau\\
&\leq c^{N-1}(N+1)^{-2}\sum_{j=1}^{N}[j^{-2} + (N+1-j)^{-2}]\{\int_t^\infty \{\exp[-(\gamma + æ - 1)b\tau]\\
&\quad + \sum_{q=2}^{\infty}(1+q)^{-6} q^{-6} \exp[-(2æ - 1)b\tau]\} d\tau \leq c^N \frac{\exp[-(\gamma + æ - 1)t]}{\gamma + æ - 1}.
\end{aligned}$$

Setting $\rho = \gamma + æ - 1$ we obtain (3.18).

Recalling (3.10) we deduce that as $t \to +\infty$:

$$
\begin{aligned}
&2\widehat{u}_1(t) = e^{-bt}[A_\varepsilon \cos(\sigma_1 t) + B_\varepsilon \sin(\sigma_1 t) + O(e^{-\rho bt})],\\
&A_\varepsilon = 2\sum_{N=0}^{\infty} \varepsilon^{N+1} A_\varepsilon^{(N)},\ B_\varepsilon = 2\sum_{N=0}^{\infty} \varepsilon^{N+1} B_\varepsilon^{(N)}.
\end{aligned} \tag{3.19}
$$

Finally we obtain

$$
\begin{aligned}
&u(x,t) = \widehat{u}_0(t) + 2\widehat{u}_1(t)\cos x + R_u(x,t),\\
&|R_u(x,t)| \le \exp(-æbt)\sum_{N=0}^{\infty} \varepsilon^{N+1} c^N \sum_{n=2}^{\infty} n^{-6} \le c\exp(-æbt),\ t>0.
\end{aligned} \tag{3.20}
$$

Combining (3.19) and (3.20) and denoting $\nu = æ - 1$, $0 < \nu < 1$, we obtain (2.3).

4. Construction of a solution and long-time asymptotics for the case $\beta\widehat{\psi}_0 > 0$ (proof of Theorem 1)

4.1. **Construction of a solution.** It suffices to consider $\beta > 0$, $\widehat{\psi}_0 > 0$ because the case $\beta < 0$, $\widehat{\psi}_0 < 0$ can be obtained by changing u to $-u$ in (2.1). The sign of $\varphi(x)$ doesn't matter in our considerations.

Now instead of (3.5) we get the following problem for the zero Fourier coefficient

$$
\begin{gathered}
\widehat{u}_0''(t) = 0,\ t > 0,\\
\widehat{u}_0(0) = \varepsilon^2 \widehat{\varphi}_0,\ \widehat{u}_0'(0) = \varepsilon^2 \widehat{\psi}_0,
\end{gathered}
$$

the solution of which is a linear function of time, namely

$$
\widehat{u}_0(t) = \varepsilon^2(\widehat{\varphi}_0 + \widehat{\psi}_0 t). \tag{4.1}
$$

Therefore this time in (3.6) we have an equation with variable in time coefficients. We can reduce it to the Airy equation by means of some transformations. Setting $\widehat{u}_n(t) = w_n(t)\exp(-bn^2 t)$, $n \ge 1$, we get

$$
\begin{gathered}
w_n''(t) + (a_n + \gamma_n t) w_n(t) = 0,\\
a_n = kn^4 + n^2(1 + 2\varepsilon^2 \beta \widehat{\varphi}_0) > 0,\\
k = \alpha - b^2 > 0,\ \gamma_n = \varepsilon^2 n^2 2\beta\widehat{\psi}_0 > 0.
\end{gathered} \tag{4.2}
$$

Introducing a new variable

$$
\begin{aligned}
&\xi_n(t) = h_n + \gamma_n^{1/3} t > 0,\\
&h_n = a_n \gamma_n^{-2/3} = \frac{kn^4 + n^2(1 + 2\varepsilon^2\beta\widehat{\varphi}_0)}{n^{4/3}\varepsilon^{4/3}(2\beta\widehat{\psi}_0)^{2/3}} > 0
\end{aligned} \tag{4.3}
$$

we transform (4.2) to the Airy equation

$$
\frac{d^2 w_n}{d\xi_n^2} + \xi_n(t) w_n = 0,
$$

where we have used the same notation for the function $w_n(\xi_n)$. Its fundamental solutions are the Airy functions of a negative argument, namely $Ai(-\xi_n)$ and $Bi(-\xi_n)$ (see [2]), which have the following representations:

$$Ai(-z) = \frac{\sqrt{z}}{3}[J_{-1/3}(\varsigma) + J_{1/3}(\zeta)],\ Bi(-z) = \sqrt{\frac{z}{3}}[J_{-1/3}(\zeta) + J_{1/3}(\zeta)],$$

where $z > 0$, $\varsigma = (2/3)z^{2/3}$, and $J_\nu(z)$ are the Bessel functions of the index ν. We shall also need the estimates as $z \to +\infty$

$$\begin{aligned} |Ai(-z)| \le cz^{-1/4},\ |Bi(-z)| \le cz^{-1/4}, \\ |Ai'(-z)| \le cz^{1/4},\ |Bi'(-z)| \le cz^{1/4}. \end{aligned} \tag{4.4}$$

Integrating (3.6) with $\widehat{u}_0(t)$ defined by (4.1) with respect to t we reduce this problem to the integral equation

$$\begin{aligned} \widehat{u}_n(t) = \varepsilon \exp(-bn^2 t)[U_n^{(1)} Ai(-\xi_n(t)) - U_n^{(2)} Bi(-\xi_n(t))] \\ + \beta n^2 \int_0^t \exp[-bn^2(t-\tau)] g_n(t,\tau) Q(\widehat{u}(\tau)) d\tau,\ n \ge 1, \end{aligned} \tag{4.5}$$

where

$$U_n^{(1)} = \pi[bn^2\gamma_n^{-1/3} Bi(-h_n) + Bi'(-h_n)]\Phi_n + Bi(-h_n)\gamma_n^{-1/3}\Psi_n,$$
$$U_n^{(2)} = \pi[bn^2\gamma_n^{-1/3} Ai(-h_n) + Ai'(-h_n)]\Phi_n + Ai(-h_n)\gamma_n^{-1/3}\Psi_n,$$
$$\Phi_n = \varepsilon\widehat{\varphi}_n,\ \Psi_n = \varepsilon\widehat{\psi}_n,$$
$$g_n(t,\tau) = \pi\gamma_n^{-1/3}[Ai(-\xi_n(\tau))Bi(-\xi_n(t)) - Ai(-\xi_n(t))Bi(-\xi_n(\tau))],$$
$$0 < \tau < t,\ t > 0.$$

Here we have used the expression for the Wronskian of the Airy functions [2]

$$\widetilde{W}\{Ai(z),\ Bi(z)\} = \frac{1}{\pi}.$$

Representing $\widehat{u}_n(t)$ as a formal series (3.10) and equating coefficients of equal powers of ε we find that for $n \ge 1$

$$\begin{aligned} \widehat{v}_n^{(0)}(t) = \exp(-bn^2 t)[U_n^{(1)} Ai(-\xi_n(t)) - U_n^{(2)} Bi(-\xi_n(t))], \\ \widehat{v}_n^{(N)}(t) = \beta n^2 \int_0^t \exp[-bn^2(t-\tau)] g_n(t,\tau) F_N(\widehat{v}(\tau)) d\tau,\ N \ge 1, \end{aligned} \tag{4.6}$$

with $F_N(\widehat{v}(t))$ defined by(3.11). Now $u(x,t)$ can be expressed by the formula (3.14) with $\widehat{v}_n^{(N)}(t)$ defined by (4.6).

In order to prove that the function represented by (3.14), (4.6) is really a classical solution of (2.1) we need to obtain some estimates of $\widehat{v}_n^{(N)}(t)$ and its derivatives. Since the arguments of the Airy functions depend not only on n, but also on ε we need to take into account the following estimates as $\varepsilon \to +0$:

$$r_n\gamma_n^{-1/3}|B(-h_n)| \le cn^{-2/3}\varepsilon^{-1/3},\ |Bi'(-h_n)| \le ch_n^{1/4} \le cn^{-2/3}\varepsilon^{1/3},$$
$$|Ai(-\xi_n(t))| \le ch_n^{-1/4} \le cn^{2/3}\varepsilon^{1/3},\ |Bi'(-\xi_n(t))| \le cn^{-2/3}\varepsilon^{1/3},$$
$$|Ai(-\xi_n(t))Bi(-\xi_n(t))| \le ch_n^{-1/2} \le c\varepsilon^{2/3}n^{-4/3}.$$

They follow from (4.4). Hence $|\widehat{v}_n^{(0)}(t)| \leq n^{-6}\exp(-bn^2 t)$ and

$$|g_n(t,\tau)| \leq \frac{c}{n^2} \text{ as } \varepsilon \to +0$$

uniformly in t, τ. Taking into account these estimates and conducting the same arguments as in the previous section we can deduce the inequalities (3.12) for $\widehat{v}_n^{(N)}(t)$ with $n \geq 1$ and (3.13) for $\widehat{v}_n^{(N)}(t)$ with $n \geq 2$. Differentiating (4.6) with respect to t we get for $n \geq 1,\ t > 0$

$$\begin{aligned}
\partial_t \widehat{v}_n^{(0)}(t) &= \exp(-bn^2 t)\{(-bn^2)[U_n^{(1)} Ai(-\xi_n(t)) - U_n^{(2)} Bi(-\xi_n(t))] \\
&\quad - \gamma_n^{1/3}[U_n^{(1)} Ai'(-\xi_n(t)) - U_n^{(2)} Bi'(-\xi_n(t))]\}, \\
\partial_t^2 \widehat{v}_n^{(0)}(t) &= \exp(-bn^2 t)\{U_n^{(1)}[(bn^2)^2 Ai(-\xi_n(t)) + bn^2\gamma_n^{1/3} Ai'(-\xi_n(t)) \\
&\quad + \gamma_n^{2/3}\xi_n(t) Ai(-\xi_n(t))] - U_n^{(2)}[(bn^2)^2 Bi(-\xi_n(t)) \\
&\quad + bn^2\gamma_n^{1/3} Bi'(-\xi_n(t)) + \gamma_n^{2/3}\xi_n(t) Bi(-\xi_n(t))]\}, \\
\partial_t \widehat{v}_n^{(N)}(t) &= \beta n^2[(-bn^2)\int_0^t \exp[-bn^2(t-\tau)] g_n(t,\tau) F_N(\widehat{v}(\tau))d\tau \\
&\quad + \int_0^t \exp[-bn^2(t-\tau)]\partial_t g_n(t,\tau) F_N(\widehat{v}(\tau))d\tau],\ N \geq 1, \\
\partial_t^2 \widehat{v}_n^{(N)}(t) &= \beta n^2\{(bn^2)^2 \int_0^t \exp[-bn^2(t-\tau)] g_n(t,\tau) F_N(\widehat{v}(\tau))d\tau \\
&\quad - 2bn^2 \int_0^t \exp[-bn^2(t-\tau)]\partial_t g_n(t,\tau) F_N(\widehat{v}(\tau))d\tau \\
&\quad + \int_0^t \exp[-bn^2(t-\tau)]\partial_t^2 g_n(t,\tau) F_N(\widehat{v}(\tau))d\tau - F_N(\widehat{v}(t))\},\ N \geq 1,
\end{aligned}$$

where

$$g_n(t,t) = 0,$$

$$\partial_t g_n(t,\tau) = -\pi[Ai(-\xi_n(\tau))Bi'(-\xi_n(t)) - Ai'(-\xi_n(t))Bi(-\xi_n(\tau))],$$

$$\partial_t g_n(t,t) = -\pi\widetilde{W}\{Ai(-\xi_n(t))\,, Bi(-\xi_n(t))\} = -1,$$

$$\partial_t^2 g_n(t,\tau) = -\gamma_n^{2/3}\xi_n(t) g_n(t,\tau).$$

Hence it follows that $\partial_t^k \widehat{v}_n^{(N)}(t)$, $n \geq 1,\ N \geq 0$, $k = 1, 2$, satisfy (3.16) and consequently (3.15) is valid for $\widehat{u}_n(t)$. Then it can be verified that (3.14), (4.6) represent a classical solution of (2.1).

4.2. **Long-time asymptotics.** Now we shall obtain a subtle asymptotic estimate of $\widehat{u}_1(t)$.We can write that

$$
\begin{aligned}
\widehat{v}_1^{(0)}(t) &= e^{-bt}[D^{(0)}Bi(-\xi_1(t)) + E^{(0)}Ai(-\xi_1(t))],\\
D^{(0)} &= -U_1^{(2)},\ E^{(0)} = U_1^{(1)};\\
\widehat{v}_1^{(N)} &= e^{-bt}\{[D^{(N)} + R_D^{(N)}]Bi(-\xi_1(t)) - [E^{(N)} + R_E^{(N)}]Ai(-\xi_1(t)),\\
D^{(N)} &= \beta\gamma_1^{-1/3}\pi\int_0^\infty e^{b\tau}Ai(-\xi_1(\tau))M_N(\tau)d\tau,\\
E^{(N)} &= -\beta\gamma_1^{-1/3}\pi\int_0^\infty e^{b\tau}Bi(-\xi_1(\tau))M_N(\tau)d\tau,\\
M_N(\tau) &= 2\sum_{q=1}^{\infty}\sum_{j=1}^{N}\widehat{v}_{1+q}^{(j-1)}(\tau)\widehat{v}_q^{(N-j)}(\tau),\ N \geq 1,
\end{aligned}
\tag{4.7}
$$

where the functions $\widehat{v}_n^{(s)}(t)$, $s = 0, 1, ..., N-1$, for the calculation of $M_N(t)$ must be taken from (4.6).

As in the previous section, using (3.12), (3.13), and the boundedness of the Airy functions of a negative argument we can prove that for $N \geq 1$, $t > 0$, $0 < \rho < 3/2$

$$|R_{D,E}^{(N)}(t)| \leq c^N \exp(-\rho bt).$$

Thus, we have

$$
\begin{aligned}
2\widehat{u}_1(t) &= e^{-bt}[D_\varepsilon Bi(-\xi_1(t)) + E_\varepsilon Ai(-\xi_1(t)) + O(e^{-\rho bt})],\\
D_\varepsilon &= 2\sum_{N=0}^{\infty}\varepsilon^{N+1}D^{(N)},\ E_\varepsilon = 2\sum_{N=0}^{\infty}\varepsilon^{N+1}E^{(N)},
\end{aligned}
\tag{4.8}
$$

and the formula (3.20) is valid for $u(x,t)$ with the same estimate of the remainder. Combining (3.20) and (4.8) we obtain (2.4).

5. Uniqueness of the solutions (proof of Theorem 1)

In this section we shall prove the uniqueness of the solutions for both cases in question. We assume the contrary, that is that there exist two classical solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem (2.1) and make an even continuation of these functions in x to the segment $[-\pi, 0]$. Setting $W(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t)$ we notice that $W(x,t)$ belongs to the space $L_2(-\pi,\pi)$ for each fixed $t > 0$ and therefore $||W(t)|| < +\infty$.

Expanding $W(x,t)$ into the complex Fourier series on $[-\pi,\pi]$ we can write

$$W(x,t) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\widehat{W}_n(t)e^{inx},\ \widehat{W}_{-n}(t) = \widehat{W}_n(t) \text{ for } n \geq 1.$$

Here $\widehat{W}_0(t) = 0$ since for both functions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ the zero Fourier coefficients satisfy the same problem (3.5) (or (3.5) with inhomogeneous initial conditions when $\beta\widehat{\psi}_0 > 0$). Then for all $n \in Z$, $n \neq 0$

$$\widehat{W}_n(t) = -\frac{\beta n^2}{\widetilde{\sigma}_n}\int_0^t \exp[-bn^2(t-\tau)] \sum_{\substack{q=-\infty \\ q\neq 0,n}}^{\infty} [\widehat{u}^{(1)}_{n-q}(\tau) - \widehat{u}^{(2)}_{n-q}][\widehat{u}^{(1)}_q(\tau) + \widehat{u}^{(2)}_q(\tau)]d\tau,$$

and $\widetilde{\sigma}_n = n\sqrt{kn^2+1}$ correspond to σ_n in (3.8) with $\widehat{\varphi}_0 = 0$.

Using the Cauchy-Schwartz inequality and Parceval's equation we can deduce that

$$|W_n(t)| \leq c\int_0^t \exp[-bn^2(t-\tau)]||W(\tau)||d\tau.$$

Squaring both sides of this inequality and summing the result from $-\infty$ to ∞ in n we get

$$||W(t)||^2 \leq c\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\int_0^t \exp[-bn^2(t-\tau)]||W(\tau)||d\tau)^2.$$

Hence for some $T_1 > 0$

$$\begin{aligned}(\sup_{t\in[0,T_1]} ||W(t)||)^2 &\leq c(\sup_{t\in[0,T_1]} ||W(t)||)^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{1-\exp(-bn^2t)}{bn^2}\right]^2 \\ &\leq c(T_1)(\sup_{t\in[0,T_1]} ||W(t)||)^2,\end{aligned}$$

where the constant $c(T_1)$ can be made less than one by the appropriate choice of T_1. This contradiction allows to complete the proof of the uniqueness for $t \in [0,T_1]$. Continuing this process for the segments $[T_1,T_2]$, $[T_2,T_3]$,..., $[T_n,T_{n+1}]$,... , with $\{T_n\} \to +\infty$ we obtain the same result for all $t > 0$. The proof of Theorem 1 is complete.

6. Proof of Corollary 1

In the case $\beta\widehat{\psi}_0 < 0$ we use the same scheme of construction of the solution as in Section 4. Setting in (3.6) $\widehat{u}_n(t) = w_n(t)\exp(-bn^2t)$ we obtain the transformed equation in the form

$$w_n''(t) + (a_n - |\gamma_n|t)w_n(t) = 0$$

with the same a_n and γ_n as in (4.2) and $n \geq 1$. Then we reduce it to the Airy equation

$$\frac{d^2w_n}{d\xi_n^2} + \xi_n(t)w_n = 0$$

with $\xi_n(t) = h_n - |\gamma_n|^{1/3}t$ which can change its sign. Indeed, for each fixed $n \geq 1$ there is such

$$t_0(n) = \frac{kn^2 + 1 + 2\varepsilon^2\beta\widehat{\varphi}_0}{\varepsilon^2 \mid 2\beta\widehat{\psi}_0 \mid}$$

that for $t > t_0(n)$ the argument of the Airy functions becomes positive, that is

$$-\xi_n(t) = \gamma_n^{1/3}[t - t_0(n)] > 0.$$

Therefore the Airy function $Bi(-\xi_n(t))$ is bounded for $t \in [0, T], T < +\infty$, while $Ai(-\xi_n(t))$ is bounded for all $t > 0$. As a consequence, instead of (3.16) we shall have for $n \geq 1$, $N \geq 0$, $k = 0, 1, 2$ and $t \in [0, T]$

$$|\partial_t^k \widehat{v}_n^{(N)}(t)| \leq [c(T)]^N (N+1)^{-2} n^{2k-6}.$$

In order to guarantee the absolute and uniform convergence of the series (2.2) we have to satisfy the condition $\varepsilon c(T) < 1$, or $\varepsilon_0(T) < [c(T)]^{-1}, 0 < \varepsilon \leq \varepsilon_0(T)$.

To see what happens if T is sufficiently large we have to remember the properties of the Airy functions of a positive argument: $Ai(z)$ decreases and $Bi(z)$ increases with the increase of z and the asymptotics of these functions are (see [2])

$$Ai(z) \sim \frac{\exp[-(2/3)z^{3/2}]}{2\sqrt{\pi} z^{1/4}}, \ Bi(z) \sim \frac{\exp[(2/3)z^{3/2}]}{\sqrt{\pi} z^{1/4}}, \ z \to +\infty.$$

Therefore even $\widehat{v}_n^{(0)}(t)$ tends to infinity as $t \to +\infty$ with a speed greater than exponential and for $\widehat{v}_n^{(N)}(t)$, $N \geq 1$, the speed is even greater because of the presence of the term

$$\sum_{j=1}^{N} \widehat{v}_{1+q}^{(j-1)}(\tau) \widehat{v}_q^{(N-j)}(\tau)$$

in the integrand (see (3.11) and (4.6)). Thus, $c(T) \to +\infty$ and $\varepsilon_0(T) \to 0$ as $T \to +\infty$.

If ε is fixed, then there exists such point $t = \overline{T}$ that the necessary condition of the convergence of the series (2.2) is broken and $u(x,t) \to +\infty$ as $t \to \overline{T}$. Naturally, the derivatives of (2.2) do not exist either and this function ceases to be a solution of (2.1).

The uniqueness of the solution on the interval $(0, T]$, $T < \overline{T}$, can be proved by means of the same arguments as in Section 5. The proof of the corollary is complete.

7. Proof of Theorem 2

First we need to prove that for all the three cases in question the series (3.14) with $\widehat{v}_n^{(N)}(t)$ defined by (3.11) or (4.6) converges absolutely and uniformly with respect to $b \in [0, b_0]$ for some $b_0 > 0$ and $t \in [0, T]$. To this end it suffices to show that for $n \geq 1$, $N \geq 0$, $t \in [0, T]$, and $b \in [0, b_0]$

$$|\widehat{v}_n^{(N)}(t)| \leq [c(T)]^N (N+1)^{-2} n^{-6}, \tag{7.1}$$

where $c(T)$ is independent of b.

We shall use the induction on the number N. For $N = 0$ we have from (3.11) and (4.5) for suffficiently small ε

$$|\widehat{v}_n^{(0)}(t)| \le n^{-6},\ n \ge 1.$$

Assuming that (7.1) holds for $\widehat{v}_n^{(s)}(t)$, $0 \le s \le N-1$, using the uniform in b boundedness of the Airy functions for $t \in [0, T]$ and the inequality

$$S_N(t) = \int_0^t \exp[-bn^2(t-\tau)]d\tau \le T$$

we establish (7.1) for all $N \ge 0$.

Passing to the limit $b \to +0$ in (3.14) we get for $t \in [0, T]$

$$\lim_{b \to +0} u(x,t) = \widehat{u}_0(t) + 2\sum_{n=1}^{\infty} \cos(nx) \sum_{N=0}^{\infty} \varepsilon^{N+1} \overline{v}_n^{(N)}(t), \tag{7.2}$$

where for the case $\widehat{\psi}_0 = 0$

$$\overline{v}_n^{(0)}(t) = \cos(\overline{\sigma}_n t)\Phi_n + \frac{\sin(\overline{\sigma}_n t)}{\overline{\sigma}_n}\Psi_n,\ \overline{\sigma}_n = n\sqrt{\alpha n^2 + 1 + 2\varepsilon^2\beta\widehat{\varphi}_0},$$

$$\overline{v}_n^{(N)} = -\frac{\beta n^2}{\overline{\sigma}_n}\int_0^t \sin[\overline{\sigma}_n(t-\tau)]F_N(\overline{v}(\tau))d\tau,\ N \ge 1,$$

and for $\beta\widehat{\psi}_0 \ne 0$

$$\begin{aligned}\overline{v}_n^{(0)}(t) = \pi[&Bi'(-\overline{h}_n)\Phi_n + Bi(-\overline{h}_n)\gamma_n^{-1/3}\Psi_n]Ai(-\overline{\xi}_n(t)) \\ &- [Ai'(-h_n)\Phi_n + Ai(-\overline{h}_n)\gamma_n^{-1/3}\Psi_n]Bi(-\overline{\xi}_n(t)),\end{aligned}$$

$$\overline{v}_n^{(N)}(t) = \beta n^2 \int_0^t \overline{g}_n(t,\tau)F_N(\overline{v}(\tau))d\tau,\ N \ge 1,$$

where $\overline{h}_n$, $\overline{\xi}_n(t)$, and $\overline{g}_n(t,\tau)$ are obtained from the corresponding formulas by putting there $b = 0$, and F_N is defined by (3.11).

By means of the estimates

$$|\partial_t^k \overline{v}_n^{(N)}(t)| \le [c(T)]^N (N+1)^{-2} n^{2k-6}$$

valid for $t \in (0, T]$, $n \ge 1$, $N \ge 0$, $k = 0, 1, 2$, it can be verified straightforwardly that (7.2) represents a classical solution of the problem (2.1*). The proof is complete.

8. Final remarks

There are no major difficulties in using the method proposed above for the overdamping case $0 < \alpha < b^2$, but one must consider several subcases. For example, when $\widehat{\psi}_0 = 0$ the eigenvalues of the linear operator of the equation are

$$\lambda_{1,2}(n) = bn^2 \mp n\sqrt{|k|[n^2 - \chi(\varepsilon)]},\ |k| = b^2 - \alpha > 0,\ \chi(\varepsilon) = \frac{1 + 2\varepsilon^2\beta\widehat{\varphi}_0}{|k|},$$

and it is necessary to consider the following possibilities:

1) $n^2 > \chi(\varepsilon)$, when $\lambda_{1,2} > 0$;

2) $n^2 < \chi(\varepsilon)$, when $\lambda_{1,2}$ are complex conjugate.

There is no need to consider multiple eigenvalues when $n^2 = \chi(\varepsilon)$ since the term $2\beta n^2\varepsilon^2\widehat{\varphi}_0\widehat{u}_n(t)$ can be transfered to the right-hand side of the equation.

We would like to emphacize the fact that the main idea of obtaining the time estimates of $\widehat{u}_n(t)$ for $\beta\widehat{\psi}_0 \neq 0$ is based on separating all the Fourier coefficients of the index n in the left-hand side of (3.6) while all the other ones are left in the convolution term in the right-hand side of the equation. The attempt to construct a classical solution of (2.1) by means of leaving the term $2\beta n^2\widehat{u}_0\widehat{u}_n(t)$ in the right-hand side is doomed to failure. Indeed, the presence of the linear function $\widehat{\varphi}_0 + \widehat{\psi}_0 t$ in the integrand in the expression for $\widehat{v}_n^{(N)}(t)$, $n \geq 1$, $N \geq 1$, will cause the appearance of a rapidly increasing function of N and t in the estimate of $|\widehat{v}_n^{(N)}(t)|$.

In order to compare the solution of (2.1) with that of the corresponding linear problem we have to construct the latter. It is

$$u_{lin}(x,t) = \varepsilon^2\{\widehat{\varphi}_0 + \widehat{\psi}_0 t + 2\sum_{n=1}^{\infty} e^{-bn^2t}[(\cos\widetilde{\sigma}_n t + bn^2\frac{\sin\widetilde{\sigma}_n t}{\widetilde{\sigma}_n})\widehat{\varphi}_n + \frac{\sin\widetilde{\sigma}_n t}{\widetilde{\sigma}_n}\widehat{\psi}_n]\},\ \widetilde{\sigma}_n = n\sqrt{kn^2+1}. \tag{8.1}$$

For the case $\widehat{\psi}_0 = 0$ passing to the limit $\beta \to 0$ in (3.11), (3.14) we can obtain (8.1). But when $\beta\widehat{\psi}_0 \neq 0$ the very choice of the fundamental solutions of the homogeneous equation associated with (3.6) prevents from passing to this limit in the constructed solution.

In conclusion, we would like to point out the main difference in the long-time behavior of the solution in question when $\beta\widehat{\psi}_0 > 0$ and $\beta\widehat{\psi}_0 < 0$. In both cases the zero term of the series (3.14) contains a linear function of t, which is not surprising since even the solution of the linear problem includes it. However, in the first case this series converges absolutely and uniformly together with its derivatives for all $t \geq 0$, while in the second case it diverges as $t \to \overline{T}$ and the solution can not be extended beyond this point.

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