CONVERGENCE OF GENERIC INFINITE PRODUCTS OF AFFINE OPERATORS

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We establish several results concerning the asymptotic behavior of random infinite products of generic sequences of affine uniformly continuous operators on bounded closed convex subsets of a Banach space. In addition to weak ergodic theorems we also obtain convergence to a unique common fixed point and more generally, to an affine retraction.

1. Introduction

Our goal in this paper is to study the asymptotic behavior of random infinite products of generic sequences of affine uniformly continuous operators on bounded closed convex subsets of a Banach space. Infinite products of operators find application in many areas of mathematics (cf. [1, 2, 3, 8, 9, 10] and the references therein). More precisely, we show that in appropriate spaces of sequences of operators there exists a subset which is a countable intersection of open everywhere dense sets such that for each sequence belonging to this subset the corresponding random infinite products converge. Results of this kind for powers of a single nonexpansive operator were already established by De Blasi and Myjak [6] while such results for infinite products have recently been obtained in [13]. The approach used in these papers and in the present paper is common in global analysis and in the theory of dynamical systems [7, 11]. Recently it has also been used in the study of the structure of extremals of variational and optimal control problems [14, 15, 16]. Thus, instead of considering a certain convergence property for a single sequence of affine operators, we investigate it for a space of all such sequences equipped with some natural metric, and show that this property holds for most of these sequences. This allows us to establish convergence without restrictive assumptions on the space and on the operators themselves. We remark in passing that common fixed point theorems for families of affine mappings (e.g., those of Markov-Kakutani and Ryll-Nardzewski) have applications in various mathematical areas. See, for example, [5] and the references therein.

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Let $(X, \|\cdot\|)$ be a Banach space and let K be a nonempty bounded closed convex subset of X with the topology induced by the norm $\|\cdot\|$.

Denote by $\mathfrak{A}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$, where each $A_t : K \to K$ is a continuous operator, $t = 1, 2, \dots$. Such a sequence will occasionally be denoted by a boldface $\mathbf{A}$.

We equip the set $\mathfrak{A}$ with the metric $\rho_s : \mathfrak{A} \times \mathfrak{A} \to [0, \infty)$ defined by

$$\begin{gathered}\rho_s\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) = \sup\big\{\big\|A_t x - B_t x\big\| : x \in K,\ t = 1, 2, \dots\big\}, \\ \{A_t\}_{t=1}^{\infty},\ \{B_t\}_{t=1}^{\infty} \in \mathfrak{A}.\end{gathered} \tag{1.1}$$

It is easy to see that the metric space $(\mathfrak{A}, \rho_s)$ is complete. We always consider the set $\mathfrak{A}$ with the topology generated by the metric ρ_s.

We say that a set E of operators $A : K \to K$ is uniformly equicontinuous (ue) if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|Ax - Ay\| \le \epsilon$ for all $A \in E$ and all $x, y \in K$ satisfying $\|x - y\| \le \delta$.

An operator $A : K \to K$ is called uniformly continuous if the singleton $\{A\}$ is a (ue) set.

Define

$$\mathfrak{A}_{\mathrm{ue}} = \big\{\{A_t\}_{t=1}^{\infty} \in \mathfrak{A} : \{A_t\}_{t=1}^{\infty} \text{ is a (ue) set}\big\}. \tag{1.2}$$

Clearly $\mathfrak{A}_{\mathrm{ue}}$ is a closed subset of $\mathfrak{A}$.

We endow the topological subspace $\mathfrak{A}_{\mathrm{ue}} \subset \mathfrak{A}$ with the relative topology.

We say that an operator $A : K \to K$ is affine if

$$A\big(\alpha x + (1-\alpha)y\big) = \alpha Ax + (1-\alpha)Ay \tag{1.3}$$

for each $x, y \in K$ and all $\alpha \in [0, 1]$.

Denote by $\mathfrak{M}$ the set of all uniformly continuous affine mappings $A : K \to K$. For the space $\mathfrak{M}$ we consider the metric

$$\rho(A, B) = \sup\{\|Ax - Bx\| : x \in K\}, \quad A, B \in \mathfrak{M}. \tag{1.4}$$

It is easy to see that the metric space $(\mathfrak{M}, \rho)$ is complete.

In the present paper, we analyze the convergence of infinite products of operators in $\mathfrak{M}$ and other mappings of affine type.

We begin by showing (Theorem 3.1) that for a generic operator B in the space $\mathfrak{M}$ there exists a unique fixed point x_B and the powers of B converge to x_B for all $x \in K$. We continue with a study of the asymptotic behavior of infinite products of this kind of operators. Section 2 contains necessary preliminaries and a weak ergodic theorem is established in Section 4. In Sections 5 and 7 we present several theorems on the generic convergence of infinite product trajectories to a common fixed point and to a common fixed point set, respectively. Proofs of these results are given in Sections 6 and 8. Finally, in Section 9 we establish the generic convergence of random products to a retraction onto a common fixed point set.

2. Infinite products

Denote by $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}$ such that for each integer $t \geq 1$, each $x, y \in K$ and all $\alpha \in [0,1]$,

$$A_t\big(\alpha x + (1-\alpha)y\big) = \alpha A_t x + (1-\alpha) A_t y. \tag{2.1}$$

Clearly $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ is a closed subset of $\mathfrak{A}_{\mathrm{ue}}$. We consider the topological subspace $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}} \subset \mathfrak{A}_{\mathrm{ue}}$ with the relative topology.

In this paper we show (Theorem 4.1) that for a generic sequence $\{C_t\}_{t=1}^{\infty}$ in the space $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$,

$$\big\| C_{r(T)} \cdot \cdots \cdot C_{r(1)} x - C_{r(T)} \cdot \cdots \cdot C_{r(1)} y \big\| \to 0 \tag{2.2}$$

uniformly for all $x, y \in K$ and all mappings $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$. Such results are usually called weak ergodic theorems in the population biology literature [4] (see also [12]).

Denote by $\mathfrak{A}_{\mathrm{ue}}^{0}$ the set of all $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}$ for which there exists $x_{\mathbf{A}} \in K$ such that

$$A_t x_{\mathbf{A}} = x_{\mathbf{A}}, \quad t = 1, 2, \ldots, \tag{2.3}$$

and for each $\gamma \in (0,1)$, $x \in K$ and each integer $t \geq 1$,

$$A_t\big(\gamma x_{\mathbf{A}} + (1-\gamma)x\big) = \lambda_t(\gamma, x) x_{\mathbf{A}} + \big(1 - \lambda_t(\gamma, x)\big) A_t x \tag{2.4}$$

with some constant $\lambda_t(\gamma, x) \in [\gamma, 1]$.

Denote by $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ the closure of $\mathfrak{A}_{\mathrm{ue}}^{0}$ in the space $\mathfrak{A}_{\mathrm{ue}}$. We consider the topological subspace $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ with the relative topology and show (Theorem 5.1) that for a generic sequence $\{C_t\}_{t=1}^{\infty}$ in the space $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ there exists a unique common fixed point x_* and all random products of the operators $\{C_t\}_{t=1}^{\infty}$ converge to x_* uniformly for all $x \in K$. We also show that this convergence of random infinite products to a unique common fixed point holds for a generic sequence from certain subspaces of the space $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$.

Assume now that $F \subset K$ is a nonempty closed convex set, $Q : K \to F$ is a uniformly continuous operator such that

$$Qx = x, \quad x \in F, \tag{2.5}$$

and for each $y \in K$, $x \in F$ and $\alpha \in [0,1]$,

$$Q\big(\alpha x + (1-\alpha)y\big) = \alpha x + (1-\alpha) Q y. \tag{2.6}$$

Denote by $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}$ such that

$$A_t x = x, \quad t = 1, 2, \ldots, \ x \in F, \tag{2.7}$$

and for each integer $t \geq 1$, each $y \in K$, $x \in F$ and $\alpha \in (0,1]$,

$$A_t\big(\alpha x + (1-\alpha)y\big) = \alpha x + (1-\alpha) A_t y. \tag{2.8}$$

Clearly $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ is a closed subset of $\mathfrak{A}_{\mathrm{ue}}$.

The topological subspace $\mathfrak{A}_{\mathrm{ue}}^{(F,0)} \subset \mathfrak{A}_{\mathrm{ue}}$ will be equipped with the relative topology.

We show (Theorem 7.1) that for a generic sequence of operators $\{C_t\}_{t=1}^{\infty}$ in the space $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ all its random infinite products

$$C_{r(t)} \cdot \cdots \cdot C_{r(1)} x \tag{2.9}$$

tend to the set F uniformly for all $x \in K$. Moreover, under a certain additional assumption on F these random products converge to a uniformly continuous retraction $P_r : K \to F$ uniformly for all $x \in K$ (Theorem 9.1).

For each bounded operator $A : K \to X$ we set

$$\|A\| = \sup\{\|Ax\| : x \in K\}. \tag{2.10}$$

For each $x \in K$ and each $E \subset X$ we set

$$d(x, E) = \inf\{\|x - y\| : y \in E\}, \quad \mathrm{rad}(E) = \sup\{\|y\| : y \in E\}. \tag{2.11}$$

In our study we need the following auxiliary result established in [13, Lemma 4.2].

Proposition 2.1. *Assume that E is a nonempty uniformly continuous set of operators $A : K \to K$, N is a natural number and ϵ is a positive number. Then there exists a number $\delta > 0$ such that for each sequence $\{A_t\}_{t=1}^{N} \subset E$, each sequence $\{B_t\}_{t=1}^{N}$, where the operators $B_t : K \to K$, $t = 1, \ldots, N$, (not necessarily continuous), satisfy*

$$\|B_t - A_t\| \le \delta, \quad t = 1, \ldots, N, \tag{2.12}$$

and each $x \in K$, the following relation holds:

$$\|B_N \cdot \cdots \cdot B_1 x - A_N \cdot \cdots \cdot A_1 x\| \le \epsilon. \tag{2.13}$$

3. Existence of a unique fixed point for a generic affine mapping

This section is devoted to the proof of the following result.

Theorem 3.1. *There exists a set $\mathfrak{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $A \in \mathfrak{F}$ the following assertions hold:*

(1) *there exists a unique $x_A \in K$ such that $Ax_A = x_A$;*

(2) *for each $\epsilon > 0$ there exist a neighborhood U of A in $\mathfrak{M}$ and a natural number N such that for each $\{B_t\}_{t=1}^{\infty} \subset U$ and each $x \in K$,*

$$\|B_T \cdot \cdots \cdot B_1 x - x_A\| \le \epsilon \quad \text{for all integers } T \ge N. \tag{3.1}$$

In the proof of Theorem 3.1 we need the following lemma.

Lemma 3.2. *Let $B \in \mathfrak{M}$ and $\epsilon \in (0,1)$. Then there exist $B_\epsilon \in \mathfrak{M}$, an integer $q \ge 1$, and $y_\epsilon \in K$ such that*

$$\rho(B, B_\epsilon) \le \epsilon, \quad \|B_\epsilon^t y_\epsilon - y_\epsilon\| \le \epsilon, \quad t = 1, \ldots, q, \tag{3.2}$$

and for each $z \in K$ the following relation holds:

$$\left\| B_\epsilon^q z - y_\epsilon \right\| \leq \epsilon. \tag{3.3}$$

Proof. Choose a number $\gamma \in (0,1)$ for which

$$8\gamma\big(\operatorname{rad}(K)+1\big) \leq \epsilon, \tag{3.4}$$

and then an integer $q \geq 1$ such that

$$(1-\gamma)^q\big(\operatorname{rad}(K)+1\big) \leq 16^{-1}\epsilon, \tag{3.5}$$

and a natural number N such that

$$16qN^{-1}\big(\operatorname{rad}(K)+1\big) \leq 8^{-1}\epsilon. \tag{3.6}$$

Fix $x_0 \in K$ and define a sequence $\{x_t\}_{t=0}^{\infty} \subset K$ by

$$x_{t+1} = Bx_t, \quad t = 0, 1, \ldots. \tag{3.7}$$

For each integer $k \geq 0$ define

$$y_k = N^{-1} \sum_{i=k}^{k+N-1} x_i. \tag{3.8}$$

It is easy to see that

$$By_k = y_{k+1}, \quad k = 0, 1, \ldots \tag{3.9}$$

and for each $k \in \{0, \ldots, q\}$

$$\|y_0 - y_k\| \leq 2kN^{-1}\operatorname{rad}(K) \leq 2qN^{-1}\operatorname{rad}(K). \tag{3.10}$$

Define $B_\epsilon : K \to K$ by

$$B_\epsilon z = (1-\gamma)Bz + \gamma y_0, \quad z \in K. \tag{3.11}$$

It is easy to see that

$$B_\epsilon \in \mathfrak{M} \qquad \text{and} \qquad \rho(B, B_\epsilon) < 2^{-1}\epsilon. \tag{3.12}$$

Now let z be an arbitrary point in K. We show by induction that for each integer $n \geq 1$

$$B_\epsilon^n z = (1-\gamma)^n B^n z + \sum_{i=0}^{n-1} c_{ni} y_i, \tag{3.13}$$

where

$$c_{ni} > 0, \quad i = 0, \ldots, n-1, \quad \sum_{i=0}^{n-1} c_{ni} + (1-\gamma)^n = 1. \tag{3.14}$$

It is easy to see that for $n = 1$ our assertion holds.

Assume that it is also valid for an integer $n \geq 1$. It follows from (3.11), (3.13), (3.14), (3.12), and (3.9) that

$$\begin{aligned} B_\epsilon^{n+1} z &= \gamma y_0 + (1-\gamma) B\big(B_\epsilon^n z\big) \\ &= \gamma y_0 + (1-\gamma)\Big[(1-\gamma)^n B^{n+1} z + \sum_{i=0}^{n-1} c_{ni} B y_i\Big] \\ &= (1-\gamma)^{n+1} B^{n+1} z + \gamma y_0 + (1-\gamma) \sum_{i=0}^{n-1} c_{ni} y_{i+1}. \end{aligned} \tag{3.15}$$

This implies that our assertion is also valid for $n+1$. Therefore for each integer $n \geq 1$, (3.13) holds with some constants c_{ni}, $i = 0, \ldots, n-1$, satisfying (3.14).

Now we show that

$$\big\| B_\epsilon^q z - y_0 \big\| \leq \epsilon. \tag{3.16}$$

We have shown that there exist positive numbers $c_{qi} > 0$, $i = 0, \ldots, q-1$, such that

$$\sum_{i=0}^{q-1} c_{qi} + (1-\gamma)^q = 1 \qquad \text{and} \qquad B_\epsilon^q z = (1-\gamma)^q B^q z + \sum_{i=0}^{q-1} c_{qi} y_i. \tag{3.17}$$

By (3.17), (3.10), (3.5), and (3.6),

$$\begin{aligned} \big\| B_\epsilon^q z - y_0 \big\| &\leq (1-\gamma)^q \big\| B^q z - y_0 \big\| + \sum_{i=0}^{q-1} c_{qi} \| y_0 - y_i \| \\ &\leq 2(1-\gamma)^q \operatorname{rad}(K) + 2qN^{-1} \operatorname{rad}(K) \\ &\leq 16^{-1}\epsilon + 8^{-1}\epsilon < 2^{-1}\epsilon. \end{aligned} \tag{3.18}$$

Therefore we have shown that

$$\big\| B_\epsilon^q z - y_0 \big\| \leq 2^{-1}\epsilon \quad \text{for each } z \in K. \tag{3.19}$$

Let $t \in \{1, \ldots, q\}$. To finish the proof we show that

$$\big\| B_\epsilon^t y_0 - y_0 \big\| \leq \epsilon. \tag{3.20}$$

By (3.13) and (3.14) there exist positive numbers c_{ti}, $i = 0, \ldots, t-1$, such that

$$\sum_{i=0}^{t-1} c_{ti} + (1-\gamma)^t = 1 \qquad \text{and} \qquad B_\epsilon^t y_0 = (1-\gamma)^t B^t y_0 + \sum_{i=0}^{t-1} c_{ti} y_i. \tag{3.21}$$

Together with (3.9), (3.10), and (3.6) this implies that

$$\begin{aligned} \big\| y_0 - B_\epsilon^t y_0 \big\| &= \Big\| y_0 - \sum_{i=0}^{t-1} c_{ti} y_i - (1-\gamma)^t y_t \Big\| \\ &\leq 4qN^{-1} \operatorname{rad}(K) < 8^{-1}\epsilon. \end{aligned} \tag{3.22}$$

This completes the proof of Lemma 3.2 (with $y_\epsilon = y_0$). □

Proof of Theorem 3.1. To begin the construction of the set $\mathfrak{F}$, let $B \in \mathfrak{M}$ and let $i \geq 1$ be an integer. By Lemma 3.2 there exist $C^{(B,i)} \in \mathfrak{M}$, $y(B,i) \in K$, and an integer $q(B,i) \geq 1$ such that

$$\rho\big(B, C^{(B,i)}\big) \leq 8^{-i}, \quad \big\|\big(C^{(B,i)}\big)^t y(B,i) - y(B,i)\big\| \leq 8^{-i}, \quad t = 0, \ldots, q(B,i), \tag{3.23}$$

$$\big\|\big(C^{(B,i)}\big)^{q(B,i)} z - y(B,i)\big\| \leq 8^{-i} \quad \text{for each } z \in K. \tag{3.24}$$

By Proposition 2.1 there exists an open neighborhood $U(B,i)$ of $C^{(B,i)}$ in $\mathfrak{M}$ such that for each $\{A_j\}_{j=1}^{q(B,i)} \subset U(B,i)$ and each $z \in K$,

$$\big\|A_{q(B,i)} \cdot \cdots \cdot A_1 z - \big(C^{(B,i)}\big)^{q(B,i)} z\big\| \leq 64^{-i}. \tag{3.25}$$

It follows from (3.24) and (3.25) that for each $\{A_i\}_{j=1}^{q(B,i)} \subset U(B,i)$ and each $z \in K$,

$$\big\|A_{q(B,i)} \cdot \cdots \cdot A_1 z - y(B,i)\big\| \leq 8^{-i} + 64^{-i}. \tag{3.26}$$

Define

$$\mathfrak{F} = \cap_{k=1}^{\infty} \cup \{U(B,i) : B \in \mathfrak{M},\ i = k, k+1, \ldots\}. \tag{3.27}$$

It is easy to see that $\mathfrak{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$.

Assume that $A \in \mathfrak{F}$ and $\epsilon > 0$. Choose a natural number k for which

$$64 \cdot 2^{-k} < \epsilon. \tag{3.28}$$

There exist $B \in \mathfrak{M}$ and an integer $i \geq k$ such that

$$A \in U(B,i). \tag{3.29}$$

Combined with (3.26) and (3.28) this implies that for each $z \in K$,

$$\big\|A^{q(B,i)} z - y(B,i)\big\| \leq 8^{-i} + 64^{-i} < \epsilon. \tag{3.30}$$

Since ϵ is an arbitrary positive number we conclude that there exists a unique $x_A \in K$ such that $Ax_A = x_A$. Clearly

$$\big\|x_A - y(B,i)\big\| \leq 8^{-i} + 64^{-i}. \tag{3.31}$$

Together with (3.26) and (3.28) this last inequality implies that for each $\{A_j\}_{j=1}^{\infty} \subset U(B,i)$, each $z \in K$, and each integer $T \geq q(B,i)$,

$$\big\|A_T \cdot \cdots \cdot A_1 z - x_A\big\| \leq 2\big(8^{-i} + 64^{-i}\big) < \epsilon. \tag{3.32}$$

This completes the proof of Theorem 3.1. □

4. A weak ergodic theorem for infinite products of affine mappings

In this section we establish the following result.

THEOREM 4.1. *There exists a set* $\mathfrak{F} \subset \mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ *which is a countable intersection of open everywhere dense subsets of* $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ *such that for each* $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ *and each* $\epsilon > 0$ *there exists a neighborhood* U *of* $\{B_t\}_{t=1}^{\infty}$ *in* $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ *and a natural number* N *such that for each* $\{C_t\}_{t=1}^{\infty} \in U$*, each integer* $T \geq N$*, each* $r : \{1,\dots,T\} \to \{1,2,\dots\}$*, and each* $x, y \in K$*,*

$$\left\| C_{r(T)} \cdot \dots \cdot C_{r(1)} x - C_{r(T)} \cdot \dots \cdot C_{r(1)} y \right\| \leq \epsilon. \tag{4.1}$$

Proof. Fix $y_* \in K$. Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ and $\gamma \in (0,1)$. For $t = 1,2,\dots$ define $A_{t\gamma} : K \to K$ by

$$A_{t\gamma} x = (1-\gamma) A_t x + \gamma y_*, \quad x \in K. \tag{4.2}$$

Clearly

$$\left\{A_{t\gamma}\right\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}, \qquad \rho_s\left(\left\{A_t\right\}_{t=1}^{\infty}, \left\{A_{t\gamma}\right\}_{t=1}^{\infty}\right) \leq 2\gamma \operatorname{rad}(K). \tag{4.3}$$

Let $i \geq 1$ be an integer. Choose a natural number $N(\gamma,i) \geq 4$ such that

$$(1-\gamma)^{N(\gamma,i)}\left(\operatorname{rad}(K)+1\right) < 16^{-1} 4^{-i}. \tag{4.4}$$

We show by induction that for each integer $T \geq 1$ the following assertion holds.

For each $r : \{1,\dots,T\} \to \{1,2,\dots\}$ there exists $y_{r,T} \in K$ such that

$$A_{r(T)\gamma} \cdot \dots \cdot A_{r(1)\gamma} x = (1-\gamma)^T A_{r(T)} \cdot \dots \cdot A_{r(1)} x + \left(1-(1-\gamma)^T\right) y_{r,T} \tag{4.5}$$

for each $x \in K$.

Clearly for $T = 1$ the assertion is true. Assume that it is also true for an integer $T \geq 1$. It follows from (4.5) that for each $r : \{1,\dots,T+1\} \to \{1,2,\dots\}$ and each $x \in K$,

$$\begin{aligned}
&A_{r(T+1)\gamma} \cdot \dots \cdot A_{r(1)\gamma} x \\
&\quad = A_{r(T+1)\gamma}\left[A_{r(T)\gamma} \cdot \dots \cdot A_{r(1)\gamma} x\right] \\
&\quad = A_{r(T+1)\gamma}\left[(1-\gamma)^T A_{r(T)} \cdot \dots \cdot A_{r(1)} x + \left(1-(1-\gamma)^T\right) y_{r,T}\right] \\
&\quad = \gamma y_* + (1-\gamma) A_{r(T+1)}\left[(1-\gamma)^T A_{r(T)} \cdot \dots \cdot A_{r(1)} x + \left(1-(1-\gamma)^T\right) y_{r,T}\right] \\
&\quad = (1-\gamma)^{T+1} A_{r(T+1)} \cdot \dots \cdot A_{r(1)} x + (1-\gamma)\left(1-(1-\gamma)^T\right) A_{r(T+1)} y_{r,T} + \gamma y_*.
\end{aligned} \tag{4.6}$$

This implies that the assertion is also valid for $T+1$. Therefore, we have shown that our assertion is true for any integer $T \geq 1$. Together with (4.4) this implies that the following property holds:

(a) for each integer $T \geq N(\gamma,i)$, each $r : \{1,\dots,T\} \to \{1,2,\dots\}$, and each $x, y \in K$,

$$\left\| A_{r(T)\gamma} \cdot \dots \cdot A_{r(1)\gamma} x - A_{r(T)\gamma} \cdot \dots \cdot A_{r(1)\gamma} y \right\| \leq 2(1-\gamma)^T \operatorname{rad}(K) \leq 8^{-1} \cdot 4^{-i}. \tag{4.7}$$

By Proposition 2.1 there is an open neighborhood $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ of $\{A_{t\gamma}\}_{t=1}^{\infty}$ in $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$ such that for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each $r : \{1, \dots, N(\gamma, i)\} \to \{1, 2, \dots\}$, and each $x \in K$,

$$\left\| C_{r(N(\gamma,i))} \cdot \dots \cdot C_{r(1)}x - A_{r(N(\gamma,i))\gamma} \cdot \dots \cdot A_{r(1)\gamma}x \right\| \le 64^{-1} \cdot 4^{-i}. \tag{4.8}$$

Together with property (a) this implies that the following property holds:

(b) for each integer $T \ge N(\gamma, i)$, each $r : \{1, \dots, T\} \to \{1, 2, \dots\}$, each $x, y \in K$, and each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$,

$$\left\| C_{r(T)} \cdot \dots \cdot C_{r(1)}x - C_{r(T)} \cdot \dots \cdot C_{r(1)}y \right\| \le 4^{-i-1}. \tag{4.9}$$

Define

$$\mathfrak{F} = \cap_{q=1}^{\infty} \cup \left\{ U\left(\{A_t\}_{t=1}^{\infty}, \gamma, i\right) : \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}},\ \gamma \in (0,1),\ i = q, q+1, \dots \right\}. \tag{4.10}$$

Clearly $\mathfrak{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$. Let $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and $\epsilon > 0$. Choose a natural number q for which

$$64 \cdot 2^{-q} < \epsilon. \tag{4.11}$$

There exist $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{\mathrm{af}}$, $\gamma \in (0,1)$, and an integer $i \ge q$ such that

$$\{B_t\}_{t=1}^{\infty} \in U\left(\{A_t\}_{t=1}^{\infty}, \gamma, i\right). \tag{4.12}$$

By property (b) and (4.11), for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each $T \ge N(\gamma, i)$, each $r : \{1, \dots, T\} \to \{1, 2, \dots\}$, and each $x, y \in K$,

$$\left\| C_{r(T)} \cdot \dots \cdot C_{r(1)}x - C_{r(T)} \cdot \dots \cdot C_{r(1)}y \right\| \le 4^{-i-1} < \epsilon. \tag{4.13}$$

This completes the proof of Theorem 4.1. □

5. The convergence of infinite products of affine mappings with a common fixed point

In this section we state three theorems which will be proved in Section 6.

THEOREM 5.1. *There exists a set $\mathfrak{F} \subset \bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ such that $\mathfrak{F} \subset \mathfrak{A}_{\mathrm{ue}}^{0}$ and for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ the following assertion holds.*

Let $x_{\mathbf{B}} \in K$, $B_t x_{\mathbf{B}} = x_{\mathbf{B}}$, $t = 1, 2, \dots$, and let $\epsilon > 0$. Then there exist a neighborhood U of $\mathbf{B} = \{B_t\}_{t=1}^{\infty}$ in $\bar{\mathfrak{A}}_{\mathrm{ue}}^{0}$ and a natural number N such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each integer $T \ge N$, each $r : \{1, \dots, T\} \to \{1, 2, \dots\}$, and each $x \in K$,

$$\left\| C_{r(T)} \cdot \dots \cdot C_{r(1)}x - x_{\mathbf{B}} \right\| \le \epsilon. \tag{5.1}$$

Denote by $\mathfrak{A}_{\mathrm{ue}}^{(1)}$ the set of all $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}$ for which there exists $x_{\mathbf{A}} \in K$ such that

$$A_t x_{\mathbf{A}} = x_{\mathbf{A}}, \quad t = 1, 2, \dots, \tag{5.2}$$

and for each $\alpha \in (0,1)$, $x \in K$, and an integer $t \geq 1$,

$$A_t\big(\alpha x_{\mathbf{A}} + (1-\alpha)x\big) = \alpha x_{\mathbf{A}} + (1-\alpha)A_t x. \tag{5.3}$$

Denote by $\bar{\mathfrak{A}}_{\text{ue}}^{(1)}$ the closure of $\mathfrak{A}_{\text{ue}}^{(1)}$ in the space $\mathfrak{A}_{\text{ue}}$. We equip the topological subspace $\bar{\mathfrak{A}}_{\text{ue}}^{(1)} \subset \mathfrak{A}_{\text{ue}}$ with the relative topology.

THEOREM 5.2. *Let a set $\mathfrak{F} \subset \bar{\mathfrak{A}}_{\text{ue}}^{0}$ be as guaranteed in Theorem 5.1. There exists a set $\mathfrak{F}^{(1)} \subset \mathfrak{F} \cap \mathfrak{A}_{\text{ue}}^{(1)}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{\text{ue}}^{(1)}$.*

Denote by $\mathfrak{A}_{\text{ue},0}^{\text{af}}$ the set of all $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\text{ue}}^{\text{af}}$ for which there exists $x_{\mathbf{A}} \in K$ such that (5.2) holds.

Denote by $\bar{\mathfrak{A}}_{\text{ue},0}^{\text{af}}$ the closure of $\mathfrak{A}_{\text{ue},0}^{\text{af}}$ in the space $\mathfrak{A}_{\text{ue}}$. We also consider the topological subspace $\bar{\mathfrak{A}}_{\text{ue},0}^{\text{af}} \subset \mathfrak{A}_{\text{ue}}$ with the relative topology.

THEOREM 5.3. *Let a set $\mathfrak{F}^{(1)}$ be as guaranteed in Theorem 5.2. There exists a set $\mathfrak{F}_* \subset \mathfrak{F}^{(1)} \cap \mathfrak{A}_{\text{ue},0}^{\text{af}}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{\text{ue},0}^{\text{af}}$.*

Theorems 5.2 and 5.3 show that the generic convergence established in Theorem 5.1 is also valid for certain subspaces of $\bar{\mathfrak{A}}_{\text{ue}}^{0}$.

6. Proofs of Theorems 5.1, 5.2, and 5.3

Proof of Theorem 5.1. Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\text{ue}}^{0}$ and $\gamma \in (0,1)$. There exists $x_{\mathbf{A}} \in K$ such that

$$A_t x_{\mathbf{A}} = x_{\mathbf{A}}, \quad t = 1,2,\ldots, \tag{6.1}$$

and for each integer $t \geq 1$, $x \in K$, and $\alpha \in (0,1)$,

$$A_t\big(\alpha x_{\mathbf{A}} + (1-\alpha)x\big) = \lambda_t(\alpha,x)x_{\mathbf{A}} + \big(1-\lambda_t(\alpha,x)\big)A_t x \tag{6.2}$$

with some constant $\lambda_t(\alpha,x) \in [\alpha,1]$.

For $t = 1,2,\ldots$ define $A_{t\gamma} : K \to K$ by

$$A_{t\gamma}x = (1-\gamma)A_t x + \gamma x_{\mathbf{A}}, \quad x \in K. \tag{6.3}$$

Clearly

$$\big\{A_{t\gamma}\big\}_{t=1}^{\infty} \in \mathfrak{A}_{\text{ue}}, \quad A_{t\gamma}x_{\mathbf{A}} = x_{\mathbf{A}}, \quad t = 1,2,\ldots \tag{6.4}$$

Let $x \in K$, $\alpha \in [0,1)$ and let $t \geq 1$ be an integer. Then there exists $\lambda_t(\alpha,x) \in [\alpha,1]$ such that (6.2) holds. Also, by (6.3) and (6.2),

$$\begin{aligned} A_{t\gamma}&\big(\alpha x_{\mathbf{A}}+(1-\alpha)x\big) \\ &= (1-\gamma)A_t\big(\alpha x_{\mathbf{A}}+(1-\alpha)x\big)+\gamma x_{\mathbf{A}} \\ &= \gamma x_{\mathbf{A}}+(1-\gamma)\big[\lambda_t(\alpha,x)x_{\mathbf{A}}+\big(1-\lambda_t(\alpha,x)\big)A_t x\big] \\ &= (1-\gamma)\big(1-\lambda_t(\alpha,x)\big)A_t x+\big[\gamma+(1-\gamma)\lambda_t(\alpha,x)\big]x_{\mathbf{A}} \\ &= \big(1-\lambda_t(\alpha,x)\big)A_{t\gamma}x+\big[\gamma+(1-\gamma)\lambda_t(\alpha,x)-\gamma\big(1-\lambda_t(\alpha,x)\big)\big]x_{\mathbf{A}} \\ &= \big(1-\lambda_t(\alpha,x)\big)A_{t\gamma}x+\lambda_t(\alpha,x)x_{\mathbf{A}}. \end{aligned} \tag{6.5}$$

Thus property (2.4) is satisfied and therefore

$$\big\{A_{t\gamma}\big\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{0}. \tag{6.6}$$

Let $z \in K$. We show by induction that for each integer $T \geq 1$ and each $r : \{1,\dots,T\} \to \{1,2,\dots\}$ there exists $\lambda(z,T,r) \in [0,(1-\gamma)^T]$ such that

$$A_{r(T)\gamma}\cdot\dots\cdot A_{r(1)\gamma}z = \lambda(z,T,r)A_{r(T)}\cdot\dots\cdot A_{r(1)}z+\big(1-\lambda(z,T,r)\big)x_{\mathbf{A}}. \tag{6.7}$$

Clearly for $T=1$ our assertion is valid.

Assume that it is also valid for an integer $T \geq 1$. Let $r : \{1,\dots,T+1\} \to \{1,2,\dots\}$. There exists $\lambda(z,T,r) \in [0,(1-\gamma)^T]$ such that (6.7) is valid. It follows from (6.7) and (6.5) that

$$\begin{aligned} A_{r(T+1)\gamma}&\cdot\dots\cdot A_{r(1)\gamma}z \\ &= A_{r(T+1)\gamma}\big[\lambda(z,T,r)A_{r(T)}\cdot\dots\cdot A_{r(1)}z+\big(1-\lambda(z,T,r)\big)x_{\mathbf{A}}\big] \\ &= (1-\gamma)(1-\kappa)A_{r(T+1)}A_{r(T)}\cdot\dots\cdot A_{r(1)}z+\big[\gamma+(1-\gamma)\kappa\big]x_{\mathbf{A}} \end{aligned} \tag{6.8}$$

with $\kappa \in [1-\lambda(z,T,r),1]$. Set

$$\lambda(z,T+1,r) = (1-\gamma)(1-\kappa). \tag{6.9}$$

It is easy to see that

$$\begin{gathered} 0 \leq \lambda(z,T+1,r) \leq (1-\gamma)\lambda(z,T,r) \leq (1-\gamma)^{T+1}, \\ \begin{aligned} A_{r(T+1)\gamma}&\cdot\dots\cdot A_{r(1)\gamma}z \\ &= \lambda(z,T+1,r)A_{r(T+1)}\cdot\dots\cdot A_{r(1)}z+\big(1-\lambda(z,T+1,r)\big)x_{\mathbf{A}}. \end{aligned} \end{gathered} \tag{6.10}$$

Therefore the assertion is valid for $T+1$. We have shown that for each integer $T \geq 1$ and each $r : \{1,\dots,T\} \to \{1,2,\dots\}$ there exists $\lambda(z,T,r) \in [0,(1-\gamma)^T]$ such that (6.7) holds.

Let $i \geq 1$ be an integer. Choose a natural number $N(\gamma,i)$ for which

$$64(1-\gamma)^{N(\gamma,i)}\big(\operatorname{rad}(K)+1\big) < 8^{-i}. \tag{6.11}$$

We show that for each $z \in K$, each integer $T \geq N(\gamma, i)$ and each $r : \{1,\dots,T\} \to \{1,2,\dots\}$,

$$\left\| A_{r(T)\gamma} \cdot \dots \cdot A_{r(1)\gamma} z - x_{\mathbf{A}} \right\| \leq 8^{-i-1}. \tag{6.12}$$

Let $T \geq N(\gamma, i)$ be an integer, $z \in K$, and $r : \{1,\dots,T\} \to \{1,2,\dots\}$. There exists $\lambda(z,T,r) \in [0,(1-\gamma)^T]$ such that (6.7) holds. It is easy to see that (6.7) and (6.11) imply (6.12).

By Proposition 2.1 there exists a number

$$\delta\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big) \in \big(0, 16^{-1}8^{-i}\big) \tag{6.13}$$

such that for each $\{C_t\}_{t=1}^{\infty} \in \bar{\mathfrak{A}}_{\text{ue}}^0$ satisfying

$$\rho_s\big(\{C_t\}_{t=1}^{\infty}, \{A_{t\gamma}\}_{t=1}^{\infty}\big) \leq \delta\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big), \tag{6.14}$$

each $r : \{1,\dots,N(\gamma,i)\} \to \{1,2,\dots\}$ and each $x \in K$,

$$\left\| C_{r(N(\gamma,i))} \cdot \dots \cdot C_{r(1)} x - A_{r(N(\gamma,i))\gamma} \cdot \dots \cdot A_{r(1)\gamma} x \right\| \leq 16^{-1} \cdot 8^{-i}. \tag{6.15}$$

Set

$$U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big) = \big\{\{C_t\}_{t=1}^{\infty} \in \bar{\mathfrak{A}}_{\text{ue}}^0 : \rho_s\big(\{C_t\}_{t=1}^{\infty}, \{A_{t\gamma}\}_{t=1}^{\infty}\big) < \delta\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big)\big\}. \tag{6.16}$$

It follows from (6.16), the choice of $\delta(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ (see (6.13), (6.15)) and (6.12) that the following property holds:

(a) for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each integer $T \geq N(\gamma, i)$, each $r : \{1,\dots,T\} \to \{1,2,\dots\}$, and each $x \in K$,

$$\left\| C_{r(T)} \cdot \dots \cdot C_{r(1)} x - x_{\mathbf{A}} \right\| \leq 8^{-i}. \tag{6.17}$$

Define

$$\mathfrak{F} = \cap_{q=1}^{\infty} \cup \big\{U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big) : \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\text{ue}}^0, \gamma \in (0,1),\ i = q, q+1, \dots\big\}. \tag{6.18}$$

It is easy to see that $\mathfrak{F}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{\text{ue}}^0$.

Assume now that $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and $\epsilon > 0$. Choose a natural number q such that

$$64 \cdot 2^{-q} < \epsilon. \tag{6.19}$$

There exist $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\text{ue}}^0$, $\gamma \in (0,1)$, and an integer $i \geq q$ such that

$$\big\{B_t\big\}_{t=1}^{\infty} \in U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big). \tag{6.20}$$

By property (a), (6.20), and (6.19), for each $x \in K$, each integer $T \geq N(\gamma, i)$, and each integer $\tau \geq 1$,

$$\left\| B_\tau^T x - x_{\mathbf{A}} \right\| \leq 8^{-i} < \epsilon. \tag{6.21}$$

Since ϵ is an arbitrary positive number we conclude that there exists $x_{\mathbf{B}} \in K$ such that

$$\lim_{T \to \infty} B_\tau^T x = x_{\mathbf{B}} \tag{6.22}$$

for each $x \in K$ and each integer $\tau \geq 1$. It is easy to see that

$$B_t x_{\mathbf{B}} = x_{\mathbf{B}}, \quad t = 1, 2, \ldots, \quad \left\| x_{\mathbf{B}} - x_{\mathbf{A}} \right\| \leq 8^{-i} < \epsilon. \tag{6.23}$$

It follows from property (a), (6.23), and (6.19) that for each sequence $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each integer $T \geq N(\gamma, i)$, each $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$, and each $x \in K$,

$$\left\| C_{r(T)} \cdot \cdots \cdot C_{r(1)} x - x_{\mathbf{B}} \right\| < \epsilon. \tag{6.24}$$

We show that for each integer $t \geq 1$, $x \in K$, and $\alpha \in (0, 1)$ there exists $\lambda \in [\alpha, 1]$ such that

$$B_t \big(\alpha x_{\mathbf{B}} + (1-\alpha) x \big) = \lambda x_{\mathbf{B}} + (1-\lambda) B_t x. \tag{6.25}$$

Let $t \geq 1$ be an integer, $x \in K$ and let $\alpha \in (0, 1)$. By (6.2) and (6.5) there exists $\lambda_\epsilon \in [\alpha, 1]$ such that

$$A_{t\gamma} \big(\alpha x_{\mathbf{A}} + (1-\alpha) x \big) = \lambda_\epsilon x_{\mathbf{A}} + (1-\lambda_\epsilon) A_{t\gamma} x. \tag{6.26}$$

Since ϵ is an arbitrary positive number it follows from (6.26), (6.23), (6.20), (6.16), (6.13), and (6.19) that for each $\epsilon > 0$ there exist $\lambda_\epsilon \in [\alpha, 1]$, $z_\epsilon \in K$ such that

$$\left\| z_\epsilon - x_{\mathbf{B}} \right\| \leq \epsilon, \quad \left\| B_t \big(\alpha z_\epsilon + (1-\alpha) x \big) - \big(\lambda_\epsilon x_{\mathbf{B}} + (1-\lambda_\epsilon) B_t x \big) \right\| \leq \epsilon. \tag{6.27}$$

This implies that (6.25) holds with some $\lambda \in [\alpha, 1]$. This completes the proof of Theorem 5.1. □

Proof of Theorem 5.2. Let $\mathfrak{F}$ be as constructed in the proof of Theorem 5.1. Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(1)}$, $\gamma \in (0, 1)$ and let $i \geq 1$ be an integer. There exists $x_{\mathbf{A}} \in K$ such that (6.15) holds, and for each $x \in K$, each integer $t \geq 1$, and each $\alpha \in [0, 1]$ the equality (6.2) holds with $\lambda_t(\alpha, x) = \alpha$. For $t = 1, 2, \ldots$ define $A_{t\gamma} : K \to K$ by (6.3). It is easy to see that $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(1)}$. Choose a natural number $N(\gamma, i)$ for which (6.11) holds. Let $\delta(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ be defined as in the proof of Theorem 5.1. Set

$$\begin{aligned} \mathfrak{F}^{(1)} = \Big[\cap_{q=1}^{\infty} \cup \big\{ U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i \big) \\ : \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(1)}, \gamma \in (0, 1), i = q, q+1, \ldots \big\} \Big] \cap \bar{\mathfrak{A}}_{\mathrm{ue}}^{(1)}. \end{aligned} \tag{6.28}$$

Clearly $\mathfrak{F}^{(1)}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{\mathrm{ue}}^{(1)}$ and $\mathfrak{F}^{(1)} \subset \mathfrak{F}$. Arguing as in the proof of Theorem 5.1 we can show that $\mathfrak{F}^{(1)} \subset \mathfrak{A}_{\mathrm{ue}}^{(1)}$. This completes the proof of Theorem 5.2. □

Since the proof of Theorem 5.3 is analogous to that of Theorem 5.2 we omit it.

7. The weak convergence of infinite products of affine mappings with a common set of fixed points

In this section, we present two theorems concerning the space $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ defined in Section 2. Recall that F is a nonempty closed convex subset of K for which there exists a uniformly continuous operator $Q : K \to F$ such that

$$Qx = x, \quad x \in F, \tag{7.1}$$

and for each $y \in K$, $x \in F$, and $\alpha \in [0,1]$,

$$Q\big(\alpha x + (1-\alpha)y\big) = \alpha x + (1-\alpha)Qy. \tag{7.2}$$

Now we state the first theorem.

THEOREM 7.1. *There exists a set* $\mathfrak{F} \subset \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ *which is a countable intersection of open everywhere dense sets in* $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ *and such that for each* $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ *the following assertion holds.*

For each $\epsilon > 0$ *there exist a neighborhood* U *of* $\{B_t\}_{t=1}^{\infty}$ *in the space* $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ *and a natural number* N *such that for each* $\{C_t\}_{t=1}^{\infty} \in U$*, each integer* $T \geq N$*, each* $r : \{1,2,\ldots,T\} \to \{1,2,\ldots\}$*, and each* $x \in K$*,*

$$d\big(C_{r(T)} \cdot \cdots \cdot C_{r(1)}x, F\big) \leq \epsilon. \tag{7.3}$$

Assume now that for each $x, y \in K$ and $\alpha \in [0,1]$,

$$Q\big(\alpha x + (1-\alpha)y\big) = \alpha Qx + (1-\alpha)Qy. \tag{7.4}$$

Denote by $\mathfrak{A}_{\mathrm{ue}}^{(F,1)}$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}$ such that

$$A_t x = x, \quad t = 1,2,\ldots, x \in F, \tag{7.5}$$

and for each $t \in \{1,2,\ldots\}$, each $x, y \in K$ and each $\alpha \in [0,1]$,

$$A_t\big(\alpha x + (1-\alpha)y\big) = \alpha A_t x + (1-\alpha)A_t y. \tag{7.6}$$

Clearly $\mathfrak{A}_{\mathrm{ue}}^{(F,1)}$ is a closed subset of $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$. We consider the topological subspace $\mathfrak{A}_{\mathrm{ue}}^{(F,1)} \subset \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ with the relative topology.

Here is the second theorem.

THEOREM 7.2. *Let a set* $\mathfrak{F}$ *be as guaranteed in Theorem 7.1. Then there exists a set* $\mathfrak{F}_1 \subset \mathfrak{F} \cap \mathfrak{A}_{\mathrm{ue}}^{(F,1)}$ *which is a countable intersection of open everywhere dense subsets of* $\mathfrak{A}_{\mathrm{ue}}^{(F,1)}$.

8. Proof of Theorems 7.1 and 7.2

Proof of Theorem 7.1. Let $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ and $\gamma \in (0,1)$. For $t = 1,2,\ldots$ we define $A_{t\gamma} : K \to K$ by

$$A_{t\gamma}x = (1-\gamma)A_t x + \gamma Qx, x \in K. \tag{8.1}$$

It is easy to see that

$$\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}. \tag{8.2}$$

Let $z \in K$. By induction we show that for each integer $T \geq 1$ the following assertion holds.

For each $r : \{1,\ldots,T\} \to \{1,2,\ldots\}$,

$$A_{r(T)\gamma} \cdot \cdots \cdot A_{r(1)\gamma}z = (1-\gamma)^T A_{r(T)} \cdot \cdots \cdot A_{r(1)}z + \big(1-(1-\gamma)^T\big)y_T \tag{8.3}$$

with some $y_T \in F$.

Clearly for $T = 1$ our assertion is valid. Assume that it is also valid for $T \geq 1$ and that $r : \{1, \ldots, T+1\} \to \{1, 2, \ldots\}$. Clearly (8.3) holds with some $y_T \in F$. By (8.3), (8.2), and (8.1),

$$\begin{aligned}
A_{r(T+1)\gamma} \cdot \cdots \cdot A_{r(1)\gamma} z &= A_{r(T+1)\gamma}\big[(1-\gamma)^T A_{r(T)} \cdot \cdots \cdot A_{r(1)} z + \big(1-(1-\gamma)^T\big) y_T\big] \\
&= (1-\gamma)^T A_{r(T+1)\gamma}\big[A_{r(T)} \cdot \cdots \cdot A_{r(1)} z\big] + \big(1-(1-\gamma)^T\big) y_T \\
&= (1-\gamma)^{T+1} A_{r(T+1)} \cdot \cdots \cdot A_{r(1)} z \\
&\quad + \gamma(1-\gamma)^T Q[A_{r(T)} \cdot \cdots \cdot A_{r(1)} z] + \big(1-(1-\gamma)^T\big) y_T \\
&= (1-\gamma)^{T+1} A_{r(T+1)} \cdot \cdots \cdot A_{r(1)} z + \big(1-(1-\gamma)^{T+1}\big) \\
&\quad \times \big[\big(1-(1-\gamma)^{T+1}\big)^{-1} \gamma(1-\gamma)^T Q\big[A_{r(T)} \cdot \cdots \cdot A_{r(1)} z\big] \\
&\quad + \big(1-(1-\gamma)^{T+1}\big)^{-1}\big(1-(1-\gamma)^T\big)\big) y_T\big].
\end{aligned} \tag{8.4}$$

This implies that our assertion also holds for $T+1$.

Therefore we have shown that it is valid for all integers $T \geq 1$.

Let $i \geq 1$ be an integer. Choose a natural number $N(\gamma, i)$ for which

$$64(1-\gamma)^{N(\gamma,i)}(\mathrm{rad}(K)+1) < 8^{-i}. \tag{8.5}$$

It follows from (8.3) that for each $z \in K$, each integer $T \geq N(\gamma, i)$ and each $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$,

$$d\big(A_{r(T)\gamma} \cdot \cdots \cdot A_{r(1)\gamma} z, F\big) \leq 8^{-i-1}. \tag{8.6}$$

By Proposition 2.1 there exists an open neighborhood $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ of $\{A_{t\gamma}\}_{t=1}^{\infty}$ in $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ such that the following property holds:

(a) for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each $r : \{1, \ldots, N(\gamma, i)\} \to \{1, 2, \ldots\}$, and each $x \in K$,

$$\big\| C_{r(N(\gamma,i))} \cdot \cdots \cdot C_{r(1)} x - A_{r(N(\gamma,i))\gamma} \cdot \cdots \cdot A_{r(1)\gamma} x \big\| \leq 16^{-1} 8^{-i}. \tag{8.7}$$

It follows from the definition of $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ and (8.6) that the following property is also true:

(b) for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each integer $T \geq N(\gamma, i)$, each $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$ and each $x \in K$,

$$d\big(C_{r(T)} \cdot \cdots \cdot C_{r(1)} x, F\big) \leq 8^{-i}. \tag{8.8}$$

Define

$$\mathfrak{F} = \cap_{q=1}^{\infty} \cup \big\{U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big) : \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}, \gamma \in (0,1),\ i = q, q+1, \ldots\big\}. \tag{8.9}$$

It is easy to see that $\mathfrak{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$.

Assume that $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and $\epsilon > 0$. Choose a natural number q such that

$$64 \cdot 2^{-q} < \epsilon. \tag{8.10}$$

There exist $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$, $\gamma \in (0,1)$, and an integer $i \geq q$ such that $\{B_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$. By (8.10) and property (b) for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each integer $T \geq N(\gamma, i)$, each $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$, and each $x \in K$,

$$d\big(C_{r(T)} \cdot \cdots \cdot C_{r(1)} x, F\big) \leq \epsilon. \tag{8.11}$$

This completes the proof of Theorem 7.1. □

Analogously to the proof of Theorem 5.2 we can prove Theorem 7.2 by modifying the proof of Theorem 7.1.

9. The convergence of infinite products of affine mappings with a common set of fixed points

In this section, as in Section 7, we assume that F is a nonempty closed convex subset of K, and $Q : K \to F$ is a uniformly continuous retraction satisfying (7.2).

However we assume in addition that there exists a number $\Delta > 0$ such that

$$\big\{x \in X : d(x, F) < \Delta\big\} \subset K. \tag{9.1}$$

In this setting we can strengthen Theorem 7.1.

THEOREM 9.1. *Let the set $\mathfrak{F} \subset \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ be as constructed in the proof of Theorem 7.1. Then for each $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ the following assertions hold:*

(1) *For each $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ there exists a uniformly continuous operator $P_r : K \to F$ such that*

$$\lim_{T \to \infty} B_{r(T)} \cdot \cdots \cdot B_{r(1)} x = P_r x \quad \textit{for each } x \in K. \tag{9.2}$$

(2) *For each $\epsilon > 0$ there exist a neighborhood U of $\{B_t\}_{t=1}^{\infty}$ in the space $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ and a natural number N such that for each $\{C_t\}_{t=1}^{\infty} \in U$, each $r : \{1, 2 \ldots\} \to \{1, 2, \ldots\}$ and each integer $T \geq N$,*

$$\big\|C_{r(T)} \cdot \cdots \cdot C_{r(1)} x - P_r x\big\| \leq \epsilon \quad \forall x \in K. \tag{9.3}$$

Proof. As in Section 8, given $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$, $\gamma \in (0,1)$, and an integer $i \geq 1$, we define $\{A_{t\gamma}\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ (see (8.1)), a natural number $N(\gamma, i)$ (see (8.5)) and an open neighborhood $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ of $\{A_{t\gamma}\}_{t=1}^{\infty}$ in $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ (see property (a)). Again as in Section 8 we define a set $\mathfrak{F}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{A}_{\mathrm{ue}}^{(F,0)}$ by

$$\mathfrak{F} = \cap_{q=1}^{\infty} \cup \big\{U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big) : \{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)},\ \gamma \in (0,1),\ i = q, q+1, \ldots\big\}. \tag{9.4}$$

Assume that $\{B_t\}_{t=1}^{\infty} \in \mathfrak{F}$ and $\epsilon \in (0,1)$. Choose a number ϵ_0 such that

$$\epsilon_0 < 64^{-1}\big(\min\{\epsilon, \Delta\}\big), \qquad 8\epsilon_0\Delta^{-1}\big(\mathrm{rad}(K)+1\big) < 8^{-1}\epsilon. \tag{9.5}$$

Choose a natural number q such that

$$64 \cdot 2^{-q} < \epsilon_0. \tag{9.6}$$

There exist $\{A_t\}_{t=1}^{\infty} \in \mathfrak{A}_{\mathrm{ue}}^{(F,0)}$, $\gamma \in (0,1)$, and an integer $i \geq q$ such that

$$\big\{B_t\big\}_{t=1}^{\infty} \in U\big(\{A_t\}_{t=1}^{\infty}, \gamma, i\big). \tag{9.7}$$

It was shown in Section 8 (see (8.6)) that the following property holds:

(c) for each $z \in K$, each integer $T \geq N(\gamma, i)$, and each $r : \{1, \ldots, T\} \to \{1, 2, \ldots\}$,

$$d\big(A_{r(T)\gamma} \cdot \cdots \cdot A_{r(1)\gamma}z, F\big) \leq 8^{-i-1}. \tag{9.8}$$

By the definition of $U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ (see Section 8 and property (a)) the following property holds:

(d) for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each $r : \{1, \ldots, N(\gamma, i)\} \to \{1, 2, \ldots\}$, and each $x \in K$,

$$\big\| C_{r(N(\gamma,i))} \cdot \cdots \cdot C_{r(1)}x - A_{r(N(\gamma,i))\gamma} \cdot \cdots \cdot A_{r(1)\gamma}x \big\| \leq 16^{-1} \cdot 8^{-i}. \tag{9.9}$$

Assume that $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$. Then by property (c), for each $x \in K$ there exists $f_r(x) \in K$ such that

$$\big\| A_{r(N(\gamma,i))\gamma} \cdot \cdots \cdot A_{r(1)\gamma}x - f_r(x) \big\| \leq 2 \cdot 8^{-i-1}. \tag{9.10}$$

We show that for each $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$, each integer $T \geq N(\gamma, i)$, and each $x \in K$,

$$\big\| C_{r(T)} \cdot \cdots \cdot C_{r(1)}x - f_r(x) \big\| \leq 8^{-1}\epsilon. \tag{9.11}$$

Let $\{C_t\}_{t=1}^{\infty} \in U(\{A_t\}_{t=1}^{\infty}, \gamma, i)$ and let $x \in K$. By (9.10) and property (d),

$$\big\| C_{r(N(\gamma,i))} \cdot \cdots \cdot C_{r(1)}x - f_r(x) \big\| \leq 8^{-i}\big(16^{-1} + 4^{-1}\big). \tag{9.12}$$

Set

$$z = f_r(x) + 8^{i}\Delta\big[C_{r(N(\gamma,i))} \cdot \cdots \cdot C_{r(1)}x - f_r(x)\big]. \tag{9.13}$$

It follows from (9.12), (9.13), and the definition of Δ that $z \in K$ and

$$C_{r(N(\gamma,i))} \cdot \cdots \cdot C_{r(1)}x = 8^{-i}\Delta^{-1}z + \big(1 - 8^{-i}\Delta^{-1}\big)f_r(x). \tag{9.14}$$

It follows from (9.14), (9.5), and (9.6) that for each integer $T > N(\gamma, i)$,

$$C_{r(T)} \cdot \cdots \cdot C_{r(1)}x = 8^{-i}\Delta^{-1}C_{r(T)} \cdot \cdots \cdot C_{r(N(\gamma,i)+1)}z + \big(1 - 8^{-i}\Delta^{-1}\big)f_r(x). \tag{9.15}$$

Together with (9.14) and (9.5) this implies that for each integer $T \geq N(\gamma, i)$,

$$\big\| C_{r(T)} \cdot \cdots \cdot C_{r(1)}x - f_r(x) \big\| \leq 2\,\mathrm{rad}(K)8^{-i}\Delta^{-1} < 8^{-1}\epsilon. \tag{9.16}$$

Therefore we have shown that for each $r:\{1,2,\ldots\}\to\{1,2,\ldots\}$ and each $x\in K$ there exists $f_r(x)\in F$ such that the following property holds:

(e) for each $\{C_t\}_{t=1}^{\infty}\in U(\{A_t\}_{t=1}^{\infty},\gamma,i)$, each integer $T\ge N(\gamma,i)$, and each $x\in K$ the inequality (9.11) is valid.

Since ϵ is an arbitrary positive number this implies that for each $r:\{1,2,\ldots\}\to\{1,2,\ldots\}$ there exists an operator $P_r:K\to K$ such that

$$\lim_{T\to\infty} B_{r(T)}\cdot\cdots\cdot B_{r(1)}x = P_r x,\quad x\in K. \tag{9.17}$$

Let $r:\{1,2,\ldots\}\to\{1,2,\ldots\}$. By (9.17), property (e), and (9.11),

$$\left\| P_r x - f_r(x)\right\| \le 8^{-1}\epsilon,\quad x\in K, \tag{9.18}$$

and for each $\{C_t\}_{t=1}^{\infty}\in U(\{A_t\}_{t=1}^{\infty},\gamma,i)$, each integer $T\ge N(\gamma,i)$, and each $x\in K$,

$$\left\| C_{r(T)}\cdot\cdots\cdot C_{r(1)}x - P_r(x)\right\| \le 4^{-1}\epsilon. \tag{9.19}$$

This completes the proof of Theorem 9.1. □

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