NONLOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER ABSTRACT ELLIPTIC DIFFERENTIAL EQUATION

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We establish conditions that guarantee Fredholm solvability in the Banach space L_p of nonlocal boundary value problems for elliptic abstract differential equations of the second order in an interval. Moreover, in the space L_2 we prove in addition the coercive solvability, and the completeness of root functions (eigenfunctions and associated functions). The obtained results are then applied to the study of a nonlocal boundary value problem for Laplace equations in a cylindrical domain.

1. Introduction

Fredholm property of boundary value problems is investigated in [1, 2, 3] for elliptic partial differential equations, and in [4, 13, 14, 15] for abstract differential equations.

In this paper, we establish conditions guaranteeing that nonlocal boundary value problems for elliptic partial differential equations of the second order in an interval are Fredholm solvable in the Banach spaces L_p. For the solution of the considered problem we prove the noncoercive estimates. But in the space L_2 we prove the coercive estimate for both the variable space and spectral parameter, in contrast to [15, 16] where we have a defect coerciveness for the spectral parameter. A coercive estimate, in the case when the problem is regular elliptic, was proved in [2, 3]. The considered problem is not regular, since the boundary value conditions are nonlocal and they do not belong to the same class of boundary value conditions treated in [15, 16]. Moreover, we prove the completeness of root functions. The completeness of root functions of regular boundary value problems was proved in [1, 5, 7, 10, 13]. The obtained results are then applied to the study of a nonlocal boundary value problem for Laplace equation in a cylindrical domain.

2. Necessary notations and definitions

Let H be a Hilbert space, A a linear closed operator in H and $D(A)$ its domain. We denote by $B(H)$ the space of bounded operators acting in H, with the usual operator

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norm, and by $L_p((0,1),H)$ the Banach space of strongly measurable functions $x \mapsto u(x):(0,1)\to H$, whose pth power is summable, with the norm

$$\|u\|_{0,p}^p = \|u\|_{L_p((0,1),H)}^p = \int_0^1 \|u(x)\|_H^p\,dx < \infty,\ p\in(1,\infty). \tag{2.1}$$

Now, introduce the $L_p((0,1),H)$ vector-valued Sobolev spaces

$$\begin{gathered}\mathbf{W}_p^2\big((0,1),H(A),H\big) = \big\{u : u'' \in L_p\big((0,1),H\big) \text{ and } Au \in L_p\big((0,1),H\big)\big\},\\ \|u\|_{W_p^2((0,1),H(A),H)} = \|Au\|_{L_p((0,1),H)} + \|u''\|_{L_p((0,1),H)} < \infty.\end{gathered} \tag{2.2}$$

We also set

$$H(A) = \big\{u \in D(A);\ \|u\|_{H(A)}^2 = \|u\|_H^2 + \|Au\|_H^2 < \infty\big\}, \tag{2.3}$$

that is, $H(A)$ is the domain of A with a Hilbert graph norm.

Let $-A$ be the generator of the semigroup $\exp(-xA)$ analytic for $x>0$, decreasing at infinity, and strongly continuous for $x \ge 0$. We define the interpolation space [12, page 96]

$$\begin{aligned}&\big(H,H\big(A^n\big)\big)_{\theta,p}\\ &\quad=\left\{u : u\in H,\ \|u\|_{\theta,p}^p = \int_0^1 t^{-n(1-\theta)p-1}\|A^n\exp(-tA)u\|^p dt + \|u\|^p < \infty\right\},\end{aligned} \tag{2.4}$$

$0<\theta<1$; $n\in\mathbb{N}$, $1\le p<\infty$ and $\|\cdot\|_{\theta,p}$ its norm.

Let H and H_1 be Hilbert spaces such that the continuous embedding $H_1 \subset H$ is fulfilled and $\overline{H_1} = H$. Then, $(H,H_1)_{\theta,2}$ is a Hilbert space [12, page 142]. Denote $(H,H_1)_\theta = (H,H_1)_{\theta,2}$. It is known that $(H,H_1)_\theta = H(S^\theta)$, where S is a selfadjoint positive-definite operator in H [11, Chapter 1, Section 2.1].

Let $Ff = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} e^{i\sigma x} f(x)\,dx$ be the Fourier transform.

Definition 2.1. The mapping $\sigma \to T(\sigma):\mathbb{R}\to B(H)$ is said to be a Fourier multiplier of the type (p,q) if for all $f\in L_p(\mathbb{R},H)$ we have

$$\big\|F^{-1}TFf\big\|_{L_q(\mathbb{R},H)} \le c\|f\|_{L_p(\mathbb{R},H)} \quad \text{for } f\in L_p(\mathbb{R},H). \tag{2.5}$$

We get the following characterization for Fourier multipliers.

Theorem 2.2 (Mikhlin-Schwartz [6, page 1181]). *If the mapping $T:\mathbb{R}\to B(H):\sigma\mapsto T(\sigma)$ is continuously differentiable and the inequality*

$$\|T(\sigma)\| \le C, \qquad \left\|\frac{\partial T(\sigma)}{\partial\sigma}\right\| \le \frac{C}{|\sigma|}, \tag{2.6}$$

holds for all $\sigma\in\mathbb{R}$, $\sigma\ne 0$, then $T(\sigma)$ is a Fourier multiplier of type (p,p).

LEMMA 2.3 (see [16, page 300]). *Let A be a selfadjoint and positive-definite operator in H. Then*

(1) $\exists\omega > 0$, $\|A^{\alpha}\exp[-x(A+\lambda I)^{1/2}]\| \le C\exp(-\omega x|\lambda|^{1/2})$ *for all* $\alpha \in \mathbb{R}$, $x \ge x_0 > 0$, $|\arg\lambda| \le \varphi < \pi$, *where C does not depend on x and λ;*

(2) $\int_0^1 \|(A+\lambda I)^{\alpha}\exp[-x(A+\lambda I)^{1/2}]u\|^2\,dx \le C(\|A^{\alpha-1/4}u\|^2 + |\lambda|^{2\alpha-1/2}\|u\|^2)$ *for all* $\alpha \ge 1/4$, $|\arg\lambda| \le \varphi < \pi$, $u \in D(A^{\alpha-1/4})$, *where C does not depend on u and λ;*

(3) $\|A^{\alpha}(A+\lambda I)^{-\beta}\| \le C(1+|\lambda|)^{\alpha-\beta}$ *for all* $0 \le \alpha \le \beta$, $|\arg\lambda| \le \varphi < \pi$, *where C does not depend on λ.*

3. Solvability of the principal problem

Consider in the Hilbert space H a boundary value problem in $[0,1]$ for the second order elliptic differential-operator equation

$$L(\lambda, D)u = -u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u(x) + \lambda u(x) = f(x), \tag{3.1}$$

$$L_k u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j \le n_{k1}} T_{k1}u'(x_{k1j}) + \sum_{j \le n_{k0}} T_{k0}u(x_{k0j}) = f_k, \tag{3.2}$$

$k = \overline{1,2}$, where x, $x_{kij} \in [0,1]$; $m_k \in \{0,1\}$; α_k, β_k are complex numbers; A, $B_1(x)$, $B_2(x)$, T_{ki} are, generally speaking, unbounded operators in H; $D = d/dx$.

First, consider the principal part of problem (3.1), (3.2)

$$L_0(\lambda, D)u = -u''(x) + (A+\lambda I)u(x) = 0, \tag{3.3}$$

$$L_{k0}u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = \overline{1,2}. \tag{3.4}$$

THEOREM 3.1. *Let the following conditions be satisfied:*

(1) *A is a selfadjoint and positive-definite operator in H;*

(2) $\theta = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \ne 0$.

Then, problem (3.3), (3.4) for $f_k \in (H(A),H)_{\theta_k,p}$, where $\theta_k = m_k/2 + 1/2p$, $p \in (1,\infty)$, and $|\lambda|$ sufficiently large such that $|\arg\lambda| \le \varphi < \pi$ has a unique solution that belongs to the space $W_p^2((0,1),H(A),H)$, in addition for this solution we have the noncoercive estimate

$$\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} \le C(\lambda)\sum_{k=1}^{2}\|f_k\|_{(H(A),H)_{\theta_k,p}}, \tag{3.5}$$

where $C(\lambda)$ does not depend on u.

Proof. We prove that any solution to (3.3), belonging to $\mathbf{W}_p^2((0,1),H(A),H)$, has the form

$$u(x) = \exp\Big[-x(A+\lambda I)^{1/2}\Big]g_1 + \exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]g_2, \tag{3.6}$$

where $g_k \in (H(A),H)_{1/2p,p}$. To show this, let $u \in \mathbf{W}_p^2((0,1),H(A),H)$ be a solution to (3.3). Then, from (3.3) we have

$$\Big[D-(A+\lambda I)^{1/2}\Big]\Big[D+(A+\lambda I)^{1/2}\Big]u(x)=0. \tag{3.7}$$

Denote

$$v(x)=\Big[D+(A+\lambda I)^{1/2}\Big]u(x). \tag{3.8}$$

Then, by virtue of Theorem 1.7 [13, page 168], $v \in \mathbf{W}_p^1((0,1),H(A^{1/2}),H)$, and

$$\Big[D-(A+\lambda I)^{1/2}\Big]v(x)=0. \tag{3.9}$$

Hence,

$$v(x)=\exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]v(1), \tag{3.10}$$

where, in view of [12, page 44], $v(1) \in (H(A^{1/2}),H)_{1/p,p}$. From (3.8) and (3.10), we have

$$\begin{aligned}
u(x)&=\exp\Big[-x(A+\lambda I)^{1/2}\Big]u(0)\\
&\quad+\int_0^x \exp\Big[-(x-y)(A+\lambda I)^{1/2}\Big]\exp\Big[-(1-y)(A+\lambda I)^{1/2}\Big]v(1)dy\\
&=\exp\Big[-x(A+\lambda I)^{1/2}\Big]u(0)\\
&\quad+\frac{1}{2}(A+\lambda I)^{-1/2}\Big\{\exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]\\
&\qquad-\exp\Big[-x(A+\lambda I)^{1/2}\Big]\exp\Big[-(A+\lambda I)^{1/2}\Big]v(1)\Big\},
\end{aligned} \tag{3.11}$$

where, by virtue of [12, page 44], $u(0) \in (H(A),H)_{1/2p,p}$. In view of [12, page 101], the operator $A^{1/2}$ is an isomorphism from $(H(A),H)_{1/2p,p}=(H,H(A))_{1-1/2p,p}$ onto $(H,H(A))_{(p-1)/2p,p}=(H,H(A^{1/2}))_{1-1/p,p}=(H(A^{1/2}),H)_{1/p,p}$. Thus, (3.11) has the desired form (3.6).

Let us now prove the converse, that is, the function $u(x)$ of the form (3.6), where $g_k \in (H(A),H)_{1/2p,p}$, belongs to the space $\mathbf{W}_p^2((0,1),H(A),H)$. Using the properties of the interpolation spaces [12, page 96], and from (3.6) we have

$$\begin{aligned}
&\|u\|_{W_p^2((0,1),H(A),H)}\\
&\le\Big(\big\|A(A+\lambda I)^{-1}\big\|+1\Big)\Bigg\{\bigg(\int_0^1\Big\|(A+\lambda I)\exp\Big[-x(A+\lambda I)^{1/2}\Big]g_1\Big\|^p dx\bigg)^{1/p}\\
&\qquad+\bigg(\int_0^1\Big\|(A+\lambda I)\exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]g_2\Big\|^p dx\bigg)^{1/p}\Bigg\}\\
&\le C\Big(\|g_1\|_{(H(A+\lambda),H)_{1/2p,p}}+\|g_2\|_{(H(A+\lambda I),H)_{1/2p,p}}\Big)\\
&\le C(\lambda)\Big(\|g_1\|_{(H(A),H)_{1/2p,p}}+\|g_2\|_{(H(A),H)_{1/2p,p}}\Big).
\end{aligned} \tag{3.12}$$

A function $u(x)$ of the form (3.6) satisfies the boundary condition (3.4) if

$$(-1)^{m_k}\Big\{\alpha_k+\beta_k\exp\Big[-(A+\lambda I)^{1/2}\Big]\Big\}(A+\lambda I)^{m_k/2}v_1 \\ +\Big\{\alpha_k\exp\Big[-(A+\lambda I)^{1/2}\Big]+\beta_k\Big\}(A+\lambda I)^{m_k/2}v_2=f_k,\quad k=\overline{1,2}. \tag{3.13}$$

Denote

$$v_1=(A+\lambda I)^{m/2}g_1,\qquad v_2=(A+\lambda I)^{m/2}g_2, \tag{3.14}$$

where $m=\max\{m_1,m_2\}$. Then, (3.13) gives

$$(-1)^{m_k}\Big\{\alpha_k+\beta_k\exp\Big[-(A+\lambda I)^{1/2}\Big]\Big\}(A+\lambda I)^{m_k/2-m/2}v_1 \\ +\Big\{\alpha_k\exp\Big[-(A+\lambda I)^{1/2}\Big]+\beta_k\Big\}(A+\lambda I)^{m_k/2-m/2}v_2=f_k,\quad k=\overline{1,2}. \tag{3.15}$$

All coefficients in system (3.15) are linear combinations of the bounded operators I, $(A+\lambda I)^{-1}$, $\exp[-(A+\lambda I)^{1/2}]$, and $(A+\lambda I)^{-1}\exp[-(A+\lambda I)^{1/2}]$ which commute with one another. Therefore system (3.15) can be solved as in the scalar case. By virtue of Lemma 2.3, the determinant of the system (3.15) has the form

$$D(\lambda)=\theta(A+\lambda I)^{m_1/2+m_2/2-m}[I+R(\lambda)], \tag{3.16}$$

where $R(\lambda)=C_1\exp[-(A+\lambda I)^{1/2}]+C_2\exp[-2(A+\lambda I)^{1/2}]$, then by virtue of Lemma 2.3 $\|R(\lambda)\|\to 0$, for $|\arg\lambda|\le\varphi<\pi$ and $|\lambda|\to\infty$. Then the second condition in our hypothesis implies that system (3.15) has a unique solution for $|\arg\lambda|\le\varphi<\pi$ and $|\lambda|$ is sufficiently large, and the solution can be expressed in the form

$$v_k=\Big[C_{1k}(A+\lambda I)^{m/2-m_1/2}+R_{1k}(\lambda)\Big]f_1 \\ +\Big[C_{2k}(A+\lambda I)^{m/2-m2/2}+R_{2k}(\lambda)\Big]f_2,\quad k=\overline{1,2}, \tag{3.17}$$

where C_{jk} are complex numbers and $\|R_{jk}(\lambda)\|\to 0$, $|\lambda|\to\infty$. Consequently,

$$g_k=\Big[C_{1k}(A+\lambda I)^{-m_1/2}+(A+\lambda I)^{-m/2}R_{1k}(\lambda)\Big]f_1 \\ +\Big[C_{2k}(A+\lambda I)^{-m2/2}+(A+\lambda I)^{-m/2}R_{2k}(\lambda)\Big]f_2,\quad k=\overline{1,2}. \tag{3.18}$$

Since $f_k\in(H(A),H)_{m_k/2+1/2p,p}$ and the operator $(A+\lambda I)^{m_k/2}$ is an isomorphism from the space $(H(A),H)_{1/2p,p}=(H,H(A))_{1-1/2p,p}$ onto the space $(H,H(A))_{1-m_k/2-1/2p,p}=(H(A),H)_{m_k/2+1/2p,p}$. We have $g_k\in(H(A),H)_{1/2p,p}$. Hence from (3.18) we have the estimate

$$\|g_k\|_{(H(A),H)_{1/2p,p}}\le C(\lambda)\sum_{k=1}^{n}\|f_k\|_{(H(A),H)_{\theta_k,p}}. \tag{3.19}$$

Substituting (3.19) in (3.12), we obtain the noncoercive estimate (3.5). □

Consider now the principal part of problem (3.1), (3.2) for a nonhomogeneous equation and with a parameter

$$L_0(\lambda, D)u = -u''(x) + (A + \lambda I)u(x) = f(x), \tag{3.20}$$

$$L_{k0}u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = \overline{1,2}. \tag{3.21}$$

THEOREM 3.2. *Let the conditions of Theorem 3.1 be satisfied. Then, the operator $u \mapsto (L_0(\lambda, D)u, L_{10}u, L_{20}u)$ for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda|$ sufficiently large is an isomorphism from the space $\mathbf{W}_p^2((0,1), H(A), H)$ onto the space $L_p((0,1), H) \oplus (H(A), H)_{\theta_1,p} \oplus (H(A), H)_{\theta_2,p}$, where $\theta_k = m_k/2 + 1/2p$, $k = 1, 2$, $p \in (1, \infty)$, and for this solution we have the noncoercive estimate*

$$\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} \le C(\lambda)\left(\|f\|_{L_p((0,1),H)} + \sum_{k=1}^{2}\|f_k\|_{(H(A),H)_{\theta_k,p}}\right), \tag{3.22}$$

where $C(\lambda)$ does not depend on u.

Proof. By Theorem 3.1, we get the unicity. Now, let us define $\tilde f(x) = f(x)$ if $x \in [0,1]$ and $\tilde f(x) = 0$ if $x \notin [0,1]$. We now show that a solution to problem (3.20), (3.21) belonging to $\mathbf{W}_p^2((0,1), H(A), H)$ can be represented as a sum of the form $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the restriction on $[0,1]$ of the solution $\tilde u_1(x)$ to the equation

$$L_0(\lambda, D)\tilde u_1 = \tilde f(x), \quad x \in \mathbb{R}, \tag{3.23}$$

and $u_2(x)$ is a solution to the problem

$$L_0(\lambda, D)u_2 = 0, \qquad L_{k0}u_2 = f_k - L_{k0}u_1, \quad k = \overline{1,2}. \tag{3.24}$$

The solution to (3.23) is given by the formula

$$\tilde u_1(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\mu x} L_0(\lambda, i\mu)^{-1} F\tilde f(\mu)\, d\mu, \tag{3.25}$$

where $F\tilde f$ is the Fourier transform of the function $\tilde f(x)$, $L_0(\lambda, S)$ is a characteristic operator pencil of (3.23), that is,

$$L_0(\lambda, S) = -S^2 I + A + \lambda I. \tag{3.26}$$

From (3.25), it follows that

$$\begin{aligned}
\|\tilde u_1\|_{\mathbf{W}_p^2(\mathbb{R}, H(A), H)} &= \|\tilde u_1\|_{L_p(\mathbb{R}, H(A))} + \|\tilde u_1''\|_{L_p(\mathbb{R}, H)} \\
&\le \left\|F^{-1}L_0(\lambda, i\mu)^{-1}F\tilde f(\mu)\right\|_{L_p(\mathbb{R}, H(A))} \\
&\quad + \left\|F^{-1}(i\mu)^2 L_0(\lambda, i\mu)^{-1}F\tilde f(\mu)\right\|_{L_p(\mathbb{R}, H)} \\
&\le \left\|F^{-1}AL_0(\lambda, i\mu)^{-1}F\tilde f(\mu)\right\|_{L_p(\mathbb{R}, H)} \\
&\quad + \left\|F^{-1}(i\mu)^2 L_0(\lambda, i\mu)^{-1}F\tilde f(\mu)\right\|_{L_p(\mathbb{R}, H)}.
\end{aligned} \tag{3.27}$$

Let us show that the functions

$$T_{k+1}(\lambda,\mu)=(i\mu)^{2k}A^{1-k}L_0(\lambda,i\mu)^{-1},\quad k=\overline{0,1}, \tag{3.28}$$

are Fourier multipliers in the space $L_p(\mathbb{R},H)$. By virtue of Lemma 2.3, for $|\arg\lambda|\leq\varphi<\pi$, $|\lambda|$ sufficiently large and $\mu\in\mathbb{R}$ we have

$$\left\|L_0(\lambda,i\mu)^{-1}\right\|=\left\|\left(A+\lambda I+\mu^2 I\right)^{-1}\right\|\leq C\left(1+\left|\lambda+\mu^2\right|\right)^{-1}\leq C\left|\mu\right|^{-2}, \tag{3.29}$$

$$\left\|AL_0(\lambda,i\mu)^{-1}\right\|=\left\|A\left(A+\lambda I+\mu^2 I\right)^{-1}\right\|\leq C. \tag{3.30}$$

From (3.29) and (3.30), we get

$$\left\|T_1(\lambda,\mu)\right\|_{B(H)}\leq C\left\|AL_0(\lambda,i\mu)^{-1}\right\|_{B(H)}\leq C, \tag{3.31}$$

$$\left\|T_2(\lambda,\mu)\right\|_{B(H)}\leq C|\mu|^2\left\|L_0(\lambda,i\mu)^{-1}\right\|_{B(H)}\leq C. \tag{3.32}$$

Since

$$\begin{aligned}\frac{\partial}{\partial\mu}T_{k+1}(\lambda,\mu)&=2ki^{2k}\mu^{2k-1}A^{1-k}L_0(\lambda,i\mu)^{-1}\\&\quad-i^{2k+1}\mu^{2k}A^{1-k}L_0(\lambda,i\mu)^{-1}\frac{\partial}{\partial\mu}L_0(\lambda,i\mu)L_0(\lambda,i\mu)^{-1},\end{aligned} \tag{3.33}$$

then,

$$\left\|\frac{\partial}{\partial\mu}T_{k+1}(\lambda,\mu)\right\|\leq|\mu|^{-1}. \tag{3.34}$$

Applying the Michlin-Schwartz Theorem 2.2, it follows from (3.31), (3.32), and (3.34) that the functions $\mu\to T_{k+1}(\lambda,\mu)$ are Fourier multipliers in the space $L_p(\mathbb{R},H)$. Then, using (3.27), we obtain

$$\left\|\tilde{u}_1\right\|_{\mathbf{W}_p^2(\mathbb{R},H(A),H)}\leq C\left\|\tilde{f}_1\right\|_{L_p(\mathbb{R},H)}. \tag{3.35}$$

So, $u_1\in\mathbf{W}_p^2((0,1),H(A),H)$. By virtue of [12, page 44] and inequality (3.35), we have $u_1^{m_k}(0)\in(H(A),H)_{m_k/2+1/2p,p}$. Hence, $L_{k0}u_1\in(H(A),H)_{\theta_k,p}$. Then by virtue of Theorem 3.1, problem (3.20), (3.21) has a unique solution $u_2(x)$ that belongs to $\mathbf{W}_p^2((0,1),H(A),H)$. And, again, by Theorem 3.1 and estimate (3.35), we obtain the inequality (3.22). □

4. Fredholm solvability of general problem

Consider problem (3.1), (3.2). Now we can find conditions for the Fredholm solvability of problem (3.1), (3.2). It is convenient to formulate the theorem in terms of the Fredholmness of some unbounded operator which acts from one Banach space into another.

Let us set the operator $\mathbb{L}$ from $\mathbf{W}_p^2((0,1),H(A),H)$ into $L_p((0,1),H)\oplus(H(A),H)_{\theta_1,p}\oplus(H(A),H)_{\theta_2,p}$, by the equalities

$$
\begin{gathered}
D(\mathbb{L})=\{u/u\in\mathbf{W}_p^2\big((0,1),H(A),H\big),L(D)u\in L_p\big((0,1),H\big),\\
L_ku\in\big(H(A),H\big)_{\theta_1,p},k=1,2\},\\
\mathbb{L}u=\big(L(D)u,L_1u,L_2u\big),
\end{gathered}
\tag{4.1}
$$

where

$$
L(D)u=-u''(x)+Au(x)+B_1(x)u'(x)+B_2(x)u(x),
\tag{4.2}
$$

L_1 and L_2 have been defined by equalities (3.2).

THEOREM 4.1. *Suppose that in addition to conditions (1), (2) stated in Theorem 3.1, the following conditions are also satisfied:*

(1) *the embedding $H(A)\subset H$ is compact;*
(2) *linear operators $B_k(x)$ from $H(A^{k/2})$ into H act compactly for almost all $x\in[0,1]$; for any $\varepsilon>0$ and for almost all $x\in[0,1]$,*

$$
\|B_k(x)u\|\le\varepsilon\|A^{k/2}u\|+c(\varepsilon)\|u\|,\quad u\in D\big(A^{k/2}\big);
\tag{4.3}
$$

for each $u\in D(A^{k/2})$, the function $B_k(x)u$ is measurable on $[0,1]$ in H;
(3) *the linear operators T_{ki} from $(H(A),H)_{(iP+1)/2p,p}$ into $(H(A),H)_{\theta_k,p}$ are compact, where $\theta_k=m_k/2+1/2p$, $p\in(1,\infty)$.*

Then,

(1) *for any function $u(x)\in\mathbf{W}_p^2((0,1),H(A),H)$, we have the noncoercive estimate*

$$
\begin{aligned}
&\|u''\|_{L_p((0,1),H)}+\|Au\|_{L_p((0,1),H)}\\
&\quad\le C(\lambda)\left(\|L(D)u\|_{L_p((0,1),H)}+\sum_{k=1}^{2}\|L_ku\|_{(H(A),H)_{\theta_k,p}}+\|u\|_{L_p((0,1),H)}\right),
\end{aligned}
\tag{4.4}
$$

where $C(\lambda)$ does not depend on u;
(2) *the operator $\mathbb{L}:u\mapsto(L(D)u,L_1u,L_2u)$, from $\mathbf{W}_p^2((0,1),H(A),H)$ into $L_p((0,1),H)\oplus(H(A),H)_{\theta_1,p}\oplus(H(A),H)_{\theta_2,p}$, is Fredholm.*

Proof. (1) Let $u(x)$ be a solution to problem $L(D)u=f$, $L_ku=f_k$, $k=1,2$, belonging to $\mathbf{W}_p^2((0,1),H(A),H)$. Then, $u(x)$ is a solution to the problem

$$
\begin{gathered}
L_0(\lambda,D)u=f(x)+\lambda u(x)-B_1(x)u'(x)-B_2(x)u(x),\\
L_{k0}u=f_k-\sum_{j\le n_{k1}}T_{k1}u'\big(x_{k1j}\big)-\sum_{j\le n_{k0}}T_{k0}u\big(x_{k0j}\big);\quad k=1,2,
\end{gathered}
\tag{4.5}
$$

where $L_0(\lambda, D)$ and L_{k0} have been defined by (3.20) and (3.21). By virtue of Theorem 3.2, for some λ_0, we have the estimate

$$\begin{aligned}
&\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)} \\
&\quad \leq C(\lambda_0)\Bigg(\|f(\cdot)+\lambda_0 u(\cdot) - B_1(\cdot)u'(\cdot) - B_2(\cdot)u(\cdot)\|_{L_p((0,1),H)} \\
&\qquad + \sum_{k=1}^{2}\Big\| f_k - \sum_{j\leq n_{k1}} T_{k1}u'(x_{k1j}) - \sum_{j\leq n_{k0}} T_{k0}u(x_{k0j})\Big\|_{(H(A),H)_{\theta_k,p}}\Bigg).
\end{aligned} \tag{4.6}$$

By virtue of Theorem 5.1.7 [13, page 168], the operator $u \mapsto u^{(i)}(x)$ from the space $\mathbf{W}_p^2((0,1),H(A),H)$ into the space $L_p((0,1),H(A^{1-i/2}))$ is bounded. Then, by condition (2) of Theorem 4.1 and Lemma 5.1.2 [13, page 162], the operator

$$u \longmapsto \lambda_0 u - B_1(x)u'(x) - B_2(x)u_2(x) \tag{4.7}$$

from $\mathbf{W}_p^2((0,1),H(A),H)$ into the space $L_p((0,1),H)$ is compact. Consequently, by Lemma 2.2.7 [13, page 53], for any $\varepsilon > 0$ we have

$$\begin{aligned}
&\|f(\cdot)+\lambda_0 u(\cdot) - B_1(\cdot)u'(\cdot) - B_2(\cdot)u(\cdot)\|_{L_p((0,1),H)} \\
&\quad \leq C(\lambda_0)\Big[\|f\|_{L_p((0,1),H)} + \varepsilon\Big(\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)}\Big) \\
&\qquad + C(\varepsilon)\|u\|_{L_p((0,1),H)}\Big].
\end{aligned} \tag{4.8}$$

By virtue of [12, page 44], we have the operator $u \mapsto u^{(i)}(x_0)$ from $\mathbf{W}_p^2((0,1), H(A),H)$ into $(H(A),H)_{(iP+1)/2p,p}$ is bounded. Then, by virtue of condition (3), the operator $u \mapsto \sum_{j\leq n_{k1}} T_{k1}u'(x_{k1j}) + \sum_{j\leq n_{k0}} T_{k0}u(x_{k0j})$ from $\mathbf{W}_p^2((0,1),H(A),H)$ into $(H(A),H)_{\theta_k,p}$ is compact. Consequently, by [13], for any $\varepsilon > 0$ we have

$$\begin{aligned}
&\sum_{k=1}^{2}\Big\| f_k - \sum_{j\leq n_{k1}} T_{k1}u'(x_{k1j}) - \sum_{j\leq n_{k0}} T_{k0}u(x_{k0j})\Big\|_{(H(A),H)_{\theta_k,p}} \\
&\quad \leq C(\lambda_0)\Bigg[\sum_{k=1}^{2}\|f_k\|_{(H(A),H)_{\theta_k,p}} + C(\varepsilon)\|u\|_{L_p((0,1),H)} \\
&\qquad + \varepsilon\Big(\|u''\|_{L_p((0,1),H)} + \|Au\|_{L_p((0,1),H)}\Big)\Bigg].
\end{aligned} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.6) we have (4.4).

(2) The operator $\mathbb{L}$ can be rewritten in the form $\mathbb{L} = \mathbb{L}_{0\lambda} + \mathbb{L}_{1\lambda}$, where

$$\mathbb{L}_{0\lambda}u = \big(L_0(\lambda, D)u, L_{10}u, L_{20}u\big), \tag{4.10}$$

$L_0(\lambda, D)u, L_{10}u, L_{20}u$ have been defined by equalities (3.20) and (3.21) and

$$\mathbb{L}_{1\lambda} = \big(-\lambda u(x) + B_1(x)u'(x) + B_2(x)u(x), T_1u, T_2u\big), \tag{4.11}$$

where

$$T_k u = \sum_{j \le n_{k1}} T_{k1} u'\big(x_{k1j}\big) + \sum_{j \le n_{k0}} T_{k0} u\big(x_{k0j}\big), \quad k = 1, 2. \tag{4.12}$$

Conditions (2) and (3) imply that $D(\mathbb{L}) = D(\mathbb{L}_{0\lambda})$. We can conclude, from Theorem 3.2, that the operator $\mathbb{L}_{0\lambda}$ from $\mathbf{W}_p^2((0,1), H(A), H)$ into $L_p((0,1), H) \oplus (H(A), H)_{\theta_1,p} \oplus (H(A), H)_{\theta_2,p}$ has an inverse for λ is sufficiently large. It follows from the proof of part (1) that the operator $\mathbb{L}_{1\lambda}$ from $\mathbf{W}_p^2((0,1), H(A), H)$ into $L_p((0,1), H) \oplus (H(A), H)_{\theta_1,p} \oplus (H(A), H)_{\theta_2,p}$ is compact. By applying the perturbation theorem of Fredholm operators [9, page 238] to the operator $\mathbb{L}$, we end the proof. □

5. Coercive solvability in $L_2((0,1), H)$

Consider a particular case of problem (3.20), (3.21) in $L_2((0,1), H)$

$$L(\lambda, D)u = -u''(x) + Au(x) + \lambda u(x) = f(x), \tag{5.1}$$

$$L_k u = \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \tag{5.2}$$

THEOREM 5.1. *Let the following conditions be satisfied:*

(1) *A is a selfadjoint and positive-definite operator in a Hilbert space H;*
(2) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$.

Then, problem (5.1), (5.2) for $f \in L_2((0,1), H)$, $f_k \in D(A^{3/4+m_k/2})$ for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda|$ is sufficiently large has a unique solution that belongs to the space $\mathbf{W}_p^2((0,1), H(A), H)$ and for this solution we have the coercive estimate

$$\begin{aligned} &|\lambda| \|u\|_{L_2((0,1),H)} + \left\|u''\right\|_{L_2((0,1),H)} + \|Au\|_{L_2((0,1),H)} \\ &\quad \le C\left(\|f\|_{L_2((0,1),H)} + \sum_{k=1}^{2}\left(\left\|A^{-m_k/2+3/4} f_k\right\| + |\lambda|^{-m_k/2+3/4}\left\|f_k\right\|\right)\right), \end{aligned} \tag{5.3}$$

where C does not depend on u, f, f_k, and λ.

Proof. We seek a solution in the form $u = u_1 + u_2$, where u_1 is the restriction on $[0,1]$ of the solution $\hat{u}_1(x)$ to (3.23) and u_2 is a solution of problem (3.24). From Theorem 3.1, we have

$$\begin{aligned} u_2(x) = &\Big\{\Big[C_{11}(A+\lambda I)^{-m_1/2} + (A+\lambda I)^{-m/2} R_{11}(\lambda)\Big] \exp\Big[-x(A+\lambda I)^{1/2}\Big] \\ &+ \Big[C_{12}(A+\lambda I)^{-m_1/2} + (A+\lambda I)^{-m/2} R_{12}(\lambda)\Big] \exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]\Big\} f_1 \\ &+ \Big\{\Big[C_{21}(A+\lambda I)^{-m_2/2} + (A+\lambda I)^{-m/2} R_{21}(\lambda)\Big] \exp\Big[-x(A+\lambda I)^{1/2}\Big] \\ &+ \Big[C_{22}(A+\lambda I)^{-m2/2} + (A+\lambda I)^{-m/2} R_{22}(\lambda)\Big] \exp\Big[-(1-x)(A+\lambda I)^{1/2}\Big]\Big\} f_2. \end{aligned} \tag{5.4}$$

Then, for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda|$ is sufficiently large, we have

$$\begin{aligned}|\lambda|\|u_2\|_{L_2((0,1),H)} &+ \|u_2''\|_{L_2((0,1),H)} + \|Au_2\|_{L_2((0,1),H)} \\ &\le C\left(\|f\|_{L_2((0,1),H)} + \sum_{k=1}^{2}\left\|A^{-m_k/2+3/4}f_k\right\| + |\lambda|^{-m_k/2+3/4}\|f_k\|\right).\end{aligned} \tag{5.5}$$

From (3.25) and the Plancherel equality, we have

$$\begin{aligned}|\lambda|\|u_1\|_{L_2((0,1),H)} &+ \|u_1''\|_{L_2((0,1),H)} + \|Au_1\|_{L_2((0,1),H)} \\ &\le |\lambda|\|\hat{u}_1\|_{L_2(\mathbb{R},H)} + \|\hat{u}_1''\|_{L_2(\mathbb{R},H)} + \|A\hat{u}_1\|_{L_2(\mathbb{R},H)} \\ &= |\lambda|\left\|L_0(\lambda,i\mu)^{-1}\left(F\tilde{f}\right)(\mu)\right\|_{L_2(\mathbb{R},H)} + \left\|(i\mu)^2L_0(\lambda,i\mu)^{-1}\left(F\tilde{f}\right)(\mu)\right\|_{L_2(\mathbb{R},H)} \\ &\quad + \left\|AL_0(\lambda,i\mu)^{-1}\left(F\tilde{f}\right)(\mu)\right\|_{L_2(\mathbb{R},H)}.\end{aligned} \tag{5.6}$$

From condition (1) of Theorem 5.1, for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda|$ is sufficiently large, we have

$$\begin{gathered}|\lambda|\left\|L_0(\lambda,i\mu)^{-1}\right\| = |\lambda|\left\|(A+\lambda I+\mu^2I)^{-1}\right\| \le c|\lambda|\left(1+\left|\lambda+\mu^2\right|\right)^{-1} \le C, \\ |\mu|^2\left\|L_0(\lambda,i\mu)^{-1}\right\| = |\mu|^2\left\|\left(A+\lambda I+\mu^2I\right)^{-1}\right\| \le c|\mu|^2\left(1+\left|\lambda+\mu^2\right|\right)^{-1} \le C, \\ \left\|AL_0(\lambda,i\mu)^{-1}\right\| = \left\|A\left(A+\lambda I+\mu^2I\right)^{-1}\right\| \le C.\end{gathered} \tag{5.7}$$

Then, from (5.6), it follows that

$$|\lambda|\|u_1\|_{L_2((0,1),H)} + \|u_1''\|_{L_2((0,1),H)} + \|Au_1\|_{L_2((0,1),H)} \le c\|f\|_{L_2((0,1),H)}. \tag{5.8}$$

From [12, page 44], we have

$$\left\|A^{-m_k/2+3/4}u_1^{(m_k)}(0)\right\| \le C\|u_1\|_{\mathbf{W}_p^2((0,1),H(A),H)} \le C\|f\|_{L_2((0,1),H)}, \tag{5.9}$$

we also use the inequality [11, Chapter 1, Section 3.2]

$$\left\|u^{(j)}(0)\right\|_H \le C\left(h^{1-\chi}\|u\|_{\mathbf{W}_p^2((0,1),H(A),H)} + h^{-\chi}\|u\|_{L_2((0,1),H)}\right), \tag{5.10}$$

where $0 \le j \le 1$, $0 < h < h_0$, $\chi = j+(1/2)/2$. Then

$$\begin{aligned}|\lambda|^{-m_k/2+3/4}\left\|u^{(m_k)}(0)\right\| \le C|\lambda|^{-m_k/2+3/4}\Big(&h^{1-(m_k/2+1/4)}\|u\|_{\mathbf{W}_p^2((0,1),H(A),H)} \\ &+ h^{-m_k/2-1/4}\|u\|_{L_2((0,1),H)}\Big),\end{aligned} \tag{5.11}$$

by taking $h = |\lambda|^{-1}$, and so from (5.8) and (5.9), we have

$$\begin{aligned}|\lambda|^{-m_k/2+3/4}\left\|u^{(m_k)}(0)\right\| &\le C\left(\|u\|_{\mathbf{W}_p^2((0,1),H(A),H)} + |\lambda|\|u\|_{L_2((0,1),H)}\right) \\ &\le \|f\|_{L_2((0,1),H)}.\end{aligned} \tag{5.12}$$

From (5.6), (5.9), and (5.12) we get

$$|\lambda|\left\|u_2\right\|_{L_2((0,1),H)}+\left\|u_2''\right\|_{L_2((0,1),H)}+\left\|Au_2\right\|_{L_2((0,1),H)} \\ \leq C\left(\sum_{k=1}^{2}\left(\left\|A^{-m_k/2+3/4}f_k\right\|+|\lambda|^{-m_k/2+3/4}\|f_k\|\right)\right). \tag{5.13}$$

Hence, (5.3) follows from (5.8) and (5.13). □

Consider in $L_2((0,1),H)$, the following problem:

$$L(\lambda,D)u=\lambda u(x)-u''(x)+Au(x)+B(x)u=f(x), \tag{5.14}$$

$$L_k u=\alpha_k u^{(m_k)}(0)+\beta_k u^{(m_k)}(1)=f_k, \quad k=1,2. \tag{5.15}$$

THEOREM 5.2. *Let the following conditions be satisfied:*

(1) *A is a selfadjoint and positive-definite operator in a Hilbert space H;*
(2) *the embedding $H(A)\subset H$ is compact;*
(3) $(-1)^{m_1}\alpha_1\beta_2-(-1)^{m_2}\alpha_2\beta_1\neq 0$*;*
(4) $\|Bu\|_{L_2((0,1),H)}\leq\varepsilon\|Au\|_{L_2((0,1),H)}+C(\varepsilon)\|u\|_{L_2((0,1),H)}$.

Then, problem (5.14), (5.15) for $f\in L_2((0,1),H)$, $f_k\in D(A^{m_k/2+3/4})$ for $|\arg\lambda|\leq\varphi<\pi$ and $|\lambda|$ is sufficiently large has a unique solution that belongs to the space $\mathbf{W}_p^2((0,1),H(A),H)$ and for this solution we have the coercive estimate

$$|\lambda|\|u\|_{L_2((0,1),H)}+\left\|u''\right\|_{L_2((0,1),H)}+\|Au\|_{L_2((0,1),H)} \\ \leq C\left(\|f\|_{L_2((0,1),H)}+\sum_{k=1}^{2}\left(\left\|A^{-m_k/2+3/4}f_k\right\|+|\lambda|^{-m_k/2+3/4}\|f_k\|\right)\right), \tag{5.16}$$

where C does not depend on u, f, f_k, and λ.

Proof. Let $u\in\mathbf{W}_p^2((0,1),H(A),H)$ be a solution to problem (5.14), (5.15). Then, by virtue of Theorem 5.1, we have

$$|\lambda|\|u\|_{L_2((0,1),H)}+\left\|u''\right\|_{L_2((0,1),H)}+\|Au\|_{L_2((0,1),H)} \\ \leq C\left(\|f-Bu\|_{L_2((0,1),H)}+\sum_{k=1}^{2}\left(\left\|A^{-m_k/2+3/4}f_k\right\|+|\lambda|^{-m_k/2+3/4}\|f_k\|\right)\right), \tag{5.17}$$

using condition (4) of Theorem 5.2, we have

$$\big(|\lambda|-C\cdot C(\varepsilon)\big)\|u\|_{L_2((0,1),H)}+\left\|u''\right\|_{L_2((0,1),H)}+(1-C\cdot\varepsilon)\|Au\|_{L_2((0,1),H)} \\ \leq C\left(\|f\|_{L_2((0,1),H)}+\sum_{k=1}^{2}\left(\left\|A^{-m_k/2+3/4}f_k\right\|+|\lambda|^{-m_k/2+3/4}\|f_k\|\right)\right), \tag{5.18}$$

by choosing ε such that $C\cdot\varepsilon<1$, (5.16) is easily obtained from (5.18). □

6. Completeness of root functions

We define the operator $\mathscr{L}$ by

$$\mathscr{L}u = -u''(x) + Au(x), \qquad \mathscr{D}(\mathscr{L}) = \mathbf{W}_p^2\big((0,1); H(A), H, L_k u = 0, k = 1,2\big), \quad (6.1)$$

LEMMA 6.1. *Suppose that $S_j(I, H(A), H) \sim Cj^{-q}$, then*

$$S_j\big(I, \mathbf{W}_2^2\big((0,1), H(A), H\big), L_2\big((0,1), H\big)\big) \sim Cj^{-1/(1/2+1/q)}. \quad (6.2)$$

Proof. Consider the operator S_1 defined in $L_2(0,1)$ such that $S_1 = S_1^* \geq \gamma^2 I$, $D(S_1) = H(S_1) = \mathbf{W}_2^2(0,1)$. From [11, Chapter 1, Section 2.1], we know that if $H_1 \subset H$, $\overline{H}_1 = H$, then there exists $S = S^*$ such that $D(S_1) = H_1$. And let the operator S_2 in H be defined by $S_2 = S_2^* \geq \gamma^2 I$, $D(S_2) = H(A)$. If we define the operator S on $L_2(0,1) \otimes H = L_2((0,1), H)$ by

$$S = S_1 \otimes I_2 + I_1 \otimes S_2, \quad (6.3)$$

where I_1 (respectively, I_2) is the identity operator in $L_2(0,1)$ (respectively, in H), we have

$$\begin{gathered} S_j\big(S_1^{-1}; L_2(0,1), L_2(0,1)\big) \sim S_j\big(I; H\big(S_1\big), L_2(0,1)\big) \sim Cj^{-2}, \\ S_j\big(S_2^{-1}; H, H\big) \sim S_j\big(I; H(A), H\big) \sim Cj^{-q}, \end{gathered} \quad (6.4)$$

and so, from [8], we obtain

$$S_j\big(S^{-1}\big) \sim Cn^{-1/(1/2+1/q)}. \quad (6.5)$$

This ends the proof. □

THEOREM 6.2. *Let conditions (1) and (3) of Theorem 5.2 hold along with $A^{-1} \in \sigma_q(H), q > 0$. Then, the system of root functions of operator $\mathscr{L}$ is complete in $L_2((0,1), H)$.*

Proof. From Theorem 5.2, we have $\|R(\lambda, \mathscr{L})\| \leq C|\lambda|^{-1}$ for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large. Using Lemma 6.1, we have $R(\lambda, \mathscr{L}) \in \sigma_p(L_2(0,1), H)$, for $p > 1/2 + 1/q$, so, for the operator $\mathscr{L}$, all conditions of Theorem 2.3 [13, page 50] have been checked. This completes the proof of the theorem. □

THEOREM 6.3. *Suppose that the conditions of Theorem 6.2 are satisfied, as well as the condition $D(B(x)) \supset D(A)$, and for all $\varepsilon > 0$*

$$\|B(x)u\| \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A), \quad (6.6)$$

then the system of root functions of operator $\mathscr{L} + B$ is complete in $L_2((0,1), H)$.

Proof. We consider in the space $L_2((0,1), H)$ the operator B defined by

$$(Bu)(x) = B(x)u(x), \qquad D(B) = L_2\big((0,1), H(A)\big). \quad (6.7)$$

It is clear that

$$\|Bu\|_{L_2((0,1),H)} \leq \varepsilon \|Au\|_{L_2((0,1),H)} + C(\varepsilon)\|u\|_{L_2((0,1),H)}, \tag{6.8}$$

since by Theorem 5.2, we have

$$\|Au\|_{L_2((0,1),H)} \leq C\|f\|_{L_2((0,1),H)} = \|(\mathscr{L}-\lambda I)u\|_{L_2((0,1),H)}, \tag{6.9}$$

hence,

$$\|Bu\|_{L_2((0,1),H)} \leq \varepsilon \|(\mathscr{L}-\lambda I)u\|_{L_2((0,1),H)} + C(\varepsilon)\|u\|_{L_2((0,1),H)}, \tag{6.10}$$

and so, for $|\lambda|$ sufficiently large, and $|\arg\lambda| \leq \varphi < \pi$, then $R(\lambda, \mathscr{L}+B) \in \sigma_p(L_2(0,1), H)$, and from Theorem 5.2, $\|R(\lambda, \mathscr{L}+B_2)\| \leq C|\lambda|^{-1}$ for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large. The system of root functions is complete in $L_2((0,1),H)$. □

7. Application

We consider in the cylindrical domain $\Omega = [0,1] \times G$, where $G \subset \mathbb{R}^r$ is a bounded domain, nonlocal boundary value problems for the Laplace equation with a parameter

$$L(\lambda)u = \lambda u(x,y) - \Delta u(x,y) + b(x,y)u(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{7.1}$$

$$L_k u = \alpha_k u^{(m_k)}(0,y) + \beta_k u^{(m_k)}(1,y) = f_k(y), \quad y \in G,\ k = 1,2, \tag{7.2}$$

$$Pu = u\big(x,y'\big) = 0, \quad \big(x,y'\big) \in [0,1] \times \Gamma, \tag{7.3}$$

where α_k, β_k are complex numbers, $y = (y_1,\dots,y_r)$, and $\Gamma = \partial G$ is the boundary of G.

A number λ_0 is called an eigenvalue of problem (7.1), (7.2), and (7.3) if the problem

$$L(\lambda_0)u = 0, \quad L_1 u = 0, \quad L_2 u = 0, \quad Pu = 0 \tag{7.4}$$

has a nontrivial solution that belongs to $W_2^2(\Omega)$. The nontrivial solution $u_0(x,y)$ of problem (7.4) that belongs to $W_2^2(\Omega)$ is called eigenfunction of problem (7.1), (7.2), and (7.3) and corresponds to the eigenvalue λ_0. A solution $u_k(x)$ to the problem

$$L\big(\lambda_0\big)u_k + u_{k-1} = 0, \quad L_1 u_k = 0, \quad L_2 u_k = 0, \quad Pu_k = 0, \tag{7.5}$$

belongs to $W_2^2(\Omega)$, and is called an associated function of the kth rank to the eigenfunction $u_0(x)$ of problem (7.1), (7.2), and (7.3).

Eigenfunctions and associated functions of problem (7.1), (7.2), and (7.3) are gathered under the general name, root functions of problem (7.1), (7.2), and (7.3).

Theorem 7.1. *Let $b(x,y) \in W_\infty^{0,1}(\Omega)$, $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$, $\Gamma \in C^2$. Then,*

(1) *Problem (7.1), (7.2), and (7.3) for $f \in W_2^{0,1}(\Omega; Pu = 0)$, $f_k \in W_2^{-m_k/2+3/4}(G; Pu = 0)$ for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda|$ is sufficiently large has a unique solution that belongs to the space $W_2^2(\Omega)$, and for this solution we have the coercive estimate*

$$|\lambda|\|u\|_{L_2(\Omega)} + \|u\|_{W_2^2(\Omega)} \leq C\Big(\|f\|_{L_2(\Omega)} + \sum_{k=1}^{2}\Big(\|f_k\|_{W_2^{-m_k/2+3/4}} + |\lambda|^{-m_k/2+3/4}\|f_k\|\Big)\Big), \tag{7.6}$$

where the constant C is independent of u and λ.

(2) *Root functions of problem (7.1), (7.2), and (7.3) are complete in the space $L_2(\Omega)$.*

Proof. Consider in the space $H = L_2(G)$ operators A, $B(x)$ defined by

$$\begin{aligned} Au &= -\Delta u(y) + \lambda_0 u(y), \qquad D(A) = W_2^2(G; Pu = 0), \\ B(x)u &= b(x,y)u(y) - \lambda_0 u(y), \qquad D\big(B(x)\big) = W_2^1(G; Pu = 0, m = 0). \end{aligned} \tag{7.7}$$

Then, problem (7.1), (7.2), and (7.3) can be rewritten in the form

$$\lambda u(x) - u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u = f(x), \tag{7.8}$$

$$\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \quad k = 1, 2. \tag{7.9}$$

We have the compact embedding [12, page 258] $W_2^2(\Omega) \subset L_2(\Omega)$. On the other hand, (cf. [12, page 350])

$$S_j\big(I; W_2^2(\Omega), L_2(\Omega)\big) \sim j^{-2/(r+1)}. \tag{7.10}$$

By virtue of Lemma 3.1 [13, page 60]

$$S_j\big(I; H(A), L_2(\Omega)\big) = S_j\big(A^{-1}; L_2(\Omega), L_2(\Omega)\big). \tag{7.11}$$

Since $H(A) \subset W_2^2(\Omega)$, then, from (7.10), (7.11), and Lemma 3.3 [13, page 61], it follows that

$$A^{-1} \in \sigma_p\big(L_2(\Omega), L_2(\Omega)\big), \quad p > \frac{r+1}{2}. \tag{7.12}$$

From (7.6) it follows that

$$\big\|R(\lambda, A)\big\| \le C|\lambda|^{-1}, \quad |\arg\lambda| \le \varphi < \pi, \quad |\lambda| \text{ is sufficiently large.} \tag{7.13}$$

So, all conditions of Theorem 6.3 have been checked. This ends the proof of the theorem. □

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