QUASILINEAR ELLIPTIC SYSTEMS OF RESONANT TYPE AND NONLINEAR EIGENVALUE PROBLEMS

PABLO L. DE NÁPOLI AND M. CRISTINA MARIANI

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This work is devoted to the study of a quasilinear elliptic system of resonant type. We prove the existence of infinitely many solutions of a related nonlinear eigenvalue problem. Applying an abstract minimax theorem, we obtain a solution of the quasilinear system $-\Delta_p u = F_u(x,u,v)$, $-\Delta_q v = F_v(x,u,v)$, under conditions involving the first and the second eigenvalues.

1. Introduction

1.1. The problem and some previous results. We consider a gradient elliptic system

$$-\Delta_p u = F_u(x,u,v), \qquad -\Delta_q v = F_v(x,u,v). \tag{1.1}$$

Elliptic problems involving the p-Laplacian have been studied by several authors (cf. [3, 7, 8, 10, 11]). We recall some results from the work of Boccardo and de Figueiredo [4].

It is well known that the solutions of (1.1) in $W = W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ are the critical points of the functional

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega F(x,u,v) \tag{1.2}$$

under the following three assumptions:

(1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p,q < N$, so that the following continuous embeddings hold:

$$W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega), \qquad W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega); \tag{1.3}$$

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(2) $F : \bar{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is C^1 and verifies the following growth condition:

$$\left|F(x,s,t)\right| \le c\left(1+|s|^{p^*}+|t|^{q^*}\right) \quad \forall x\in\bar{\Omega};\ s,t\in\mathbb{R}; \tag{1.4}$$

(3) in order to have $\Phi\in C^1(W,\mathbb{R})$, we assume

$$\begin{aligned} \left|F_s(x,s,t)\right| &\le C\left(1+|s|^{p^*-1}+|t|^{q^*(p^*-1)/p^*}\right) \quad \forall x\in\bar{\Omega};\ s,t\in\mathbb{R},\\ \left|F_t(x,s,t)\right| &\le C\left(1+|t|^{q^*-1}+|s|^{p^*(q^*-1)/q^*}\right) \quad \forall x\in\bar{\Omega};\ s,t\in\mathbb{R}. \end{aligned} \tag{1.5}$$

The geometry of Φ depends strongly on the values of α and β in the estimate

$$\left|F(x,s,t)\right| \le c\left(1+|s|^{\alpha}+|t|^{\beta}\right) \quad \forall x\in\bar{\Omega};\ s,t\in\mathbb{R}, \tag{1.6}$$

where $\alpha\le p^*$, $\beta\le q^*$. In this work we are interested in the case $\alpha = p$, $\beta = q$ (systems of resonant type).

In our case, it is quite adequate to assume the following condition on F: consider the function

$$L(x,s,t) = \frac{1}{p}F_s(x,s,t)s+\frac{1}{q}F_t(x,s,t)t-F(x,s,t). \tag{1.7}$$

Assume that

$$\lim_{\|(s,t)\|\to\infty} L(x,s,t) = \pm\infty \quad \text{uniformly for } x\in\Omega. \tag{1.8}$$

This assumption implies that Φ satisfies the following compactness Cerami condition.

Definition 1.1. Let X be a Banach space and $\Phi\in C^1(X,\mathbb{R})$. Given $c\in\mathbb{R}$, we say that Φ satisfies condition (C_c), if

(1) any bounded sequence $(u_n)\subset X$ such that $\Phi(u_n)\to c$ and $\Phi'(u_n)\to 0$ has a convergent subsequence;
(2) there exist constants $\delta, R, \alpha > 0$ such that

$$\left\|\Phi'(u)\right\|\|u\| \ge \alpha \quad \forall u\in\Phi^{-1}\left([c-\delta,c+\delta]\right) \text{ with } \|u\|\ge R. \tag{1.9}$$

If $\Phi\in C^1(X,\mathbb{R})$ satisfies condition (C_c) for every $c\in\mathbb{R}$, we say that Φ satisfies condition (C).

Condition (C) was introduced by Cerami [5]. It was shown in [2] that from condition (C) it is possible to obtain a deformation lemma, that is fundamental in order to get minimax theorems.

In order to avoid resonance, Boccardo and de Figueiredo [4] introduced an assumption on F involving an eigenvalue problem

$$\begin{aligned} -\Delta_p u - aG_u(u,v) &= \lambda|u|^{p-2}u,\\ -\Delta_q v - aG_v(u,v) &= \lambda|v|^{q-2}v, \end{aligned} \tag{1.10}$$

where $a = a(x) \in L^\infty(\Omega)$ and G is a C^1 even function $G : \mathbb{R} \to [0, \infty)$ such that

$$G\left(c^{1/p}s, c^{1/q}t\right) = cG(s,t) \quad \forall c > 0, \tag{1.11}$$

$$G(s,t) \le K\left(\frac{1}{p}|s|^p + \frac{1}{q}|t|^q\right). \tag{1.12}$$

We call such a G a (p,q) homogeneous function.

It is easy to see that (1.11) implies (1.12). A (p,q)-homogeneous function satisfies

$$\frac{1}{p}G_s(s,t)s + \frac{1}{q}G_t(s,t)t = G(s,t). \tag{1.13}$$

Examples of (p,q) homogeneous functions are

(1) $G(s,t) = c_1|s|^p + c_2|t|^q$,

(2) $G(s,t) = c|s|^\alpha|t|^\beta$ with $\alpha/p + \beta/q = 1$ where c, c_1, c_2 are constants.

The following results are proved in [4].

Theorem 1.2. *Problem (1.10), with G as above, has a first eigenvalue $\lambda_1(a)$, characterized variationally by*

$$\lambda_1(a) = \inf_{(u,v)\neq(0,0)} \frac{(1/p)\int_\Omega |\nabla u|^p + (1/q)\int_\Omega |\nabla v|^q - \int_\Omega aG(u,v)}{(1/p)\int_\Omega |u|^p + (1/q)\int_\Omega |v|^q} \tag{1.14}$$

which depends continuously on a in the L^∞-norm.

Theorem 1.3. *Assume (1.5), (1.6) with $\alpha = p$, $\beta = q$, and that the following conditions hold:*

(1) *there exist positive numbers c, R, μ, and ν such that*

$$\frac{1}{p}sF_s(x,s,t) + \frac{1}{q}tF_t(x,s,t) - F(x,s,t) \ge c\left(|s|^\mu + |t|^\nu\right) \quad \text{for } |s|, |t| > R, \tag{1.15}$$

(2) *there exists G as above, such that*

$$\lim \sup_{|s|,|t|\to\infty} \frac{F(x,s,t)}{G(s,t)} \le a(x) \in L^\infty(\Omega), \tag{1.16}$$

where $\lambda_1(a) > 0$.

Then the functional Φ is bounded from below and the infimum is achieved.

1.2. The existence of infinitely many eigenfunctions. Let $\mathscr{C}$ be the class of compact symmetric ($C = -C$) subsets of the space W. We recall that for $C \in \mathscr{C}$ the Krasnoselskii genus $\operatorname{gen}(C)$ is defined as the minimum integer n such that there exists an odd continuous mapping $\varphi : C \to (\mathbb{R}^n - \{0\})$ (cf. [1]). We note

$$\mathscr{C}_k = \{C \in \mathscr{C} : \operatorname{gen}(C) \ge k\}. \tag{1.17}$$

For an arbitrary symmetric subset S of $W-\{0\}$ the genus over compact sets $\gamma(S)$ is defined by

$$\gamma(S)=\sup\{\operatorname{gen}(C):C\subset S,\ C\in\mathscr{C},\ C \text{ compact}\}. \tag{1.18}$$

Now we may state our main result on the eigenvalue problem.

Theorem 1.4. *The eigenvalue problem (1.10), with G as above, has infinitely many eigenfunctions given by*

$$\lambda_k(a,G)=\inf_{C\in\mathscr{C}_k}\sup_{(u,v)\in C}\frac{(1/p)\int_\Omega|\nabla u|^p+(1/q)\int_\Omega|\nabla v|^q-\int_\Omega aG(u,v)}{(1/p)\int_\Omega|u|^p+(1/q)\int_\Omega|v|^q} \tag{1.19}$$

and $\lambda_k(a,G)\to\infty$ as $k\to\infty$. Moreover, λ_k depends continuously on a in the L^∞-norm.

Remark 1.5. Equivalently if we define

$$S=\left\{(u,v)\in W:\frac{1}{p}\int_\Omega|u|^p+\frac{1}{q}\int_\Omega|v|^q=1\right\}, \tag{1.20}$$

we have

$$\lambda_k(a,G)=\inf_{C\in\mathscr{C}_k,C\subset S}\sup_{(u,v)\in C}\frac{1}{p}\int_\Omega|\nabla u|^p+\frac{1}{q}\int_\Omega|\nabla v|^q-\int_\Omega aG(u,v). \tag{1.21}$$

We will write $\lambda_k(a)$ instead of $\lambda_k(a,G)$, when the dependence on the (p,q)-homogeneous function G is clear from the context.

1.3. The existence result for resonant systems. Applying Theorem 1.4 and an abstract minimax principle from [9], we prove the following theorem.

Theorem 1.6. *Assume that $F:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ verifies (1.5), (1.6) with $\alpha=p$, $\beta=q$, (1.8), and that $a_1,a_2\in L^\infty(\Omega)$ satisfy*

$$a_1(x)\le\lim\inf_{|s|,|t|\to\infty}\frac{F(x,s,t)}{G_1(s,t)}\le\lim\sup_{|s|,|t|\to\infty}\frac{F(x,s,t)}{G_2(s,t)}\le a_2(x) \tag{1.22}$$

with G_1 and G_2 two (p,q)-homogeneous functions and $\lambda_1(a_1,G_1)<0<\lambda_2(a_2,G_2)$, where $\lambda_1(a_1,G_1)$, $\lambda_2(a_2,G_2)$ are given by (1.19). Then problem (1.1) has at least one solution.

Remark 1.7. The conditions above could be reformulated in terms of a different eigenvalue problem, for $a\in L^\infty(\Omega)$, $a(x)>0$

$$-\Delta_p u=\mu aG_u(u,v),\qquad -\Delta_q v=\mu aG_v(u,v). \tag{1.23}$$

This problem also has infinitely many eigenvalues given by

$$\mu_k(a)=\inf_{C\in\mathscr{C}_k}\sup_{(u,v)\in C}\frac{(1/p)\int_\Omega|\nabla u|^p+(1/q)\int_\Omega|\nabla v|^p}{\int_\Omega aG(u,v)}. \tag{1.24}$$

The condition $\lambda_1(a) < 0$ is equivalent to $\mu_1(a) < 1$, and the condition $\lambda_2(a) > 0$ is equivalent to $\mu_2(a) > 1$.

Remark 1.8. As an example for Theorem 1.6, we may take

$$G_1(s,t) = G_2(s,t) = |s|^{\alpha}|t|^{\beta} \tag{1.25}$$

with $\alpha/p + \beta/q = 1$;

$$F(x,s,t) = \lambda|s|^{\alpha}|t|^{\beta} + c|s|^{\mu}|t|^{\delta}, \tag{1.26}$$

where $c \neq 0$ is a constant, and we assume that

$$\mu_1(1) < \lambda < \mu_2(1) \tag{1.27}$$

and $\mu < \alpha$, $\delta < \beta$, where $\mu_1(1)$, $\mu_2(1)$ are defined as above (with $a \equiv 1$).

2. The eigenvalue problem

2.1. The functional framework. We apply the following abstract theorem due to Amann [1].

Theorem 2.1. *Suppose that the following hypotheses are satisfied:*

- *X is a real Banach space of infinite dimension, that is uniformly convex;*
- *$A : X \to X^*$ is an odd potential operator (i.e., A is the Gateaux derivative of $\mathcal{A} : X \to \mathbb{R}$) which is uniformly continuous on bounded sets, and satisfies condition $(S)_1$: if $u_j \rightharpoonup u$ (weakly in X) and $A(u_j) \to v$, then $u_j \to u$ (strongly in X).*
- *For a given constant $\alpha > 0$, the level set*

$$M_\alpha = \{u \in X : \mathcal{A}(u) = \alpha\} \tag{2.1}$$

  is bounded and each ray through the origin intersects M_α. Moreover, for every $u \neq 0$, $\langle A(u), u\rangle > 0$ and there exists a constant $\rho_\alpha > 0$ such that $\langle A(u), u\rangle \geq \rho_\alpha$ on M_α.
- *The mapping $B : X \to X^*$ is a strongly sequentially continuous odd potential operator (with potential $\mathcal{B}$), such that $\mathcal{B}(u) \neq 0$ implies that $B(u) \neq 0$.*

Let

$$\beta_k = \sup_{C \in \mathcal{C}, C \subset M_\alpha} \inf_{u \in C} \mathcal{B}(u). \tag{2.2}$$

Then if $\beta_k > 0$, there exists an eigenfunction $u_k \in M_\alpha$ with $\mathcal{B}(u) = \beta_k$. If

$$\gamma(\{u \in M_\alpha : \mathcal{B}(u) \neq 0\}) = \infty, \tag{2.3}$$

then there exist infinitely many eigenfunctions.

We will work in the Banach space

$$W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \tag{2.4}$$

equipped with the norm

$$\|(u,v)\|_W = \sqrt{\|u\|_p^2 + \|v\|_q^2}. \tag{2.5}$$

As each factor is uniformly convex, we can conclude that W is uniformly convex (see [6]). Given $(u^*, v^*) \in W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega)$ we may think of it as an element of W^*:

$$\langle (u^*, v^*), (u,v) \rangle = \langle u^*, u \rangle + \langle v^*, v \rangle. \tag{2.6}$$

Then we have $W^* \cong W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega)$ (isometric isomorphism), where the norm in W^* is given by

$$\|(u^*, v^*)\|_{W^*} = \sqrt{\|u^*\|^2 + \|v^*\|^2}. \tag{2.7}$$

With the notations of Theorem 2.1, we define

$$\mathcal{A}_0(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q, \tag{2.8}$$

$$\mathcal{A}(u,v) = \mathcal{A}_0(u,v) - \int_\Omega aG(u,v) + M\left(\frac{1}{p}\int_\Omega |u|^p + \frac{1}{q}\int_\Omega |v|^q\right), \tag{2.9}$$

with a and G as in the statement of Theorem 1.4, and M a fixed constant such that $M > K\|a\|_{L^\infty}$, where K is the constant in (1.12).

We write $\mathcal{A}_a$ instead of $\mathcal{A}$ when we want to remark the dependence on the weight a

$$\begin{aligned} A(u,v) &= \left(-\Delta_p u - aG_u(u,v) + M|u|^{p-2}u, -\Delta_q v - aG_q(u,v) + M|v|^{q-2}v\right), \\ \mathcal{B}(u,v) &= \frac{1}{p}\int_\Omega |u|^p + \frac{1}{q}\int_\Omega |v|^q, \\ B(u,v) &= \left(|u|^{p-2}u, |v|^{q-2}v\right). \end{aligned} \tag{2.10}$$

In order to apply Theorem 2.1, we prove the following two lemmas.

Lemma 2.2. (1) *A is uniformly continuous on bounded sets.*
(2) *A verifies the $(S)_1$ condition.*

Proof. We write $A = A_1 - A_2$, where

$$\begin{aligned} A_1(u,v) &= \left(-\Delta_p u, -\Delta_q v\right), \\ A_2(u,v) &= \left(aG_u(u,v) - M|u|^{p-2}u, aG_v(u,v) - M|v|^{q-2}v\right). \end{aligned} \tag{2.11}$$

We claim that $A_2 : W \to W^*$ verifies that: if $(u_j, v_j) \rightharpoonup (u, v)$ in W, then $A_2(u_j, v_j) \to A_2(u, v)$ in W^*.

Indeed, if $(u_j, v_j) \rightharpoonup (u, v)$, then

$$(u_j, v_j) \longrightarrow (u, v) \quad \text{in } L^p(\Omega) \times L^q(\Omega) \tag{2.12}$$

and we obtain that

$$\begin{aligned} G_u(u_j, v_j) &\longrightarrow G_u(u, v) \quad \text{in } L^{p'}(\Omega), \\ G_v(u_j, v_j) &\longrightarrow G_v(u, v) \quad \text{in } L^{q'}(\Omega). \end{aligned} \tag{2.13}$$

Hence, $A_2(u_j, v_j) \to A_2(u, v)$ in W^*.

Let $(u_j, v_j) \rightharpoonup (u, v)$ in W such that

$$A(u_j, v_j) \longrightarrow (z, w). \tag{2.14}$$

Therefore $A_2(u_j, v_j) \to A_2(u, v)$ and then $A_1(u_j, v_j) \to (z, w) + A_2(u, v)$. Since A_1 verifies condition $(S)_1$, it follows that $(u_j, v_j) \to (u, v)$. □

Lemma 2.3. (1) *The set* $M_\alpha = \{(u, v) \in W : \mathcal{A} = \alpha\}$ *is bounded.*

(2) *Every ray* $t \cdot (u, v)$ *with* $(u, v) \neq 0$ *intersects* M_α.

(3) *There exists a constant* $\rho_\alpha > 0$ *such that*

$$\langle A(u, v), (u, v) \rangle \leq \rho_\alpha. \tag{2.15}$$

(4) *Condition (2.3) is satisfied.*

Proof. (1) As we have fixed $M > K\|a\|_{L^\infty}$ on M_α, then

$$\alpha = \mathcal{A}(u, v) \geq \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q \tag{2.16}$$

and the proof is complete.

(2) Let $f(c) = \mathcal{A}(c(u, v))$, $f(0) = 0$,

$$\begin{aligned} f(c) = {} & \frac{c^p}{p}\int_\Omega |\nabla u|^p + \frac{c^q}{q}\int_\Omega |\nabla v|^q \\ & - \int_\Omega aG(cu, cv) + M\left(\frac{c^p}{p}\int_\Omega |u|^p + \frac{c^q}{q}\int_\Omega |v|^q\right). \end{aligned} \tag{2.17}$$

From (1.12) and the choice of M, we have

$$f(c) \geq \frac{c^p}{p}\int_\Omega |\nabla u|^p + \frac{c^q}{q}\int_\Omega |\nabla v|^q \longrightarrow +\infty \tag{2.18}$$

as $c \to \infty$. Since f is continuous, there exists $c \in \mathbb{R}$ such that $f(c) = \alpha$.

(3) We have

$$\langle A(u,v),(u,v)\rangle = \int_\Omega |\nabla u|^p + \int_\Omega |\nabla v|^q - \int_\Omega a\big[G_u(u,v)u + G_v(u,v)v\big] + M\left(\int_\Omega |u|^p + \int_\Omega |v|^q\right). \tag{2.19}$$

Then, using (1.13)

$$\langle A(u,v),(u,v)\rangle \geq \min\{p,q\}\mathcal{A}(u,v) = \min\{p,q\}\alpha. \tag{2.20}$$

(4) In order to see that $\gamma(M_\alpha) \geq k$, it is enough to show that M_α contains subsets homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd homeomorphism. Hence, the proof is completed. □

2.2. The continuous dependence of $\lambda_k(a)$ on a. In this section we prove that the eigenvalue $\lambda_k(a)$ depends continuously on the weight a in the L^∞-norm. This result will be used for proving Lemma 3.3.

Proposition 2.4. *The eigenvalue $\lambda_k(a)$ depends continuously on a in the L^∞-norm.*

Proof. We have

$$\big|\mathcal{A}_a(u,v) - \mathcal{A}_b(u,v)\big| \leq K\,\|a-b\|_{L^\infty}\left(\frac{1}{p}\int_\Omega |u|^p + \frac{1}{q}\int_\Omega |v|^q\right), \tag{2.21}$$

where K is given by condition (1.12), with $\mathcal{A}_a$, $\mathcal{A}_b$ as above. Let $\varepsilon > 0$. Then there exists $C \in \mathscr{C}_k$, $C \subset S$ such that

$$\sup_{(u,v)\in C} \mathcal{A}_a(u,v) \leq \lambda_k(a) + \frac{\varepsilon}{2}. \tag{2.22}$$

Then for any $(u,v) \in C$, if $\|a-b\|_{L^\infty} \leq \delta = \varepsilon/2K$ we get

$$\mathcal{A}_b(u,v) \leq \mathcal{A}_a(u,b) + \frac{\varepsilon}{2} \leq \lambda_k(a) + \varepsilon. \tag{2.23}$$

It follows that

$$\sup_{(u,v)\in C} \mathcal{A}_b(u,v) \leq \lambda_2(a) + \varepsilon \tag{2.24}$$

and we obtain

$$\lambda_k(b) \leq \lambda_k(a) + \varepsilon. \tag{2.25}$$

By reversing the roles of a and b, we get $|\lambda_k(a) - \lambda_k(b)| \leq \varepsilon$. □

3. Proof of the existence theorem

3.1. A minimax principle. Our main tool for proving Theorem 1.6 will be an abstract minimax principle due to El Amrouss and Moussaoui [9].

THEOREM 3.1. *Let Φ be a C^1 functional on X satisfying condition (C), let Q be a closed connected subset of X such that $\partial Q \cap \partial(-Q) \neq \emptyset$, and let $\beta \in \mathbb{R}$. Assume that*

(1) *for every $K \in \mathscr{C}_2$ there exists $v_K \in K$ such that $\Phi(v_K) \geq \beta$ and $\Phi(-v_K) \geq \beta$,*

(2) *$a = \sup_{\partial Q} \Phi < \beta$,*

(3) *$\sup_Q \Phi < \infty$.*

Then Φ has a critical value $c \geq \beta$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in Q} \Phi(h(x)), \tag{3.1}$$

where $\Gamma = \{h \in C(X,X) : h(x) = x \text{ for every } x \in \partial Q\}$.

3.2. Compactness conditions

LEMMA 3.2. *Suppose that F satisfies (1.6), (1.8), and (1.22). Then the functional Φ, given by (1.2), satisfies the Cerami condition.*

Proof. In a similar way to [9, Lemma 3.1], we see that the first condition in Definition 1.1 holds.

We will prove that the second condition in Definition 1.1 holds, in the case $L(x,s,t) \to -\infty$ as $\|(s,t)\| \to \infty$ (the case $L(x,s,t) \to +\infty$ is similar). To do that, assume by contradiction that there exists a sequence $(u_n, v_n)_{n\in\mathbb{N}} \subset W$ such that

$$\Phi(u_n, v_n) \longrightarrow c, \qquad \varepsilon_n = \|\Phi'(u_n, v_n)\| \|(u_n, v_n)\| \longrightarrow 0, \qquad \|(u_n, v_n)\| \longrightarrow \infty. \tag{3.2}$$

Therefore,

$$\left| \frac{1}{p}\langle \Phi_u(u_n, v_n), u_n\rangle + \frac{1}{q}\langle \Phi_v(u_n, v_n), v_n\rangle - \Phi(u_n, v_n) \right| \longrightarrow c \tag{3.3}$$

or equivalently

$$\lim_{n\to\infty} \left| \int_\Omega \frac{1}{p} F_u(x, u_n, v_n) u_n + \frac{1}{q} F_v(x, u_n, v_n) v_n - F(x, u_n, v_n) \right| = c. \tag{3.4}$$

We define

$$z_n = \alpha_n^{1/p} u_n, \qquad w_n = \alpha_n^{1/q} v_n, \tag{3.5}$$

where

$$\alpha_n = \frac{1}{\mathscr{A}_0(u_n, v_n)} \longrightarrow 0 \tag{3.6}$$

with $\mathcal{A}_0$ given by definition (2.8). We have that $\mathcal{A}_0(z_n,w_n)=1$ so (z_n,w_n) is bounded in W. After passing to a subsequence, we may assume that

$$\begin{aligned} z_n &\rightharpoonup z && \text{in } W^{1,p}(\Omega),\\ w_n &\rightharpoonup w && \text{in } W^{1,q}(\Omega),\\ z_n &\longrightarrow z && \text{in } L^p(\Omega),\ \text{a.e. in } \Omega,\\ w_n &\longrightarrow w && \text{in } L^q(\Omega),\ \text{a.e. in } \Omega. \end{aligned} \tag{3.7}$$

Now we show that $(z,w)\neq(0,0)$

$$\frac{\Phi(u_n,v_n)}{\mathcal{A}_0(u_n,v_n)}=1-\frac{\int_\Omega F(x,u_n,v_n)}{\mathcal{A}_0(u_n,v_n)}. \tag{3.8}$$

From (1.22), we get that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$F(x,s,t)\le\big(a_2(x)+\varepsilon\big)G_2(s,t)+C_\varepsilon. \tag{3.9}$$

As a consequence

$$\int_\Omega F(x,u_n,v_n)\le\int_\Omega\big(a_2(x)+\varepsilon\big)G_2(u_n,v_n)+C_\varepsilon|\Omega|, \tag{3.10}$$

then

$$\frac{\int_\Omega F(x,u_n,v_n)}{\mathcal{A}_0(u_n,v_n)}\le\alpha_n\int_\Omega\big(a_2(x)+\varepsilon\big)G_2(u_n,v_n)+C_\varepsilon|\Omega|\alpha_n. \tag{3.11}$$

Since

$$\alpha_n\int_\Omega\big(a_2(x)+\varepsilon\big)G_2(u_n,v_n)=\int_\Omega\big(a_2(x)+\varepsilon\big)G_2(z_n,w_n) \tag{3.12}$$

in the limit we get

$$0\ge1-\int_\Omega\big(a_2(x)+\varepsilon\big)G_2(z,w) \tag{3.13}$$

and we conclude that $G_2(z,w)\not\equiv0$.

Let

$$L(x,s,t)=\frac{1}{p}F_s(x,s,t)s+\frac{1}{q}F_t(x,s,t)t-F(x,s,t). \tag{3.14}$$

By (1.8) (and since L is continuous), $L(x,s,t)\le-M$. It follows that

$$\int_\Omega L(x,u_n,v_n)\le\int_{\{G_2(z,w)\neq0\}}L(x,u_n,v_n)+M\big|\{x:G_2(z(x),w(x))=0\}\big|. \tag{3.15}$$

Note that

$$\alpha_nG_2(u_n,v_n)\longrightarrow G_2(z,w). \tag{3.16}$$

So in the set $\{x : G_2(z(x), w(x)) \neq 0\}$, $G_2(u_n, v_n) \to +\infty$, and then by (1.11), we have that $u_n(x), v_n(x) \to \infty$. It follows that $L(u_n, v_n) \to -\infty$ by condition (1.8). Hence the first integral tends to $-\infty$ by Fatou lemma, and we get

$$\lim_{n\to\infty} \int_\Omega L(x, u_n, v_n) = -\infty. \tag{3.17}$$

This contradicts (3.4), and the proof is completed. □

3.3. Geometric conditions. In this section we show that the functional Φ satisfies the geometric conditions of Theorem 3.1.

Lemma 3.3. *Let F satisfy the assumptions of Theorem 1.6. Then the functional Φ, given by (1.2), satisfies*

(1) *there exists $(\varphi, \psi) \in W$ such that $\Phi(c^{1/p}\varphi, c^{1/q}\psi) \to -\infty$ as $c \to +\infty$;*
(2) *for every $K \in \mathscr{C}_2$ there exists $(u_K, v_K) \in K$ and $\beta \in \mathbb{R}$ such that $\Phi(u_K, v_K) \geq \beta$ and $\Phi(-u_K, -v_K) \geq \beta$.*

Proof. (1) As $\lambda_1(a, G_1) < 0$, we may choose $\varepsilon > 0$ such that $\lambda_1(a_1 - \varepsilon, G_1) < 0$. Let (φ, ψ) be the first eigenfunction for the problem

$$\begin{aligned} -\Delta_p u - (a_1(x) - \varepsilon) G_{1u}(u, v) &= \lambda |u|^{p-2} u \qquad \text{in } \Omega, \\ -\Delta_q v - (a_1(x) - \varepsilon) G_{1v}(u, v) &= \lambda |v|^{q-2} v \qquad \text{in } \Omega, \\ u = v &= 0 \qquad \text{in } \partial\Omega, \end{aligned} \tag{3.18}$$

normalized by

$$\frac{1}{p}\int_\Omega |\varphi|^p + \frac{1}{q}\int_\Omega |\psi|^q = 1. \tag{3.19}$$

Then, using (1.13), we get

$$\frac{1}{p}\int_\Omega |\nabla\varphi|^p + \frac{1}{q}\int_\Omega |\nabla\psi|^q - \int_\Omega (a_1(x) - \varepsilon) G_1(u, v) = \lambda_1(a_1 - \varepsilon, G_1). \tag{3.20}$$

By (1.22), we have

$$F(x, s, t) \geq (a_1(x) - \varepsilon) G_1(s, t) - C_\varepsilon. \tag{3.21}$$

It follows that

$$\begin{aligned} \Phi(c^{1/p}\varphi, c^{1/q}\psi) &\leq c\bigg(\frac{1}{p}\int_\Omega |\nabla\varphi|^p + \frac{1}{q}\int_\Omega |\nabla\psi|^q \\ &\qquad - \int_\Omega (a_1(x) - \varepsilon) G_1(\varphi, \psi)\bigg) + C_\varepsilon |\Omega| \\ &\leq c\lambda_1(a_1 - \varepsilon, G_1) + C_\varepsilon |\Omega|, \end{aligned} \tag{3.22}$$

and so $\Phi(c^{1/p}\varphi, c^{1/q}\psi) \to -\infty$ as $c \to +\infty$.

(2) Since $\lambda_2(a_2, G_2) > 0$, we may choose $\varepsilon > 0$ such that $\lambda_2(a_2 + \varepsilon, G_2) > 0$. Given $K \in \mathscr{C}_2$ and this $\varepsilon > 0$, we claim that there exists $(u_K, v_K) \in K$ verifying

$$\begin{aligned} &\lambda_2(a_2+\varepsilon, G_2)\left(\frac{1}{p}\int_\Omega |u_K|^p + \frac{1}{q}\int_\Omega |v_K|^q\right) \\ &\quad \le \frac{1}{p}\int_\Omega |\nabla u_K|^p + \frac{1}{q}\int_\Omega |\nabla v_K|^q - \int_\Omega (a_2(x)+\varepsilon)G_2(u_K, v_K). \end{aligned} \tag{3.23}$$

By (1.22), we have

$$F(x,s,t) \le (a_2(x)+\varepsilon)G_2(s,t) + C_\varepsilon. \tag{3.24}$$

It follows that

$$\begin{aligned} \Phi(u_K, v_K) &\ge \frac{1}{p}\int_\Omega |\nabla u_K|^p + \frac{1}{q}\int_\Omega |\nabla v_K|^q \\ &\quad - \int_\Omega (a_2(x)+\varepsilon)G_2(u_K, v_K) - C_\varepsilon|\Omega| \\ &\ge \lambda_2(a_2+\varepsilon, G_2)\left(\frac{1}{p}\int_\Omega |u_K|^p + \frac{1}{q}\int_\Omega |v_K|^q\right) - C_\varepsilon|\Omega| \\ &\ge -C_\varepsilon|\Omega| = \beta. \end{aligned} \tag{3.25}$$

Similarly,

$$\Phi(-u_K, -v_K) \ge -C_\varepsilon|\Omega| = \beta. \tag{3.26}$$

□

3.4. Proof of Theorem 1.6. We apply Theorem 3.1. We take

$$Q = \left\{\left(|c|^{1/p-1}c\varphi, |c|^{1/q-1}c\psi\right),\ -R \le c \le R\right\}, \tag{3.27}$$

where (φ, ψ) is given by Lemma 3.3. Q is closed and compact (it is the image of $[-R, R]$ under a continuous mapping). Also $\partial Q = \partial(-Q) = \{(\pm R^{1/p}\varphi, \pm R^{1/q}\psi)\} \ne \emptyset$. By Lemma 3.3 if we choose R big enough, we have

$$\sup_{\partial Q} \Phi < \beta. \tag{3.28}$$

Also $\sup_Q \Phi < +\infty$ since Q is compact and Φ is continuous. The functional Φ verifies condition (C) by Lemma 3.2. Then all the conditions of Theorem 3.1 are fulfilled and the proof is completed. □

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Pablo L. de Nápoli: Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina

E-mail address: pdenapo@dm.uba.ar

M. Cristina Mariani: Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina

E-mail address: mcmarian@dm.uba.ar