ON DYNAMICS OF VISCOELASTIC MULTIDIMENSIONAL MEDIUM WITH VARIABLE BOUNDARY

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Received 30 January 2002

We prove the existence and uniqueness theorems for solutions of an initial-boundary value problem to the system of equations, which describes dynamics of viscoelastic continuous medium with a variable boundary and a memory along the trajectories of particles in classes of summable functions.

1. Introduction and main results

Let $\Omega_t \subset \mathbb{R}^n$, $n \geq 2$, be a family of the bounded domains with boundary Γ_t. We consider the following initial-boundary value problem:

$$\begin{gathered}
\rho(t,x)\left(V_t+\sum_{k=1}^{n} V_k\frac{\partial V}{\partial x_k}\right)-\mu\hat{\triangle}V-\operatorname{Div}G(B(V))+\operatorname{grad}\Phi(\rho(t,x))=\rho(t,x)F(t,x);\\
\rho_t(t,x)+\operatorname{div}\left(\rho(t,x)V(t,x)\right)=0,\quad (t,x)\in Q=\left\{(t,x):0\leq t\leq t_0,\ x\in\Omega_t\right\};\\
V(0,x)=V^0(x),\quad x\in\Omega_0;\\
V(t,x)=\Upsilon(t,x),\quad (t,x)\in\Gamma=\left\{(t,x):0\leq t\leq t_0,\ x\in\Gamma_t\right\}.
\end{gathered}\tag{1.1}$$

Here $V(t,x)=(V_1,\dots,V_n)$, $\rho(t,x)$ are a vector function and a scalar function which denote the velocity and the density of the medium, $G(Z)$ is an $n\times n$ matrix function with the coefficients $g_{ij}=g_{ij}(z_{11},\dots,z_{nn})$ whose arguments are the coefficients of an $n\times n$ matrix Z, Φ is a scalar function, G and Φ are supposed to be smooth, $\mu>0$. Next, $B(V)=z_x(0,t,x)$, where $z(\tau,t,x)$ is a solution to the Cauchy problem (in the integral form)

$$z(\tau,t,x)=x+\int_t^{\tau} V\left(s,z(s;t,x)\right)ds,\quad (t,x)\in Q;\tag{1.2}$$

$$\hat{\triangle}V=\frac{1}{2}\operatorname{Div}\left(V_x+V_x^*\right).\tag{1.3}$$

Abstract and Applied Analysis 7:9 (2002) 475–495
2000 Mathematics Subject Classification: 35Q99
URL: http://dx.doi.org/10.1155/S1085337502203097

Here V_x is the Jacobi matrix of a vector function $V(t,x)$, $\operatorname{Div} Z$ means the divergence of a matrix function Z, that is, the vector whose coefficients are the divergences of the rows of a matrix function Z, Z^* is the adjoint matrix to Z.

Problem (1.1) describes the dynamics of a viscoelastic continuous medium which occupies at the moment t the position Ω_t. Particles $y \in \Omega_0$ of the medium move in the field of velocities $V(t,x)$ along trajectories determined by the function

$$x = z(t,0,y) \equiv u(t,y), \qquad y \in \Omega_0, \tag{1.4}$$

with the velocity $v(t,y) = u_t(t,y)$. Thus, the field of velocities $V(t,x)$ is associated with velocities of particles $v(t,y)$ by the relations

$$v(t,y) = V\big(t,u(t,y)\big), \qquad V(t,x) = v\big(t,U(t,x)\big), \qquad U(t,x) = z(0,t,x). \tag{1.5}$$

Recall that the variables x and y are called Euler and Lagrange coordinates, respectively. It is supposed that the stress tensor T_s of the medium has the form

$$T_s = \mu T_v + \nu T_e + I\Phi(\rho). \tag{1.6}$$

Here $T_v = (1/2)(V_x + V_x^*)$ is the rate of strain tensor; the strain tensor has the form (see [3, Chapter 1, page 78])

$$T_e = \varpi_1 I + \varpi_2 U_x U_x^* + \varpi_3 \big(U_x U_x^*\big)^2, \tag{1.7}$$

where the functions $\varpi_i = \varpi_i(I_1, I_2, I_3)$ are smooth functions of the principal invariants I_i of the matrix $U_x U_x^*$, $U(t,x) = z(0,t,x)$. We consider a more general situation assuming that $T_e = G(U_x U_x^*)$.

The study of nonlinear viscous continuous mediums with a memory along trajectories of particles was initiated in [6] for the stationary case. The dynamics case was considered in [2, 15]. The viscoelastic problem (1.1) and its generalization in the case of cylindrical domain was studied in [8, 9, 10, 11]. The existence and uniqueness theorems and the stability of solutions in classes of summable functions under necessary conditions on the data were established. Dynamics problems in the case of a variable boundary for viscous medium (the case of a noncylindrical domain) was studied in [7]. An important moment in these works was proceeding from the Euler to Lagrange coordinates. This approach allows to investigate viscoelastic mediums in the case of variable boundary as well as free boundary problems (cf. [14]).

Now we give the main results. It is convenient to exclude $\rho(t,x)$ setting

$$\rho(t,x)=R(V), \quad R(V)=\left|U_x(t,x)\right|=\left|z_x(0,t,x)\right|, \tag{1.8}$$

and consider the density as an operator $R(V)$. Hereinafter $|Z|$ denotes the determinant of a matrix Z. Then the original problem takes the form

$$\begin{gathered} R(V)\left(V_t+\sum_{k=1}^{n} V_k \frac{\partial V}{\partial x_k}\right)-\mu \hat{\triangle} V-\operatorname{Div} G(B(V)) \\ +\operatorname{grad} \Phi(R(V))=R(V) F(t,x), \quad (t,x) \in Q; \end{gathered} \tag{1.9}$$

$$V(0,x)=V^0(x), \quad x \in \Omega_0; \qquad V(t,x)=\Upsilon(t,x), \quad (t,x) \in \Gamma. \tag{1.10}$$

Let the function $v(t,x) \in W_q^{1,2}(Q_0)$, $Q_0=\{(t,x): 0 \leq t \leq t_0,\ x \in \Omega_0\}$, $\Gamma_0 \in C^2$. Then, the trace of the function $v(t,x)$ on $S_0=[0,t_0] \times \Gamma_0$ belongs to $W_q^{1-1/q,2-2/q}(S_0) \equiv W$. We assume that the domains Ω_t, $0 \leq t \leq t_0$, are defined as

$$\Omega_t=\{x: x=u(t,y),\ y \in \Omega_0\}, \tag{1.11}$$

where $u(t,y)=y+\int_0^t v(s,y)\,ds,\ y \in \Omega_0$, $\Gamma_t=\{x: x=u(t,y),\ y \in \Gamma_0\}$, respectively.

The value Υ of the function V on Γ is defined as

$$\Upsilon(t,x)=v(t,U(t,x)), \tag{1.12}$$

where $U(t,x)$ for each t is the inverse to $u(t,y)$ map. The solution to problem (1.9), (1.10) is called summable in Q with the power q function $V(t,x)$, having generalized derivatives $V_t(t,x)$ and $V_{x_i,x_j}(t,x)$ whose qth power are summable, satisfying (1.9) and conditions (1.10). It follows necessarily from (1.10) that

$$V^0(y)=v(0,y), \quad (t,y) \in \Gamma_0. \tag{1.13}$$

The basic result reads as follows.

Theorem 1.1. *Let $q \in (n,\infty)$, $F \in L_q(Q_0)$, $V^0 \in W_q^{2-2/q}(\Omega_0)$, $v \in W_q^{1,2}(Q_0)$. Let (1.12) and (1.13) hold. Then problem (1.9), (1.10) has a unique solution $V(t,x)$, if t_0 is small enough.*

It will be convenient to study problem (1.9), (1.10) by means of proceeding to Lagrange coordinates. For this purpose we make the change of variable (1.4) in (1.9). Then problem (1.9), (1.10) takes the form

$$
\begin{aligned}
v_t(t,y) - \frac{1}{2}\mu \operatorname{Div}\Bigg(\left|I+\int_0^t v_y(s,y)\,ds\right| \\
\times\Bigg(v_y(s,y)\left(I+\int_0^t v_y(s,y)\,ds\right)^{-1} \\
+\left(I+\int_0^t v_y^*(s,y)\,ds\right)^{-1} v_y^*(t,y)\Bigg)\left(I+\int_0^t v_y^*(s,y)\,ds\right)^{-1}\Bigg) \\
-\operatorname{Div}\Bigg(\left|I+\int_0^t v_y(s,y)\,ds\right| G\left(I+\int_0^t v_y(s,y)\,ds\right) \\
\times\left(I+\int_0^t v_y^*(s,y)\,ds\right)^{-1}\Bigg) \\
-\left(I+\int_0^t v_y^*(s,y)\,ds\right)^{-1} \operatorname{grad}\Phi\left(\left|I+\int_0^t v_y(s,y)\,ds\right|^{-1}\right) \\
= f(t,y), \quad (t,x)\in Q_0;
\end{aligned}
\tag{1.14}
$$

$$
v(0,y)=V^0(y), \quad y\in\Omega_0; \qquad v(t,y)=\upsilon(t,y), \quad (t,y)\in S_0. \tag{1.15}
$$

Here

$$
f(t,y)=F\left(t,y+\int_0^t v(s,y)\,ds\right). \tag{1.16}
$$

A function $v(t,y)\in W_q^{1,2}(Q_0)$ is called a solution to problem (1.14), (1.15), and (1.16) if it satisfies a.e. (1.14) and conditions (1.15).

Theorem 1.2. *Let $F\in L_q(Q_0)$, $V^0\in W_q^{2-2/q}(\Omega_0)$, $\upsilon\in W_q^{1,2}(Q_0)$, $q\in(n,\infty)$. Let (1.13) hold. Then problem (1.14), (1.15), and (1.16) has a unique solution $V(t,x)$, if t_0 is small enough.*

For the proof of Theorem 1.2, we consider the case when $f\in L_q(Q_0)$ in (1.14) is an arbitrary function but not the function defined by (1.16).

Theorem 1.3. *Let the conditions of Theorem 1.2 hold. Then for the sufficiently small t_0, problem (1.14), (1.15) has a unique solution.*

Thus, having Theorems 1.2 and 1.3 proved, we get Theorem 1.1. The paper is organized as follows: the proofs of Theorems 1.3, 1.2, and 1.1 are given in Sections 2, 3, and 4, respectively. In Section 5, the necessity of conditions on the data of the problems is discussed.

Below, $\|\cdot\|_{k,m}$, $\|\cdot\|_0$, $|\cdot|_m$, $|\cdot|_0$, and $]\cdot[$ stand for the norms in $W_q^{k,m}(Q_0)$, $L_q(Q_0)$, $W_q^m(\Omega_0)$, $L_q(\Omega_0)$, and W, respectively, $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$. The function in this paper are scalar, vector, or matrix-valued and we do not distinguish them in our notations; the situation is clear from the context. For definitions and properties of Sobolev spaces W see [1, Chapter 3, page 123].

2. Proof of Theorem 1.3

We give a number of the known facts for linear problems. The linear problem

$$\begin{gathered} \mathfrak{L}(v) \equiv v_t - \mu\hat{\triangle} = \phi(t,y), \quad (t,y) \in Q_0; \\ v(0,y) = 0, \quad y \in \Omega_0; \\ v(t,y) = 0, \quad (t,y) \in S_0, \end{gathered} \tag{2.1}$$

has a unique solution for every $\phi \in L_q(Q_0)$ (see [12, 13]) and the estimate

$$\|v\|_{1,2} + \sup_t |v(t,y)|_{2-2/q} \le M\|\phi\|_0 \tag{2.2}$$

holds. If $\phi = \operatorname{Div}\psi$, where ψ is a matrix function, and $\psi \in W_q^{0,1}(Q_0)$, then (2.2) implies $\|v(t,y)\|_{0,1} \le M\|\psi\|_0$. Let $v = L\phi$ be a solution to problem (2.1). Thus, L is the inverse to $\mathfrak{L}$ operator. Consider the problem

$$\begin{gathered} \tilde{v}_t - \mu\hat{\triangle}\tilde{v} = 0, \quad (t,y) \in Q_0; \\ \tilde{v}(0,y) = V^0(y), \quad y \in \Omega_0; \\ \tilde{v}(t,y) = v(t,y), \quad (t,y) \in S_0. \end{gathered} \tag{2.3}$$

Problem (2.3) has a unique solution (see [4, Chapter 4, page 388] and [12]), and for it, the estimate

$$\|\tilde{v}\|_{1,2} + \sup_t |\tilde{v}(t,y)|_{2-2/q} \le M(|v_0|_{2-2/q} +]v[) \tag{2.4}$$

holds. Set $v = \tilde{v} + w$. Then w is a solution to the problem

$$\begin{gathered} \mathfrak{L}(w) \equiv w_t - \mu\hat{\triangle}w = \bar{\phi} + K(w); \\ w(0,y) = 0, \quad y \in \Omega_0; \\ w(t,y) = 0, \quad (t,y) \in S_0. \end{gathered} \tag{2.5}$$

Here

$$\bar{\phi}(t,y) = f(t,y) - \mu\hat{\triangle}\tilde{v}, \qquad K(w) = \mu K_1(w) + K_2(w) + K_3(w), \tag{2.6}$$

where by the notations

$$\begin{gathered} \hat{w}(t,y) = \int_0^t \tilde{v}_y(s,y)\,ds + \int_0^t w_y(s,y)\,ds, \\ \tilde{w} = I + \hat{w}, \quad \tilde{w}_* = (\tilde{w})^*, \quad \tilde{w}_*^{-1} = (\tilde{w}^{-1})^*, \end{gathered} \tag{2.7}$$

we set

$$\begin{aligned} K_1(w) &= \operatorname{Div}\left(|\tilde{w}|\left((\tilde{v}_y + w_y)\tilde{w}^{-1} + \tilde{w}_*^{-1}(\tilde{v}_y^* + w_y^*)\right)\tilde{w}_*^{-1}\right) \\ &\quad - \operatorname{Div}\left(w_y + w_y^* + \tilde{v}_y + \tilde{v}_y^*\right); \\ K_2(w) &= \operatorname{Div}\left(|w| G(\tilde{w})\tilde{w}_*^{-1}\right); \\ K_3(w) &= \tilde{w}_*^{-1}\operatorname{grad}\Phi\left(|I+\tilde{w}|^{-1}\right). \end{aligned} \tag{2.8}$$

Applying the operator L to both parts of (2.5), we have

$$w = \tilde{K}(w), \quad \tilde{K}(w) = L\bar{\phi} + LK(w). \tag{2.9}$$

Let $S(R) = \{w : \|w\|_{1,2} \le R\}$. Consider (2.9) as an operator equation and show that on the ball $S(R)$ it has a unique solution. For this purpose, we apply the fixed point theorem for contraction. Note that it cannot be applied to $S(R)$ endowed with the $\|\cdot\|_{1,2}$-metric. However, considering $S(R)$ as the metric space $\mathfrak{M}$ endowed with the $\|\cdot\|_{0,1}$-metrics, it is possible to apply it.

Lemma 2.1. *The metric space $\mathfrak{M}$ is complete.*

Proof. Let a sequence w_n, $n = 1,2,\ldots$, be a Cauchy sequence in $\mathfrak{M}$. Since $\|w_n\|_{1,2} \le R$, then w_n (or its subsequence) weakly converges in $W_q^{1,2}(Q_0)$ to $\bar{w} \in W_q^{1,2}(Q_0)$ and $\|\bar{w}\|_{1,2} \le \liminf_{n\to\infty} \|w_n\|_{1,2} \le R$. Hence, $\bar{w} \in \mathfrak{M}$. From the compact embedding $W_q^{1,2}(Q_0) \subset W_q^{0,1}(Q_0)$, it follows that w_n converges strongly to $\bar{w}$ in $W_q^{0,1}(Q_0)$. Hence, w_n converges to $\bar{w} \in \mathfrak{M}$ in $\mathfrak{M}$. $\square$

Lemma 2.2. *Let R be large enough and let t_0 be small enough. Then the operator $\tilde{K}$ transforms $\mathfrak{M}$ into itself.*

Proof. Let $w \in \mathfrak{M}$. It follows from (2.2) that

$$\begin{aligned} \|\tilde{K}(w)\|_{0,2} &\le \|L\bar{\phi}\|_{1,2} + \|LK(w)\|_{1,2} \\ &\le M\left(\|\bar{\phi}\|_0 + \|K(w)\|_0\right) \\ &\le M\left(\|\bar{\phi}\|_0 + \sum_{i=1}^{3}\|K_i(w)\|_0\right). \end{aligned} \tag{2.10}$$

We denote by M constants, which values are not important

$$\|K_i(w)\|_0 \le M(R)t_0^{1-1/q}, \quad i = 1,2,3. \tag{2.11}$$

First, establish (2.11) for $i = 1$. Letting

$$\begin{aligned} &A_0 = \tilde{w}, \qquad A_1 = |\tilde{w}|I, \qquad A_2 = w_y + w_y^* + \tilde{v}_y + \tilde{v}_y^*, \qquad A_3 = \tilde{w}^{-1}, \\ &A_4 = \tilde{w}_*^{-1}, \qquad A_5 = (\tilde{v}_y^* + w_y^*)\hat{w} + \hat{w}^*(\tilde{v}_y + w_y), \end{aligned} \tag{2.12}$$

rewrite $K_1(w)$ in the form

$$\begin{aligned} K_1(w) = {} & \operatorname{Div}\left(A_1A_4A_2A_3(A_4-I)\right) + \operatorname{Div}\left(A_1A_4A_2(A_3-I)\right) \\ & + \operatorname{Div}\left((A_1-I)A_4A_2\right) + \operatorname{Div}\left((A_4-I)A_2\right) \\ & + \operatorname{Div}(A_1A_4A_5A_3A_4) \\ = {} & \sum_1^5 S_i. \end{aligned} \tag{2.13}$$

Here every term involves a small multiplier at small t_0. We need the following fact to estimate S_i. Let a_i be an arbitrary coefficient of a matrix A_i.

Proposition 2.3. *If $t_0 = t_0(R)$ is small enough, then the following inequalities hold:*

$$\max_{t,y} \left| a_i(t,y) \right| \le M(R), \quad i = 0,1,2,3,4,5; \tag{2.14}$$

$$\max_{t,y} \left| a_i(t,y) - 1 \right| \le M(R) t_0^{1-1/q}, \quad i = 0,1,3,4; \tag{2.15}$$

$$\max_{t} \left| \frac{\partial}{\partial y_k} a_i(t,y) \right|_0 \le M(R) t_0^{1-1/q}, \quad i \ne 2,\ 1 \le k \le n. \tag{2.16}$$

Proof. First, establish (2.14) for $i = 0$. Let $\|\tilde v\|_{1,2} + \|\phi\|_0 \le \delta$. Using the continuous embedding

$$W_q^1(\Omega) \subset C(\bar\Omega), \quad q > n, \tag{2.17}$$

and Hölder's inequality, we have

$$\begin{aligned} ||\hat w||_{C(\Omega)} & \le \int_0^t ||\tilde v_y(s,y)||_{C(\Omega)}\, ds + \int_0^t ||w_y(s,y)||_{C(\Omega)}\, ds \\ & \le M\left(\int_0^{t_0} \left| \tilde v_y(s,y) \right|_2 ds + \int_0^{t_0} \left| w_y(s,y) \right|_2 ds \right) \\ & \le M t_0^{1-1/q} \left(\|\tilde v\|_{1,2} + \|w\|_{0,2} \right) \\ & \le M t_0^{1-1/q} (R + \delta). \end{aligned} \tag{2.18}$$

Since $\tilde w = I + \hat w$, estimate (2.14) for $i = 0$ follows from (2.18). It follows from (2.7) that the matrix $\tilde w$ at small $t_0 = t_0(R)$ has the inverse matrix $\tilde w^{-1} = A_3$ and for every coefficient a_3 inequality (2.14) holds. Since the matrix A_4 is adjoint to A_3, then estimate (2.14) for $i = 4$ follows from (2.14) for $i = 3$. Since nonzero coefficient a_1 of the matrix A_1 is the determinant of the matrix $\tilde w$, estimate (2.14) for $i = 1$ follows from (2.18) and (2.7). Estimates (2.14) are proved.

Since $A_0 = I + \hat{w}$, estimate (2.15) for $i = 0$ follows from (2.18). It follows from (2.12) that $A_3 - I = -\hat{w}(t,y)\tilde{w}^{-1}(t,y)$. Estimate (2.15) for $i = 3$ follows from (2.14) and (2.15) for $i = 0$. Estimates (2.15) for $i = 1,4$ follows from estimate (2.14) for $i = 3$. Estimates (2.15) are proved.

Taking an y_k derivative, we have

$$\frac{\partial}{\partial y_k}(\tilde{w}^{-1}) = -\tilde{w}^{-1}\left(\frac{\partial}{\partial y_k}\tilde{w}\right)\tilde{w}^{-1} = -\tilde{w}^{-1}\left(\frac{\partial}{\partial y_k}\hat{w}\right)\tilde{w}^{-1}. \tag{2.19}$$

It is easy to see that

$$\begin{aligned} \left|\frac{\partial}{\partial y_k}\tilde{w}\right|_0 &= \left|\frac{\partial}{\partial y_k}\hat{w}\right|_0 \\ &\le M\left(\int_0^t |\tilde{v}(s,y)|_2\,ds + \int_0^t |w(s,y)|_2\,ds\right) \\ &\le M t_0^{1-1/q}\left(\|\tilde{v}\|_{0,2} + \|w\|_{0,2}\right) \\ &\le M t_0^{1-1/q}(R+\delta). \end{aligned} \tag{2.20}$$

Estimate (2.16) for $i = 0$ follows from the above relation. It follows from (2.20) and inequality (2.14) for $i = 3$ that estimate (2.16) for a coefficient a_3 of the matrix $A_3 = \tilde{w}^{-1}$ holds. Estimate (2.16) for $i = 4$ is proved in a similar way.

Since nonzero coefficients of the matrix A_1 are determinants of the matrix A_0, then in force of (2.7) and (2.20), we get

$$\left|\frac{\partial}{\partial y_k}a_1(t,y)\right|_0 \le M\max_k\left|\frac{\partial}{\partial y_k}\hat{w}\right|_0 \le M t_0^{1-1/q}. \tag{2.21}$$

Estimate (2.16) for $i = 1$ is established. This completes the proof of Proposition 2.3. □

Now estimate the terms S_i from the right-hand side of (2.13). First, consider S_1. After differentiation, S_1 has a form of a sum of vectors whose coefficients look like

$$\begin{aligned} &S_{11} = \left(\frac{\partial}{\partial y_k}a_1\right)a_4a_2a_3(a_4-1), \qquad S_{12} = a_1\left(\frac{\partial}{\partial y_k}a_4\right)a_2a_3(a_4-1), \\ &S_{13} = a_1a_4\left(\frac{\partial}{\partial y_k}a_2\right)a_3(a_4-1), \qquad S_{14} = a_1a_4a_2\left(\frac{\partial}{\partial y_k}a_3\right)(a_4-1), \\ &S_{15} = a_1a_4a_2a_3\left(\frac{\partial}{\partial y_k}a_4\right). \end{aligned} \tag{2.22}$$

Using a continuity on y of a_i and the imbedding (2.17), we have

$$\begin{aligned} |S_{13}(t,y)|_0 &\le ||a_1||_{C(\Omega)}||a_4||_{C(\Omega)}\left|\frac{\partial}{\partial y_k}a_2\right|_0||a_3||_{C(\Omega)}||a_4-1||_{C(\Omega)} \\ &\le ||a_1||_{C(\Omega)}||a_4||_{C(\Omega)}|a_2|_1||a_3||_{C(\Omega)}||a_4-1||_{C(\Omega)}. \end{aligned} \tag{2.23}$$

It follows from estimates (2.14), (2.15), (2.16), and the form of a_2 that

$$|S_{13}(t,y)|_0 \le Mt_0^{1-1/q}(|\tilde{v}|_2+|w|_2). \tag{2.24}$$

Integration over $[0,t_0]$ yields

$$||S_{13}||_0 \le Mt_0^{1-1/q}(\|\tilde{v}\|_{0,2}+\|w\|_{0,2}) \le Mt_0^{1-1/q}(R+\delta). \tag{2.25}$$

In a similar way, we get

$$\begin{aligned} |S_{15}(t,y)|_0 &\le ||a_1||_{C(\Omega)}||a_4||_{C(\Omega)}||a_2||_{C(\Omega)}||a_3||_{C(\Omega)}\left|\frac{\partial}{\partial y_k}a_4\right|_0 \\ &\le Mt_0^{1-1/q}(R+\delta). \end{aligned} \tag{2.26}$$

It follows from here that $\|S_{15}\|_0 \le Mt_0^{1-1/q}(R+\delta)$. The similar estimates hold for other S_{1i}. It follows from the estimates of S_{1i} that

$$||S_1||_0 \le Mt_0^{1-1/q}(R+\delta). \tag{2.27}$$

The analogous estimates for S_2, S_3, S_4, and S_5 are established as in (2.27).

Inequality (2.11) for $i=1$ follows from the estimates of S_i.

Establish estimate (2.11) for $i=2$. Let $B_1=|\tilde{w}|I$, $B_2=G(\tilde{w})$, and $B_3=\tilde{w}^{-1}$. Let b_i denote an arbitrary coefficient of the matrix B_i. Differentiation of $K_2(w)$ gives an expressions of the form $Z_i=b_i(\partial b_j/\partial y_k)b_l$. First, establish the estimates

$$|B_i(t,y)| \le M(R), \tag{2.28}$$

$$\left|\frac{\partial}{\partial y_k}b_i(t,y)\right|_0 \le M(R)t_0^{1-1/q}. \tag{2.29}$$

Since $A_i=B_i$ for $i=2$, estimates (2.28), (2.29) for $i=2$ follow from estimates (2.14), (2.15), and (2.16) for $i=2$. Estimate (2.28) for $i=2$ follows from the boundedness of coefficients of the matrix $\tilde{w}$ and the continuity of G. Since $(\partial/\partial y_k)\tilde{w}=(\partial/\partial y_k)\hat{w}$, estimate (2.29) for $i=2$ follows from a continuity of the derivatives of G, boundedness of $\tilde{w}$ and estimate (2.18). Inequalities (2.28) and (2.29) are established.

This implies that $\|z_i\|_0 \le Mt_0^{1-1/q}(R+\delta)$ whence (2.11) for $i=2$ follows. Estimate (2.11) for $i=3$ is handled similarly.

It follows from estimates (2.11) and relation (2.10) that if R is large enough and t_0 is small enough, then the operator $\tilde{K}$ transforms $\mathfrak{M}$ into itself. This completes the proof of Lemma 2.2. $\square$

Lemma 2.4. *If t_0 is small enough, then there exists $\varkappa \in (0,1)$ such that for every $w^1, w^2 \in \mathfrak{M}$ the following inequality holds:*

$$\|\tilde{K}(w^1) - \tilde{K}(w^2)\|_{0,1} \leq \varkappa \|w^1 - w^2\|_{0,1}. \tag{2.30}$$

Proof. It follows from (2.2) that

$$\begin{aligned}
\|\tilde{K}(w^1) - \tilde{K}(w^2)\|_{0,1} &= \|L(K(w^1) - K(w^2))\|_{0,1} \\
&\leq M\sum_{i=1}^{3} \|L(K_i(w^1) - K_i(w^2))\|_{0,1} \\
&\equiv M\sum_{i=1}^{3} \|L\mathrm{Div}(Z_i(w^1, w^2))\|_{0,1} \\
&\leq M\sum_{i=1}^{3} \|Z_i(w^1, w^2)\|_{0,1}.
\end{aligned} \tag{2.31}$$

We show that

$$\|Z_i(w^1, w^2)\|_0 \leq M(R)t_0^{1-1/q}\|w^1 - w^2\|_{0,1}, \quad i = 1,2,3. \tag{2.32}$$

Denote by A_i^1 the matrices (2.12) for $w = w^1$, the similar matrices for $w = w^2$ are denoted by A_i^2. Elementary calculations yield

$$\begin{aligned}
Z_1(w^1, w^2) &= A_1^1A_4^1A_2^1A_3^1 - A_1^2A_4^2A_2^2A_3^2 - (A_2^1 - A_2^2) \\
&= A_1^1A_4^1A_2^1(A_3^1 - A_3^2) + A_1^1(A_4^1 - A_4^2)A_2^1A_3^2 + (A_1^1 - A_1^2)A_4^2A_2^1A_3^2 \\
&\quad + A_1^2(A_4^2 - I)(A_2^1 - A_2^2)A_3^2 + (A_1^2 - I)(A_2^1 - A_2^2)A_3^2 \\
&\quad + (A_2^1 - A_2^2)A_3^2 + A_1^1A_4^1A_5^1(A_3^1 - A_3^2) + A_1^1(A_4^1 - A_4^2)A_5^1A_3^2 \\
&\quad + (A_1^1 - A_1^2)A_4^2A_5^1A_3^2 + A_1^2A_4^2(A_5^1 - A_5^2)A_3^2.
\end{aligned} \tag{2.33}$$

We need the following relations to estimate the norms of $Z_1(w^1, w^2)$:

$$\|A_i^1 - A_i^2\|_0 \leq M(R)t_0^{1-1/q}\|w^1 - w^2\|_{0,1}, \quad i \neq 2, \tag{2.34}$$

$$\|A_i^k(t,y) - I\| \leq M(R)t_0^{1-1/q}, \quad i \neq 2. \tag{2.35}$$

In fact,

$$\begin{aligned}
|\hat{w}^1(t,y) - \hat{w}^2(t,y)|_0 &\leq \int_0^t |w_y^1(s,y) - w_y^2(s,y)|_0\, ds \\
&\leq Mt_0^{1-1/q}\|w^1 - w^2\|_{0,1}.
\end{aligned} \tag{2.36}$$

It is easy to see that

$$\begin{aligned} A_3^1 - A_3^2 &= (\tilde{w}^1)^{-1} - (\tilde{w}^2)^{-1} \\ &= (\tilde{w}^1)^{-1}(\tilde{w}^2 - \tilde{w}^1)(\tilde{w}^1)^{-1} \\ &= (\tilde{w}^1)^{-1}(\hat{w}^2 - \hat{w}^1)(\tilde{w}^1)^{-1}. \end{aligned} \tag{2.37}$$

It follows from (2.14) and (2.36) that

$$\begin{aligned} |A_3^1 - A_3^2|_0 &\le \max\left(||\tilde{w}^1(t,y)||_{C(\Omega)}, ||\tilde{w}^2(t,y)||_{C(\Omega)}\right) |\hat{w}^2(t,y) - \hat{w}^1(t,y)|_0 \\ &\le M(R+\delta)t_0^{1-1/q}||w^1 - w^2||_{0,1}. \end{aligned} \tag{2.38}$$

Integration over $[0,t_0]$ yields

$$||A_3^1(t,y) - A_3^2(t,y)||_0 \le M(R+\delta)t_0^{1-1/q}||w^1 - w^2||_{0,1}. \tag{2.39}$$

Estimate (2.34) for $i = 3$ is established.

Estimate (2.34) for $i = 1,4,5$ follows from estimate (2.34) for $i = 3$ in view of the facts that A_1^1 and A_1^2 are products of the determinants of matrices $(\partial/\partial y)\tilde{w}^1$ and $(\partial/\partial y)\tilde{w}^2$, A_4^1 and A_4^2 are adjoint to A_3^1 and A_3^2, respectively, A_5^1, A_5^2 are integrals of w_y^1, w_y^2. Inequality (2.34) is established.

Inequality (2.35) follows from inequality (2.15) and the definition of matrices A_i^1 and A_i^2.

Applying inequalities (2.34) and (2.35) to the right-hand side of (2.33), we get estimate (2.33) for $i = 1$. By this, the multipliers $A_i^1 - A_i^2$ are estimated in $L_q(Q)$-norm while others in $C(\Omega)$-norm.

For proving inequality (2.32) for Z_2, note that it can be rewritten in the form (2.33), where A_i^1, A_i^2, $i \ne 2$, are from (2.33) while $A_2^1 = G(w^1)$, $A_2^2 = G(w^2)$. It follows from the Newton-Leibnitz formula that

$$A_2^1 - A_2^2 = \int_0^1 G_z\left(\hat{w}^1(t,y) + \lambda(\hat{w}^2(t,y) - \hat{w}^1(t,y))\right)(\hat{w}^2(t,y) - \hat{w}^1(t,y))\,d\lambda. \tag{2.40}$$

Then, the boundedness of $\|\hat{w}^i(t,y)\|$ and continuity of $G_z(Z)$ imply that

$$||A_2^1 - A_2^2|| \le M(R)||w^1 - w^2||. \tag{2.41}$$

Bearing this in mind, inequality (2.32) for $i = 2$ as well as for $i = 3$ is proved in a similar way to the case where $i = 1$.

It follows from inequality (2.32) that

$$||\tilde{K}(w^1) - \tilde{K}(w^2)||_{0,1} \le M(R)t_0^{1-1/q}||w^1 - w^2||_{0,1}, \tag{2.42}$$

whence at small t_0 Lemma 2.4 follows.

It follows from Lemmas 2.1, 2.2, and 2.4 that at small t_0, the operator $\tilde{K}$ satisfies on $\mathfrak{M}$ the conditions of the fixed point theorem for contractions. This completes the proof of Theorem 1.3. □

Remark 2.5. It follows from the proof of Theorem 1.3 that if

$$f(t,y) \in S_0(R_0) = \{\phi : \phi \in L_q(Q),\ \|\phi\|_0 \le R_0\}, \tag{2.43}$$

then at sufficiently small t_0, depending only on R_0, problem (1.14), (1.15) has a unique solution v. Moreover, we may assume that $v \in \mathfrak{M}$ and $\|v\|_{1,2} \le R$, if R is large enough and t_0 is small enough.

If t_0 is those, then for each $f(t,y) \in S_0(R_0)$, problem (1.14), (1.15) has a unique solution. Denote by $\mathfrak{N}$ the operator assigning in accordance to function $F \in S_0(R_0)$ the solution v to problem (1.14), (1.15) so that

$$v = \mathfrak{N}(f). \tag{2.44}$$

3. Proof of Theorem 1.2

3.1. Proof of existence. Denoting $w = F(t, y + \int_0^t v(s,y)\,ds)$ and supposing w is known, we obtain (by the assumption $w \in S_0(R_0)$) that $v = \mathfrak{N}(w)$. Substituting $v = \mathfrak{N}(w)$ by w, we get

$$w = F\left(t, y + \int_0^t \mathfrak{N}(w)\,ds\right) \equiv \mathfrak{S}(w). \tag{3.1}$$

Thus, problem (1.14), (1.15) is reduced to the operator (3.7) in $L_q(Q_0)$.

Lemma 3.1. *Let R_0 be large enough and let t_0 be small enough. Then the operator $\mathfrak{S}$ transforms $S_0(R_0)$ into itself and is compact on $S_0(R_0)$.*

Proof. First, we show that operator $\mathfrak{S}$ transforms $S_0(R_0)$ into itself. Let $w \in S_0(R_0)$ and let $v = \mathfrak{N}(w)$ be a solution to problem (1.14), (1.15), $v \in \mathfrak{M}$. Consider the map of Ω_0 onto Ω_t

$$x = u(t,y) = x + \int_0^t v(s,y)\,ds. \tag{3.2}$$

It is easy to see that

$$||I + u_y(t,y)||_{C(\Omega_0)} \le \int_0^t ||v_y(s,y)||_{C(\Omega_0)}\,ds \le M_1 t_0^{1-1/q} \|v\|_{1,2} \le M_1 R t_0^{1-1/q}. \tag{3.3}$$

Assuming that t_0 is small, we have the inequalities

$$\frac{1}{2} \le ||u_y(t,y)||_{C(\Omega_0)}, \qquad ||U_x(t,x)||_{C(\Omega_t)} \le 2. \tag{3.4}$$

Making the change of variable (3.2), we get

$$\begin{aligned}
\|\mathfrak{S}(w)\|_0^q &= \int_0^{t_0}\int_{\Omega_0}\left\|F\left(t,y+\int_0^t v(s,y)\,ds\right)\right\|^q dy\,dt \\
&= \int_0^{t_0}\int_{\Omega_t}\|F(t,x)\|^q\,|U_x(t,x)|\,dx\,dt \\
&\le 2\|F\|_0^q.
\end{aligned} \tag{3.5}$$

Supposing that $R_0 \ge 2\|F\|_0$, we obtain that $\|\mathfrak{S}(w)\|_0 \le R_0$.

Establish now a compactness of $\mathfrak{S}(w)$. Specify a sequence of smooth finite functions $F_n(t,x)$ in Q such that $\|F - F_n\|_0 \to 0$ at $n \to \infty$. Without loss of generality, we may assume that $\|F_n\|_0 \le 2\|F\|_0$ for all n. Therefore, operators $\mathfrak{S}_n$, which for every F_n put in correspondence the solutions to problem (1.14), (1.15), transform $S_0(R_0)$ into itself. Note that the existence of such sequence $F_n(t,x)$ follows from the fact that the map $(t,x) \to (t,U(t,x))$ is a homeomorphism of Q onto Q_0 with the lateral surface $S_0 = [0,t_0]\times\Gamma_0$, $\Gamma_0 \in C^2$. Arranging a sequence of smooth finite functions $F_n(t,y)$, convergent in $L_q(Q_0)$ to $f_n(t,y) = F_n(t,\zeta(t,0y))$, we obtain the desired sequence $F_n(t,x) = f_n(t,\zeta(0,t,y))$.

We show that the operator $\mathfrak{S}_n$ is compact on $S_0(R_0)$. Let $w \in S_0(R_0)$ and $v = \mathfrak{N}(w)$. Then the function $u(t,y) = y + \int_0^t v(s,y)\,ds$ is continuously differentiable with respect to every variable and uniformly bounded together with its derivatives since $\|v\|_{1,2} \le R$. From this and the smoothness of F_n, there follow uniform boundedness and equicontinuity of the set of functions $F_n(t,y+\int_0^t v(s,y)\,ds)$ (or $\mathfrak{S}_n(S_0(R_0))$). This implies the compactness of $\mathfrak{S}_n$ in $L_q(Q_0)$.

Now we show that $\mathfrak{S}_n$ uniformly on $S_0(R_0)$ converges to $\mathfrak{S}$. In fact, making use of the change (3.2) by $v = \mathfrak{N}(w)$, $w \in S_0(R_0)$ and estimate (3.12), we have

$$\begin{aligned}
\|\mathfrak{S}_n(w) - \mathfrak{S}(w)\|_0 &= \left\|F_n\left(t,y+\int_0^t v(s,y)\,ds\right) - F\left(t,y+\int_0^t v(s,y)\,ds\right)\right\|_0 \\
&\le 2^{1/q}\|F_n(t,x) - F(t,x)\|_0.
\end{aligned} \tag{3.6}$$

The uniform convergence of $\mathfrak{S}_n$ follows from the convergence of F_n to F. The compactness of $\mathfrak{S}$ on $S_0(R_0)$ follows from the compactness of $\mathfrak{S}_n$ and uniform convergence of $\mathfrak{S}_n$ to $\mathfrak{S}$ on $S_0(R_0)$. This completes the proof of Lemma 3.1. □

Finally, the existence of a fixed point for $\mathfrak{S}$ follows from the Schauder fixed point theorem.

3.2. Proof of uniqueness. The proof of the uniqueness is accomplished by a series of auxiliary propositions for solutions to the Cauchy problem in noncylindrical domain.

Proposition 3.2. *Let* $V, V^1, V^2 \in W_q^{1,2}(Q)$, $\|V^1\|_{W_q^{1,2}(Q)}$, $\|V^2\|_{W_q^{1,2}(Q)}$, *and* $\|V\|_{W_q^{1,2}(Q)} \le R$. *Let* $z^i(\tau,t,x)$, $i = 1,2$ *and let* $z(\tau,t,x)$ *be solutions to the Cauchy problem*

$$z^i(\tau,t,x) = x + \int_t^\tau V^i(s, z(s;t,x))\,ds \tag{3.7}$$

and (1.2), respectively. Then at τ, $t \in [0,t_0]$ *the following inequalities hold:*

$$\begin{aligned}
||z_x(\tau,t,x)|| &\le M;\\
||z_x(\tau,t,x) - I||_{C(\Omega_t)} &\le \left| \int_t^\tau ||V_x(s,x)||_{C(\Omega_s)}\,ds \right|;\\
||z_{xx}(\tau,t,x)||_{L_q(\Omega_t)} &\le M\left| \int_t^\tau ||V(s,x)||_{W_q^2(\Omega_s)}\,ds \right|;\\
0 \le \left| z_x(\tau,t,x) \right| &\le \left| \int_t^\tau ||\operatorname{Spur} V_x(s,x)||_{C(\Omega_s)}\,ds \right|;\\
||z_x^1(\tau,t,x) - z_x^2(\tau,t,x)||_{L_r(\Omega_t)} &\le M\left| \int_t^\tau ||V^1(s,x) - V^2(s,x)||_{L_r(\Omega_t)}\,ds \right|,\\
&\qquad 1 \le r \le q;\\
||z_x^1(\tau,t,x) - z_x^2(\tau,t,x)||_{W_r^1(\Omega_t)} &\le M\left| \int_t^\tau ||V^1(s,x) - V^2(s,x)||_{W_r^1(\Omega_s)}\,ds \right|,\\
&\qquad 1 \le r \le q.
\end{aligned} \tag{3.8}$$

For the case of cylindrical domain $Q = Q_0$ and $V(t,x) = 0$ on Γ_0, Proposition 3.2 is proved in [10]. The proof in the general case is reduced to the case of cylindrical domain $Q = Q_0$ in the following way. In what follows $u(t,y) = x + \int_0^t v(s,y)\,ds$, where $v(t,x)$ satisfies the conditions of Theorem 1.1. The function $u(t,y)$ defines, at fixed $t \in [0,t_0]$ and small t_0, the homeomorphism $x = u(t,y)$ of domain Ω_0 onto Ω_t and we may assume that for $u(t,y)$ and for inverse to it map $y = U(t,x)$, estimates (3.4) hold. Let $z(\tau,t,x)$ be a solution to Cauchy problem (1.2). Then the function

$$\bar{z}(\tau,t,y) = U\big(\tau, z\big(\tau,t,u(t,y)\big)\big) \tag{3.9}$$

is the solution to Cauchy problem

$$\bar{z}(\tau,t,y) = y + \int_t^\tau \bar{V}(s, \bar{z}(s,t,y))\,ds, \tag{3.10}$$

where

$$\bar{V}(t,y) = -U_x\big(t,u(t,y)\big)v(t,y) + U_x\big(t,u(t,y)\big)V\big(t,u(t,y)\big). \tag{3.11}$$

Thus, the study of $z(\tau,t,x)$ in the case of noncylindrical domain Q is reduced to the study of a solution $\bar{z}(\tau,t,y)$ to Cauchy problem in the cylindrical domain $Q = Q_0$. Note that $\bar{V}(t,y) = 0$ on S_0. The statement of Proposition 3.2

for $z(\tau,t,y)$ follows from the statement of Proposition 3.2 for $\bar{z}(\tau,t,x)$ due to inequalities (3.2) and the relation $z(\tau,t,x)=u(\tau,\bar{z}(\tau,t,U(t,x)))$.

Proposition 3.3. *For each* $\Theta(x)\in W_q^1(\Omega_t)$,

$$||\Theta(x)||_{C(\Omega_t)} \leq M||\Theta(x)||_{W_q^1(\Omega_t)}, \quad M \neq M(t). \tag{3.12}$$

Proof. Let $\theta(y)=\Theta(u(t,y))$. Due to the change of variable $x=u(t,y)$ and the continuity of the imbedding (2.17) and (3.2) we have

$$||\Theta(x)||_{C(\Omega_t)} = ||\theta(y)||_{C(\Omega_0)} \leq M||\theta(y)||_{W_q^1(\Omega_0)} \leq M||\Theta(x)||_{W_q^1(\Omega_t)}, \quad M \neq M(t). \tag{3.13}$$

□

In what follows, we denote by $|\cdot|_1$ the norm in $W_2^1(\Omega_t)$. This norm depends on t; however, the situation is clear from the context. The norm in the dual to $\overset{o}{W}{}_2^1(\Omega_t)$ space $W_2^{-1}(\Omega_t)$ will be denoted by $|\cdot|_{-1}$.

Proposition 3.4. *Let* $a\in W_q^1(\Omega_t)$, $u\in W_q^1(\Omega_t)$. *Then, at every fixed t,*

$$|au|_{-1} \leq K|a|_0|u|_1, \quad K \neq K(t); \tag{3.14}$$

$$|au|_{-1} \leq K|a|_1|u|_{-1}, \quad K \neq K(t). \tag{3.15}$$

If $a\in W_q^1(\Omega_t)$, $u\in W_q^2(\Omega_t)$, *then*

$$|au_{xx}|_{-1} \leq K|a|_0|u|_2, \quad K \neq K(t). \tag{3.16}$$

Proof. Given $v\in W_2^1(\Omega_t)$, using the definition of $|\cdot|_{-1}$-norm and (2.17) we get

$$|au|_{-1} = \sup_v \frac{(au,v)}{|v|_1} \leq \sup_v \frac{(u,av)}{|av|_0} \sup_v \frac{|av|_0}{|v|_1} \leq M|a|_0|u|_1. \tag{3.17}$$

Here we have used the inequality $|av|_0 \leq M|a|_0|v|_1$. Inequality (3.14) is proved. Making use of the inequality $|av|_1 \leq M|a|_1|v|_1$, we have

$$|au|_{-1} = \sup_v \frac{(au,v)}{|v|_1} \leq \sup_v \frac{(u,av)}{|av|_1} \sup_v \frac{|av|_1}{|v|_1} \leq M|a|_1|u|_{-1}. \tag{3.18}$$

Inequality (3.15) is proved. Using (3.12), (3.14), (3.15), and the inclusion $u\in W_2^2(\Omega_t)$, we get

$$\begin{aligned} |au_{xx}|_{-1} &\leq |(au_x)_x|_{-1} + |a_xu_x|_{-1} \\ &\leq |au_x|_{-1} + |a_x|_{-1}|u_x|_1 \leq |a|_0|u_x|_2. \end{aligned} \tag{3.19}$$

Inequality (3.16) is proved. □

Proposition 3.5. *The following inequalities hold:*

$$\|R(V^i)\|, \|R^{-1}(V^i)\| \le M, \quad (t,x) \in Q, \ i = 1,2; \tag{3.20}$$

$$|R(V^i)|_1 \le Mt_0^{1/2}, \quad i = 1,2; \tag{3.21}$$

$$\|R(V^i) - I\|_{C(\Omega_t)} \le Mt_0^{1-1/q}; \tag{3.22}$$

$$|R(V^1) - R(V^2)|_0 \le Mt_0^{1/2}\|V^1 - V^2\|_{0,1}; \tag{3.23}$$

$$|B(V^1) - B(V^2)|_0 \le Mt_0^{1/2}\|V^1 - V^2\|_{0,1}. \tag{3.24}$$

The above proposition follows from Proposition 3.2 and (1.8).

Consider now the linear problem

$$\begin{aligned} V_t - \hat{\triangle} V &= \Xi(t,y), && (t,y) \in Q; \\ V(0,y) &= 0, && y \in \Omega_0; \\ V(t,y) &= 0, && (t,y) \in \Gamma. \end{aligned} \tag{3.25}$$

In what follows, $\|\cdot\|_{0,-1}$ stands for the norm in $L_2(0,t_0 : W_2^{-1}(\Omega_t))$.

Lemma 3.6. *Let $V \in W_q^{1,2}(Q)$, $q \in (0,+\infty)$, and V satisfy (3.25). Then*

$$\|V\|_{0,1} \le M\|\Xi\|_{0,-1}, \tag{3.26}$$

$$\sup_t |V(t,y)|_0 \le M\|\Xi\|_0. \tag{3.27}$$

Proof. Multiplying (3.25) in $L_2(\Omega_t)$ by $V(t,x)$, we get

$$(V_t, V) + (\hat{\triangle} V, V) = (\Xi, V). \tag{3.28}$$

Hereinafter, $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega_t)$ while $(\cdot)$ denotes the scalar product in $\mathbb{R}^n$. It is easy to see that

$$(V_t, V) = \frac{1}{2}\int_{\Omega_t} \frac{d}{dt}(V\cdot V)\,dx = \frac{1}{2}\int_{\Omega_t} \Psi_t(t,x)\,dx, \quad \Psi(t,x) = (V\cdot V). \tag{3.29}$$

Let $V^1 \in W_q^{1,2}(Q)$ be a solution to problem (1.9), (1.10). Making use of the change of variable

$$x = U(t,y), \qquad u(t,y) = y + \int_0^t V^1(s,us,y)\,ds, \tag{3.30}$$

and differentiating the integral by parameter, we have

$$\begin{aligned}
\int_{\Omega_t} \Psi_t(t,x)\,dx &= \int_{\Omega_0} \Psi_t(t,u(t,y))I(t,y)\,dy \\
&= \frac{d}{dt}\int_{\Omega_0} \Psi(t,u(t,y))I(t,y)\,dy \\
&\quad - \int_{\Omega_0} (\nabla_x \Psi(t,u(t,y)))v^1(t,y)I(t,y)\,dy \\
&\quad - \int_{\Omega_0} \Psi(t,u(t,y))\frac{d}{dt}I(t,y)\,dy.
\end{aligned} \tag{3.31}$$

Here $v^1(t,y) = V^1(t,u(t,y))$, $I(t,y) = |u_y(t,y)|$. It follows from the Ostrogradskii-Liouville formula $I(t,y) = \exp(\int_0^t \operatorname{div} V^1(s,u(s,y))\,ds)$ that

$$\frac{d}{dt}I(t,y) = I(t,y)\frac{d}{dt}\ln I(t,y) = \operatorname{div} V^1(t,u(t,y))I(t,y). \tag{3.32}$$

Making use of the inverse to (3.30) change of variable, using (3.32) and integrating in the second integral in parts, we obtain

$$\begin{aligned}
\int_{\Omega_t} \Psi_t(t,x)\,dx &= \frac{d}{dt}\int_{\Omega_t} \Psi(t,x)\,dx - \int_{\Omega_t} (\nabla_x \Psi(t,x))V^1(t,x)\,dx \\
&\quad - \int_{\Omega_t} \Psi(t,x)\operatorname{div} V^1(t,x)\,dx \\
&= \frac{d}{dt}\int_{\Omega_t} \Psi(t,x)\,dx \\
&= \frac{d}{dt}|V(t,x)|_0^2.
\end{aligned} \tag{3.33}$$

Consider the second term in (3.28). Integrating by parts, we have

$$\begin{aligned}
-(\hat{\triangle} V, V) &= \frac{1}{2}(\nabla V + (\nabla V)^*, \nabla V) \\
&= \frac{1}{4}(\nabla V + (\nabla V)^*, \nabla V + (\nabla V)^*) \\
&= \frac{1}{4}|\nabla V + (\nabla V)^*|_0^2.
\end{aligned} \tag{3.34}$$

It follows from the Corn inequality [5, Theorem 2.2, page 30] that

$$m_1|V|_1 \le |\nabla V + (\nabla V)^*|_0 \le m_2|V|_1. \tag{3.35}$$

Thus,

$$M_1|V|_1^2 \le -(\hat{\triangle} V, V) \le M_2|V|_1^2. \tag{3.36}$$

Next, using, for small $\epsilon > 0$, the inequality

$$(\Xi, V) \le |\Xi|_{-1}|V|_1 \le C(\epsilon)|\Xi|_{-1}^2 + \epsilon|V|_1^2 \tag{3.37}$$

and relations (3.28), (3.33), (3.36), and (3.37), we have

$$\frac{d}{dt}\left|V(t,x)\right|_0^2+|V|_1^2 \le M|\Xi|_{-1}^2. \tag{3.38}$$

Integrating the last inequality over $[0,t_0]$, we obtain inequality (3.26). Making use of the inequality $(\Xi,V)\le|\Xi|_0|V|_0$ and relations (3.28), (3.37), and (3.33), we get $(d/dt)|V(t,x)|_0 \le |\Xi|_0$. Integrating the last inequality on t on interval $[0,t_0]$, we obtain inequality (3.27). Thus, Lemma 3.6 is proved. □

Now we proceed directly to the proof of the uniqueness. Arguing by contradiction, suppose that there are two solutions V^1 and V^2 of (1.9), (1.10). Then, for $V=V^1-V^2$, we have

$$\begin{aligned}
V_t-\hat{\triangle}V &= -\sum_i^n V_i\frac{\partial V^1}{\partial x_i}-\sum_i^n V_i^2\frac{\partial V}{\partial x_i}+\left(1-R^{-1}(V^1)\right)\hat{\triangle}V\\
&\quad +R^{-1}(V^1)R^{-1}(V^2)\left(R^{-1}(V^1)-R^{-1}(V^2)\right)\hat{\triangle}V^2\\
&\quad +R^{-1}(V^1)R^{-1}(V^2)\left(R^{-1}(V^1)-R^{-1}(V^2)\right)\operatorname{Div}B(V^1)\\
&\quad +R^{-1}(V^2)\operatorname{Div}B(V^1)-\operatorname{Div}B(V^2)\\
&\equiv \sum_1^6 Z_i.
\end{aligned} \tag{3.39}$$

Without loss of generality, we set above that $G(Z)\equiv Z$. Using Lemma 3.6, we obtain

$$\|V\|_{0,1}\le M\sum_1^6 \|Z_i\|_{0,-1}. \tag{3.40}$$

Estimate $\|Z_i\|_{0,-1}$. Using (3.14), we have

$$\left|Z_1\right|_{-1}\le M|V|_0\left|V_x^1\right|_1\le|V|_0\left|V^1\right|_2. \tag{3.41}$$

Since it follows from the Newton-Leibnitz formula that

$$\left|V(t,x)\right|_0\le\int_0^{t_0}\left|V_t(s,x)\right|_0 ds\le M{t_0}^{1/2}\|V\|_{1,2}, \tag{3.42}$$

then we have

$$\begin{aligned}
\|Z_1\|_{0,-1}&\le M\left(\int_0^{t_0}|V|_0^2\left|V_x^1\right|_1^2 ds\right)^{1/2}\\
&\le \sup_t\left|V(t,x)\right|_0^{2/q}\left(\int_0^{t_0}\left|V(s,x)\right|_0^{2(q-2)/q}\left|V^1(s,x)\right|_2^2 ds\right)^{1/2}.
\end{aligned} \tag{3.43}$$

Making use of the Hölder inequality, we get

$$\begin{aligned} ||Z_1||_{0,-1} &\le M t_0^{1/2q}\left(\int_0^{t_0} |V_t(s,x)|_0^2\,ds\right)^{(q-2)/2q}\left(\int_0^{t_0} |V^1(s,x)|_0^2\,ds\right)^{1/2q} \\ &\le M {t_0}^{1/2q}\|V\|_{0,1}^{(q-2)/q}||V^1||^{q/2}_{W_q^{1,2}(Q)}. \end{aligned} \tag{3.44}$$

Similarly the term Z_2 is handled. For other Z_i, the following inequalities hold:

$$||Z_i||_{0,-1} \le M {t_0}^{1/2}\|V\|_{0,1}, \quad i = 3,\dots,6. \tag{3.45}$$

Prove (3.45) for $I = 3$. Using (3.16) and (3.20), we get

$$\begin{aligned} |Z_3|_{-1} &\le M|R^{-1}(V^1)R^{-1}(V^2)(R^{-1}(V^1) - R^{-1}(V^2))|_0|V^2|_2 \\ &\le M|R^{-1}(V^1) - R^{-1}(V^2)|_0|V^2|_2. \end{aligned} \tag{3.46}$$

Due to (3.23), it follows from here that

$$||Z_3||_{0,-1} \le M\sup_t |R^{-1}(V^1) - R^{-1}(V^2)|_0||V^2||_{0,2} \le M{t_0}^{1/2}\|V\|_{0,1}. \tag{3.47}$$

Making use of (3.15), (3.20), (3.21), and (3.24), we obtain

$$\begin{aligned} ||Z_6||_{0,-1} &\le M\left(\int_0^{t_0} |R^{-1}(V^2)|_1^2|\operatorname{Div}(B(V^2) - B(V^2))|_{-1}^2\,ds\right)^{1/2} \\ &\le M\left(\int_0^{t_0} |R(V^2)|_1^2|\operatorname{Div}(B(V^2) - B(V^2))|_0^2\,ds\right)^{1/2} \\ &\le M{t_0}^{1/2}\|V\|_{0,1}. \end{aligned} \tag{3.48}$$

It follows from (3.40), (3.44), and (3.45) that

$$\|V\|_{0,1} \le M{t_0}^{1/2q}\|V\|_{0,1}^{(q-2)/q} + M{t_0}^{1/2}\|V\|_{0,1}. \tag{3.49}$$

Choosing t_0 sufficiently small, we have that $\|V\|_{0,1} \le 0$. Hence, $V = 0$. The uniqueness of the solution is established. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

The proof of Theorem 1.3 follows from the proof of Theorem 1.2. We should make the change of variable $x = z(t,0,y)$—$z(\tau,t,x)$ is a solution to Cauchy problem (1.2), where V is a solutions to problem (1.14), (1.15), and (1.16)—and then make use of uniform boundedness from above and from below of the Jacobian of $z(\tau,t,x)$ which implies an isomorphism of Sobolev spaces on Q onto Sobolev spaces on Q_10.

5. Remarks

It follows from the definition of the solution and the form of the left-hand side of (1.14) that necessarily $f \in L_q(Q_0)$. Rewrite problem (1.14), (1.15), and (1.16) in the form

$$\begin{aligned} v_t - \mu\hat{\triangle} v &= w; \\ v(0,y) = V^0(y), &\quad y \in \bar{\Omega}, \\ v(t,y) = \Upsilon_1(t,y), &\quad (t,y) \in S, \end{aligned} \tag{5.1}$$

where w is defined as in the proof of Theorem 1.3. It follows from (1.14) and the definition of the solution that necessarily $w \in L_q(Q_0)$. From the properties of the solutions of the linear problem (5.1) and from [4, Chapter 4, page 388], we get that necessarily $V^0 \in W_{q,0}^{2-2/q}(\Omega_0)$, $\Upsilon \in W$. Note also that the proofs of Theorems 1.2 and 1.3 are carried out without an assumption about any smoothness of F (cf. [14]).

Acknowledgment

This work was partially supported by the Russian Foundation for Basic Research grant 01-01-00425.

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