

# FINITE DIFFERENCE SCHEMES WITH MONOTONE OPERATORS

N. C. APREUTESEI

*Received 13 October 2003 and in revised form 10 December 2003*

*To the memory of my mother, Liliana*

Several existence theorems are given for some second-order difference equations associated with maximal monotone operators in Hilbert spaces. Boundary conditions of monotone type are attached. The main tool used here is the theory of maximal monotone operators.

## 1. Introduction

In [1, 2], the authors proved the existence of the solution of the boundary value problem

$$p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), \quad \text{a.e. on } [0, T], \quad T > 0, \quad (1.1)$$

$$u'(0) \in \alpha(u(0) - a), \quad u'(T) \in -\beta(u(T) - b), \quad (1.2)$$

where  $A : D(A) \subseteq H \rightarrow H$ ,  $\alpha : D(\alpha) \subseteq H \rightarrow H$ , and  $\beta : D(\beta) \subseteq H \rightarrow H$  are maximal monotone operators in the real Hilbert space  $H$  (satisfying some specific properties),  $a, b$  are given elements in the domain  $D(A)$  of  $A$ ,  $f \in L^2(0, T; H)$ , and  $p, r : [0, T] \rightarrow \mathbb{R}$  are continuous functions,  $p(t) \geq k > 0$  for all  $t \in [0, T]$ .

Particular cases of this problem were considered before in [9, 10, 12, 15, 16]. If  $p \equiv 1$ ,  $r \equiv 0$ ,  $f \equiv 0$ ,  $T = \infty$ , and the boundary conditions are  $u(0) = a$  and  $\sup\{\|u(t)\|, t \geq 0\} < \infty$  instead of (1.2), the solution  $u(t)$  of (1.1), (1.2) defines a semigroup of nonlinear contractions  $\{S_{1/2}(t), t \geq 0\}$  on the closure  $\overline{D(A)}$  of  $D(A)$  (see [9, 10]). This semigroup and its infinitesimal generator  $A_{1/2}$  have some important properties (see [9, 10, 11, 12]).

A discretization of (1.1) is  $p_i(u_{i+1} - 2u_i + u_{i-1}) + r_i(u_{i+1} - u_i) \in k_i Au_i + g_i$ ,  $i = \overline{1, N}$ , where  $N$  is a given natural number,  $p_i, r_i, k_i > 0$ ,  $g_i \in H$ . This leads to the finite difference scheme

$$(p_i + r_i)u_{i+1} - (2p_i + r_i)u_i + p_i u_{i-1} \in k_i Au_i + g_i, \quad i = \overline{1, N}, \quad (1.3)$$

$$u_1 - u_0 \in \alpha(u_0 - a), \quad u_{N+1} - u_N \in -\beta(u_{N+1} - b), \quad (1.4)$$

where  $a, b \in H$  are given,  $(p_i)_{i=\overline{1, N}}$ ,  $(r_i)_{i=\overline{1, N}}$ , and  $(k_i)_{i=\overline{1, N}}$  are sequences of positive numbers, and  $(g_i)_{i=\overline{1, N}} \in H^N$ .

In this paper, we study the existence and uniqueness of the solution of problem (1.3), (1.4) under various conditions on  $A$ ,  $\alpha$ , and  $\beta$ .

The case  $p_i \equiv 1$ ,  $r_i \equiv 0$ ,  $g_i \equiv 0$  was discussed in [14] for the boundary conditions  $u_0 = a$  and  $u_{N+1} = b$ . These boundary conditions can be seen as a particular case of (1.4) with  $\alpha = \beta = \partial j$  (the subdifferential of  $j$ ), where  $j : H \rightarrow \mathbb{R}$  is the lower-semicontinuous, convex, and proper function:

$$j(x) = \begin{cases} 0, & x = 0, \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.5)$$

In [6, 8, 13, 14], one studies the existence, uniqueness, and asymptotic behavior of the solution of the difference equation

$$(p_i + r_i)u_{i+1} - (2p_i + r_i)u_i + p_i u_{i-1} \in k_i A u_i + g_i, \quad i \geq 1, \quad (1.6)$$

( $p_i \equiv 1$ ,  $r_i \equiv 0$  in [13, 14] and the general case in [6, 8]), subject to the boundary conditions

$$u_0 = a, \quad \sup_{i \geq 0} \|u_i\| < \infty. \quad (1.7)$$

Here  $\|\cdot\|$  is the norm of  $H$ . In [7], the author establishes the existence for problem (1.3), (1.4) under the hypothesis that  $A$  is also strongly monotone.

Other classes of difference or differential inclusions in abstract spaces are presented in [3, 4, 5].

In Section 2, we recall some notions and results that we need to show our main existence theorems. They are stated in Section 3 and represent the discrete version of some results obtained in [1, 2] for the continuous case.

## 2. Preliminary results

In this section, we recall some fundamental elements on nonlinear analysis we need in this paper.

If  $H$  is a real Hilbert space with the scalar product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ , then the operator  $A \subseteq H \times H$  (with the domain  $D(A)$  and the range  $R(A)$ ) is called a *monotone operator* if  $(x - x', y - y') \geq 0$  for all  $x, x' \in D(A)$ ,  $y \in Ax$ , and  $y' \in Ax'$ . The monotone operator  $A \subseteq H \times H$  is said to be *maximal monotone* if it is not properly enclosed in a monotone operator. A basic result of Minty (see [11, Theorem 1.2, page 9]) asserts that  $A$  is maximal monotone if and only if  $A$  is monotone and the range of  $A + \lambda I$  is the whole space  $H$  for all  $\lambda > 0$  (or equivalently, for only one  $\lambda_0 > 0$ ). It is also known that a maximal monotone and coercive operator  $A$  is surjective, that is, its range  $R(A)$  is  $H$ .

For all  $x \in D(A)$ , we denote by  $A^0 x$  the element of least norm in  $Ax$ :

$$\|A^0 x\| = \inf \{\|y\|, y \in Ax\}. \quad (2.1)$$

If  $A$  is maximal monotone and  $\|A^0 x\| \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ , then  $A$  is surjective.

The operator  $A \subseteq H \times H$  (possibly multivalued) is said to be *one to one* if  $(Ax_1) \cap (Ax_2) \neq \emptyset$  (with  $x_1, x_2 \in D(A)$ ) implies  $x_1 = x_2$ .

If  $A$  and  $B$  are maximal monotone in  $H$  and their domains satisfy the condition  $(\text{int}D(A)) \cap D(B) \neq \emptyset$ , then  $A + B$  is maximal monotone (see [11, Theorem 1.7, page 46]). If  $A : D(A) \subseteq H \rightarrow H$  is maximal monotone, then  $A$  is *demiclosed*, that is, from  $[x_n, y_n] \in A$ ,  $x_n \rightharpoonup x$  and  $y_n \rightarrow y$ , then  $[x, y] \in A$ . Here and everywhere below, we denote by “ $\rightharpoonup$ ” the weak convergence and by “ $\rightarrow$ ” the strong convergence in  $H$ .

For every maximal monotone operator  $A$  and the scalar  $\lambda > 0$ , we may consider the single-valued and everywhere-defined operators  $J_\lambda$  and  $A_\lambda$ , namely,  $J_\lambda = (I + \lambda A)^{-1}$  and  $A_\lambda = (I - J_\lambda)/\lambda$ . They are called the *resolvent* and the *Yosida approximation* of  $A$ , respectively. Obviously, we have  $J_\lambda x + \lambda A_\lambda x = x$  for all  $x \in H$  and for all  $\lambda > 0$ . Properties of these operators can be found in, for example, [11, Proposition 1.1, page 42] or [11, Proposition 3.2, page 73].

Recall now another result concerning the sum of two maximal monotone operators (see [11, Theorem 3.6, page 82]).

**THEOREM 2.1.** *If  $A : D(A) \subseteq H \rightarrow H$  and  $B : D(B) \subseteq H \rightarrow H$  are maximal monotone operators in  $H$  such that  $D(A) \cap D(B) \neq \emptyset$  and  $\langle y, A_\lambda x \rangle \geq 0$  for all  $[x, y] \in B$  and for all  $\lambda > 0$ , then  $A + B$  is maximal monotone.*

We end this section with some remarks on problem (1.3), (1.4). Denoting

$$\theta_i = \frac{p_i}{p_i + r_i}, \quad c_i = \frac{k_i}{p_i + r_i}, \quad f_i = \frac{g_i}{p_i + r_i}, \quad i = \overline{1, N}, \quad (2.2)$$

problem (1.3), (1.4) becomes

$$\begin{aligned} u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1} &\in c_i A u_i + f_i, \quad i = \overline{1, N}, \\ u_1 - u_0 &\in \alpha(u_0 - a), \quad u_{N+1} - u_N \in -\beta(u_{N+1} - b). \end{aligned} \quad (2.3)$$

If  $p_i, r_i, k_i > 0$ ,  $i = \overline{1, N}$ , then  $\theta_i \in (0, 1)$  and  $c_i > 0$  for all  $i = \overline{1, N}$ .

Let  $(a_i)_{i=\overline{1, N}}$  be the finite sequence given by

$$a_0 = 1, \quad a_i = \frac{1}{\theta_1 \cdots \theta_i}, \quad i = \overline{1, N}, \quad (2.4)$$

and let  $\mathcal{L}$  be the product space  $H^N = H \times \cdots \times H$  ( $N$  factors) endowed with the scalar product

$$\langle (u_i)_{i=\overline{1, N}}, (v_i)_{i=\overline{1, N}} \rangle = \sum_{i=1}^N a_i (u_i, v_i). \quad (2.5)$$

It is clear that  $H^N$  and  $\mathcal{L}$  coincide as sets and their norms are equivalent. Observe that

$$a_i \theta_i = a_{i-1}, \quad i = \overline{1, N}. \quad (2.6)$$

Consider the operator  $B$  in  $H^N \times H^N$ :

$$\begin{aligned} B\left((u_i)_{i=\overline{1, N}}\right) &= (-u_{i+1} + (1 + \theta_i)u_i - \theta_i u_{i-1})_{i=\overline{1, N}}, \\ D(B) &= \left\{ (u_i)_{i=\overline{1, N}} \in H^N, u_1 - u_0 \in \alpha(u_0 - a), u_{N+1} - u_N \in -\beta(u_{N+1} - b) \right\}. \end{aligned} \quad (2.7)$$

## 14 Finite difference schemes with monotone operators

This operator is not necessarily monotone in  $H^N$ , but we have the following auxiliary result (see [7, Proposition 2.1]).

**PROPOSITION 2.2.** *The operator  $B$  given above is maximal monotone in  $\mathcal{L}$ .*

Recall here an existence theorem from [7], which we use in the sequel.

**THEOREM 2.3.** *Assume that  $A$ ,  $\alpha$ , and  $\beta$  are maximal monotone operators in  $H$  with  $0 \in D(A) \cap D(\alpha) \cap D(\beta)$ ,  $A$  is also strongly monotone and*

$$(A_\lambda x - A_\lambda y, z) \geq 0 \quad (2.8)$$

*for all  $z \in \alpha(x - y)$  (with  $x - y \in D(\alpha)$ ) and for all  $z \in \beta(x - y)$  (with  $x - y \in D(\beta)$ ). If  $\theta_i \in (0, 1)$ ,  $c_i > 0$ ,  $f_i \in H$ ,  $i = \overline{1, N}$ , and  $a, b \in H$ , then problem (2.3) has a unique solution  $(u_i)_{i=\overline{1, N}} \in D(A)^N$ .*

### 3. Existence theorems

Let  $H$  be a real Hilbert space with the norm  $\|\cdot\|$  and the scalar product  $(\cdot, \cdot)$ . Consider the maximal monotone operators  $A : D(A) \subseteq H \rightarrow H$ ,  $\alpha : D(\alpha) \subseteq H \rightarrow H$ , and  $\beta : D(\beta) \subseteq H \rightarrow H$  satisfying the properties

$$0 \in D(A) \cap D(\alpha) \cap D(\beta), \quad 0 \in \alpha(0) \cap \beta(0), \quad (3.1)$$

$$(A_\lambda x - A_\lambda y, z) \geq 0 \quad \forall z \in \alpha(x - y) \text{ with } x - y \in D(\alpha), \quad (3.2)$$

$$(A_\lambda x - A_\lambda y, z) \leq 0 \quad \forall z \in -\beta(x - y) \text{ with } x - y \in D(\beta). \quad (3.3)$$

Consider the difference inclusion (1.3), (1.4). As we have already discussed, problem (1.3), (1.4) has the equivalent form (2.3).

We first study the existence of the solution to problem (1.3), (1.4) in the case  $a = b = 0$ , supposing that

$$(A_\lambda x, z) \geq 0 \quad \forall z \in \alpha(x) \text{ with } x \in D(\alpha), \quad z \in \beta(x) \text{ with } x \in D(\beta), \quad (3.4)$$

and

$$R(\alpha) \text{ is bounded,} \quad \|\beta^0(x)\| \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty, \quad (3.5)$$

or

$$R(\beta) \text{ is bounded,} \quad \|\alpha^0(x)\| \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty. \quad (3.6)$$

**THEOREM 3.1.** *Let  $A$ ,  $\alpha$ , and  $\beta$  be maximal monotone operators in the real Hilbert space  $H$  such that (3.1), (3.4), and (3.5) or (3.6) hold. If  $p_i, r_i, k_i > 0$ ,  $i = \overline{1, N}$ , and  $(g_i)_{i=\overline{1, N}} \in H^N$ , then the boundary value problem*

$$\begin{aligned} (p_i + r_i)u_{i+1} - (2p_i + r_i)u_i + p_i u_{i-1} &\in k_i A u_i + g_i, \quad i = \overline{1, N}, \\ u_1 - u_0 &\in \alpha(u_0), \quad u_{N+1} - u_N \in -\beta(u_{N+1}), \end{aligned} \quad (3.7)$$

has at least one solution  $(u_i)_{i=\overline{1,N}} \in D(A)^N$ . The solution is unique up to an additive constant. If  $A$  or  $\alpha$  is one to one, then the solution is unique. If  $A$  is, in addition, strongly monotone, then again uniqueness is obtained.

*Proof.* We use the form (2.3) of the problem (3.7), where  $a = b = 0$ . By Proposition 2.2, we know that the operator

$$\begin{aligned} B\left((u_i)_{i=\overline{1,N}}\right) &= (-u_{i+1} + (1 + \theta_i)u_i - \theta_i u_{i-1})_{i=\overline{1,N}}, \\ D(B) &= \left\{ (u_i)_{i=\overline{1,N}} \in H^N, u_1 - u_0 \in \alpha(u_0), u_{N+1} - u_N \in -\beta(u_{N+1}) \right\} \end{aligned} \quad (3.8)$$

is maximal monotone in  $\mathcal{L}$ . Denote by  $|\cdot|$  the norm in  $\mathcal{L}$ . We show that

$$\left| B\left((u_i)_{i=\overline{1,N}}\right) \right| \longrightarrow \infty \quad \text{as} \quad \left| (u_i)_{i=\overline{1,N}} \right| \longrightarrow \infty. \quad (3.9)$$

Suppose by contradiction that  $(u_i^n)_{i=\overline{1,N}} \in D(B)$  such that  $|(u_i^n)_{i=\overline{1,N}}| \rightarrow \infty$  as  $n \rightarrow \infty$  and  $|B((u_i^n)_{i=\overline{1,N}})| \leq C_1$ . If  $(a_i)_{i=\overline{1,N}}$  is the sequence given in (2.4), this means that

$$\sum_{i=1}^N a_i \|u_i^n\|^2 \longrightarrow \infty, \quad \sum_{i=1}^N a_i \|u_{i+1}^n - u_i^n - \theta_i(u_i^n - u_{i-1}^n)\|^2 \leq C_1. \quad (3.10)$$

Assume that (3.5) holds. By the boundary conditions in (3.7), we obtain that  $u_1^n - u_0^n$  is bounded, say  $\|u_1^n - u_0^n\| \leq C_2$ , for all  $n \in \mathbb{N}$  and

$$\|u_{N+1}^n - u_N^n\| \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty \quad \text{if} \quad \|u_{N+1}^n\| \longrightarrow \infty. \quad (3.11)$$

The equality  $a_i(u_{i+1}^n - u_i^n) = u_1^n - u_0^n + \sum_{k=1}^i [a_k(u_{k+1}^n - u_k^n) - a_{k-1}(u_k^n - u_{k-1}^n)]$  implies that  $\|a_i(u_{i+1}^n - u_i^n)\| \leq C_2 + C_3 |B((u_i)_{i=\overline{1,N}})|$  and in view of (3.10), we get  $\|a_i(u_{i+1}^n - u_i^n)\| \leq C_4$ ,  $i = \overline{1, N}$ ,  $n \in \mathbb{N}$ . In particular,  $\|a_N(u_{N+1}^n - u_N^n)\| \leq C_4$  for all  $n \in \mathbb{N}$  and from (3.11), we infer that  $\|u_{N+1}^n\| \leq C_5$  for all  $n \in \mathbb{N}$ .

Using the boundedness of  $u_{N+1}^n$  and  $a_k(u_{k+1}^n - u_k^n)$  and the identity

$$u_i^n = u_{N+1}^n - \sum_{k=i}^N (u_{k+1}^n - u_k^n), \quad i = \overline{1, N}, \quad (3.12)$$

one arrives at  $\|u_i^n\| \leq C_6$ , hence  $\sum_{i=1}^N a_i \|u_i^n\|^2 \leq C_7$  for all  $n \in \mathbb{N}$ . But this is in contradiction with (3.10) and therefore (3.9) is true. This shows that  $B$  is coercive.

Next we show that

$$\left\langle B\left((u_i)_{i=\overline{1,N}}\right), (A_\lambda u_i)_{i=\overline{1,N}} \right\rangle \geq 0 \quad \forall (u_i)_{i=\overline{1,N}} \in D(B), \lambda > 0. \quad (3.13)$$

Indeed,

$$\begin{aligned}
 \langle B((u_i)_{i=\overline{1,N}}), (A_\lambda u_i)_{i=\overline{1,N}} \rangle &= - \sum_{i=1}^N [a_i(u_{i+1} - u_i, A_\lambda u_i) - a_{i-1}(u_i - u_{i-1}, A_\lambda u_{i-1})] \\
 &\quad + \sum_{i=1}^N a_{i-1}(u_i - u_{i-1}, A_\lambda u_i - A_\lambda u_{i-1}) \\
 &\geq -a_N(u_{N+1} - u_N, A_\lambda u_N) + (u_1 - u_0, A_\lambda u_0).
 \end{aligned} \tag{3.14}$$

Hypothesis (3.4) for  $x = u_0$  and  $z = u_1 - u_0$  gives us  $(u_1 - u_0, A_\lambda u_0) \geq 0$ , while (3.4) for  $x = u_{N+1}$  and  $z = -u_{N+1} + u_N$  implies that  $-(u_{N+1} - u_N, A_\lambda u_N) = (u_{N+1} - u_N, A_\lambda u_{N+1} - A_\lambda u_N) - (u_{N+1} - u_N, A_\lambda u_{N+1}) \geq 0$ . Thus, by (3.14), inequality (3.13) follows.

Let  $\mathcal{A} : D(A)^N \rightarrow H^N$  be the operator

$$\mathcal{A}((u_i)_{i=\overline{1,N}}) = (c_1 v_1, \dots, c_N v_N), \quad v_i \in Au_i, \quad u_i \in D(A), \quad i = \overline{1, N}. \tag{3.15}$$

Since  $(0, \dots, 0) \in D(\mathcal{A}) \cap D(B)$  and (3.13) takes place, we deduce with the aid of Theorem 2.1 and Proposition 2.2 the maximal monotonicity of  $B + \mathcal{A}$  in  $\mathcal{L}$ . Next, we can easily show that  $\langle B((u_i)_{i=\overline{1,N}}), \mathcal{A}((u_i)_{i=\overline{1,N}}) \rangle \geq 0$ , so  $|(B + \mathcal{A})(u_i)_{i=\overline{1,N}}| \geq |B((u_i)_{i=\overline{1,N}})|$ , and from (3.9), one obtains the coercivity of  $B + \mathcal{A}$ . This shows that  $B + \mathcal{A}$  is surjective, that is, for all  $(f_i)_{i=\overline{1,N}} \in H^N$ , there exists  $(u_i)_{i=\overline{1,N}} \in D(\mathcal{A}) \cap D(B)$  such that  $(B + \mathcal{A})((u_i)_{i=\overline{1,N}}) = (-f_i)_{i=\overline{1,N}}$ . But this is the abstract form of (3.7). Thus the existence is proved.

We show now that the difference of the two solutions  $(u_i)_{i=\overline{1,N}}$  and  $(v_i)_{i=\overline{1,N}}$  of (3.7) is a constant. Put  $w_i = u_i - v_i$ ,  $i = \overline{0, N+1}$ . Subtracting the corresponding equations of (2.3) for  $u_i$  and  $v_i$ , multiplying by  $a_i w_i$ , and summing from  $i = 1$  to  $i = N$ , one arrives with the aid of the monotonicity of  $A$  at

$$\sum_{i=1}^N [a_i(w_{i+1} - w_i, w_i) - a_i \theta_i(w_i - w_{i-1}, w_i)] \geq 0 \tag{3.16}$$

or, in view of (2.6), at

$$\sum_{i=1}^N [a_i(w_{i+1} - w_i, w_i) - a_{i-1}(w_i - w_{i-1}, w_{i-1})] \geq \sum_{i=1}^N a_{i-1} \|w_i - w_{i-1}\|^2. \tag{3.17}$$

By the boundary conditions in (2.3), we have

$$\sum_{i=1}^N a_{i-1} \|w_i - w_{i-1}\|^2 \leq a_N(w_{N+1} - w_N, w_N) - (w_1 - w_0, w_0) \leq 0, \tag{3.18}$$

so  $w_0 = w_1 = \dots = w_N$ . This implies that  $u_i = v_i + C$ ,  $i = \overline{0, N}$ , where  $C \in H$  is a constant. If  $A$  or  $\alpha$  is one to one, then the uniqueness follows easily. If  $A$  is maximal monotone and strongly monotone, then we obtain

$$\sum_{i=1}^N a_i \|w_i\|^2 + \sum_{i=1}^N a_{i-1} \|w_i - w_{i-1}\|^2 \leq 0, \tag{3.19}$$

so the solution is unique and the proof is complete.  $\square$

Now we replace (3.4) by (3.2) and (3.3) and remove (3.5) and (3.6). Adding the boundedness of the domain of  $\beta$ , we can state the following result.

**THEOREM 3.2.** *Let  $A$ ,  $\alpha$ , and  $\beta$  be maximal monotone operators in  $H$  such that  $D(\beta)$  is bounded and (3.1), (3.2), (3.3) hold. If  $a, b \in H$ ,  $(g_i)_{i=\overline{1,N}} \in H^N$ , and  $p_i, r_i, k_i > 0$ ,  $i = \overline{1,N}$ , then problem (1.3), (1.4) admits at least one solution  $(u_i)_{i=\overline{1,N}} \in D(A)^N$  and the difference between two solutions is constant. If  $A$  or  $\alpha$  is one to one, then the solution is unique. If  $A$  is also strongly monotone, then again uniqueness is obtained.*

*Proof.* We use again the equivalent form (2.3) of problem (1.3), (1.4) and the maximal monotone operator  $\mathcal{A}$  given by (3.15). If  $A_\lambda$  and  $\mathcal{A}_\lambda$  are the Yosida approximations of  $A$  and  $\mathcal{A}$ , respectively, then  $\mathcal{A}_\lambda((u_i)_{i=\overline{1,N}}) = (c_1 A_\lambda u_1, \dots, c_N A_\lambda u_N)$  for all  $(u_i)_{i=\overline{1,N}} \in H^N$ . By Proposition 2.2,  $B + \mathcal{A}_\lambda$  is maximal monotone in  $\mathcal{L}$ , therefore,  $R(B + \mathcal{A}_\lambda + \lambda I) = \mathcal{L}$ , that is, for all  $(f_i)_{i=\overline{1,N}} \in H^N$ , for all  $\lambda > 0$ , the problem

$$\begin{aligned} u_{i+1}^\lambda - (1 + \theta_i)u_i^\lambda + \theta_i u_{i-1}^\lambda &= c_i A_\lambda u_i^\lambda + \lambda u_i^\lambda + f_i, \quad i = \overline{1,N}, \\ u_1^\lambda - u_0^\lambda &\in \alpha(u_0^\lambda - a), \quad u_{N+1}^\lambda - u_N^\lambda \in -\beta(u_{N+1}^\lambda - b), \end{aligned} \quad (3.20)$$

has a unique solution  $(u_i^\lambda)_{i=\overline{1,N}} \in H^N$ . (The uniqueness follows from Theorem 2.3 for the strongly monotone operator  $\mathcal{A}_\lambda + \lambda I$ .)

We first prove that  $(u_i^\lambda)_{i=\overline{1,N}}$  is bounded in  $H$  with respect to  $\lambda$ . To do this, we multiply (3.20) by  $a_i u_i^\lambda$  and sum up from  $i = 1$  to  $i = N$ . Without any loss of generality, suppose that  $0 \in A0$ . If not, we put  $\tilde{A} = A + A^0 0$  and  $\tilde{f}_i = f_i - c_i A^0 0$  instead of  $A$  and  $f_i$ , respectively, where  $A^0 x$  denotes the element of least norm in  $Ax$ . Since  $A_\lambda$  is monotone,  $A_\lambda 0 = 0$ , and  $a_i \theta_i = a_{i-1}$ , we derive

$$\begin{aligned} \sum_{i=1}^N a_i (u_{i+1}^\lambda - u_i^\lambda, u_i^\lambda) - \sum_{i=1}^N a_{i-1} (u_i^\lambda - u_{i-1}^\lambda, u_{i-1}^\lambda) \\ \geq \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 + \lambda \sum_{i=1}^N a_i \|u_i^\lambda\|^2 + \sum_{i=1}^N a_i (f_i, u_i^\lambda), \end{aligned} \quad (3.21)$$

hence

$$\sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \leq a_N (u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) - (u_1^\lambda - u_0^\lambda, u_0^\lambda) - \sum_{i=1}^N a_i (f_i, u_i^\lambda). \quad (3.22)$$

Since  $u_1^\lambda - u_0^\lambda \in \alpha(u_0^\lambda - a)$ ,  $0 \in \alpha(0)$ , and  $\alpha$  is monotone, we infer

$$-(u_1^\lambda - u_0^\lambda, u_0^\lambda) \leq -(u_1^\lambda - u_0^\lambda, a) \leq \|a\| \cdot \|u_1^\lambda - u_0^\lambda\| \quad (3.23)$$

and, similarly,

$$(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) \leq -\|u_{N+1}^\lambda - u_N^\lambda\|^2 + (u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b) + (u_{N+1}^\lambda - u_N^\lambda, b), \quad (3.24)$$

so

$$(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) \leq \|b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\|. \quad (3.25)$$

Now (3.22), (3.23), and (3.25) yield

$$\begin{aligned} \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 &\leq a_N \|b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\| + \|a\| \cdot \|u_1^\lambda - u_0^\lambda\| \\ &\quad + \left( \sum_{i=1}^N a_i \|f_i\|^2 \right)^{1/2} \left( \sum_{i=1}^N a_i \|u_i^\lambda\|^2 \right)^{1/2}. \end{aligned} \quad (3.26)$$

The hypothesis that  $D(\beta)$  is bounded and the boundary conditions imply the boundedness of  $u_{N+1}^\lambda$  with respect to  $\lambda$ . Using this, together with the estimates

$$\begin{aligned} \|u_k^\lambda\| &\leq \left( \sum_{i=1}^N a_i \|u_i^\lambda\|^2 \right)^{1/2}, \\ \|u_k^\lambda - u_{k-1}^\lambda\| &\leq \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/2} \end{aligned} \quad (3.27)$$

for  $k = \overline{1, N}$  in (3.26), one deduces

$$\sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \leq C_1 + C_2 \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/2} + C_3 \left( \sum_{i=1}^N a_i \|u_i^\lambda\|^2 \right)^{1/2}, \quad (3.28)$$

with  $C_1, C_2, C_3 > 0$  independent of  $\lambda$ .

For each  $i = \overline{1, N}$ , we have  $u_i^\lambda = u_0^\lambda + \sum_{k=1}^i (u_k^\lambda - u_{k-1}^\lambda)$ , so

$$\|u_i^\lambda\| \leq \|u_0^\lambda\| + \left( \sum_{k=1}^N \frac{1}{a_{k-1}} \right) \left( \sum_{k=1}^N a_{k-1} \|u_k^\lambda - u_{k-1}^\lambda\|^2 \right)^{1/2}, \quad i = \overline{1, N}. \quad (3.29)$$

From the boundary conditions, it follows that

$$\|u_0^\lambda\|^2 \leq \|a\| \cdot \|u_0^\lambda\| + \|a\| \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/2}, \quad (3.30)$$

and thus

$$\|u_0^\lambda\| \leq \|a\| + \|a\|^{1/2} \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/4}. \quad (3.31)$$

Inequalities (3.29) and (3.31) imply that

$$\|u_i^\lambda\| \leq C_4 + C_5 \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/2}, \quad i = \overline{1, N}, \quad (3.32)$$

which, together with (3.28), leads to the boundedness

$$\sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \leq C_6 \quad \forall \lambda > 0. \quad (3.33)$$



Now (3.31) and (3.32) show that  $\|u_i^\lambda\| \leq C_7$ ,  $i = \overline{0, N}$  and  $\lambda > 0$ , and therefore,

$$\sum_{i=1}^N a_i \|u_i^\lambda\|^2 \leq C_8 \quad \forall \lambda > 0. \quad (3.34)$$

All the constants  $C_j > 0$  ( $j = 1, \dots, 13$ ) here and below are independent of  $\lambda$ .

Multiplying (3.20) by  $a_i A_\lambda u_i^\lambda$  and summing from 1 to  $N$ , we get via (2.6)

$$\begin{aligned} & \sum_{i=1}^N [a_i (u_{i+1}^\lambda - u_i^\lambda, A_\lambda u_i^\lambda) - a_{i-1} (u_i^\lambda - u_{i-1}^\lambda, A_\lambda u_{i-1}^\lambda)] - \sum_{i=1}^N a_{i-1} (u_i^\lambda - u_{i-1}^\lambda, A_\lambda u_i^\lambda - A_\lambda u_{i-1}^\lambda) \\ &= \sum_{i=1}^N a_i c_i \|A_\lambda u_i^\lambda\|^2 + \lambda \sum_{i=1}^N a_i (u_i^\lambda, A_\lambda u_i^\lambda) + \sum_{i=1}^N a_i (f_i, A_\lambda u_i^\lambda). \end{aligned} \quad (3.35)$$

Let  $c = \inf \{c_i, i = \overline{1, N}\}$ . Then

$$c \sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 \leq a_N (u_{N+1}^\lambda - u_N^\lambda, A_\lambda u_N^\lambda) - (u_1^\lambda - u_0^\lambda, A_\lambda u_0^\lambda) - \sum_{i=1}^N a_i (f_i, A_\lambda u_i^\lambda). \quad (3.36)$$

We observe that assumptions (3.2) and (3.3) and the boundary conditions yield

$$\begin{aligned} (u_{N+1}^\lambda - u_N^\lambda, A_\lambda u_N^\lambda) &\leq \|A^0 b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\|, \\ -(u_1^\lambda - u_0^\lambda, A_\lambda u_0^\lambda) &\leq \|A^0 a\| \cdot \|u_1^\lambda - u_0^\lambda\|, \end{aligned} \quad (3.37)$$

therefore, (3.36) implies

$$\begin{aligned} c \sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 &\leq a_N \|A^0 b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\| + \|A^0 a\| \cdot \|u_1^\lambda - u_0^\lambda\| \\ &\quad + \left( \sum_{i=1}^N a_i \|f_i\|^2 \right)^{1/2} \left( \sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 \right)^{1/2}. \end{aligned} \quad (3.38)$$

In view of (3.29), (3.31), and the boundedness of  $u_{N+1}^\lambda$ , this means that

$$\sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 \leq C_9 + C_{10} \left( \sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 \right)^{1/2} + C_{11} \left( \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2 \right)^{1/2}. \quad (3.39)$$

According to (3.33), this leads to

$$\sum_{i=1}^N a_i \|A_\lambda u_i^\lambda\|^2 \leq C_{12}. \quad (3.40)$$

We prove now that  $u_i^\lambda - u_{i-1}^\lambda$  is a Cauchy sequence with respect to  $\lambda$ . Subtracting (3.20) with  $\nu$  in place of  $\lambda$  from the original equation (3.20), multiplying the result by  $a_i(u_i^\lambda - u_i^\nu)$ , and summing up from  $i = 1$  to  $i = N$ , we find, with the aid of the equality  $x = J_\lambda x + \lambda A_\lambda x$ ,

$$\begin{aligned} & \sum_{i=1}^N a_i(u_{i+1}^\lambda - u_{i+1}^\nu - u_i^\lambda + u_i^\nu, u_i^\lambda - u_i^\nu) - \sum_{i=1}^N a_{i-1}(u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu, u_{i-1}^\lambda - u_{i-1}^\nu) \\ &= \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\|^2 + \sum_{i=1}^N a_i c_i (A_\lambda u_i^\lambda - A_\nu u_i^\nu, J_\lambda u_i^\lambda - J_\nu u_i^\nu) \\ &+ \sum_{i=1}^N a_i c_i (A_\lambda u_i^\lambda - A_\nu u_i^\nu, \lambda A_\lambda u_i^\lambda - \nu A_\nu u_i^\nu) + \sum_{i=1}^N a_i (\lambda u_i^\lambda - \nu u_i^\nu, u_i^\lambda - u_i^\nu), \end{aligned} \quad (3.41)$$

hence

$$\begin{aligned} & \sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\|^2 \\ & \leq a_N (u_{N+1}^\lambda - u_{N+1}^\nu - u_N^\lambda + u_N^\nu, u_N^\lambda - u_N^\nu) - (u_1^\lambda - u_1^\nu - u_0^\lambda + u_0^\nu, u_0^\lambda - u_0^\nu) \\ & + (\lambda + \nu) \sum_{i=1}^N a_i c_i (A_\lambda u_i^\lambda, A_\nu u_i^\nu) + (\lambda + \nu) \sum_{i=1}^N a_i (u_i^\lambda, u_i^\nu). \end{aligned} \quad (3.42)$$

The boundary conditions in (3.20) and the upper bounds (3.34) and (3.40) imply

$$\sum_{i=1}^N a_{i-1} \|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\|^2 \leq C_{13}(\lambda + \nu), \quad (3.43)$$

and therefore,  $u_i^\lambda - u_{i-1}^\lambda$  is a strongly convergent sequence in  $H$ .

Let  $u_i^\lambda \rightharpoonup u_i$ ,  $i = \overline{1, N}$  (on a subsequence denoted again by  $\lambda$ ). Then  $u_i^\lambda - u_{i-1}^\lambda \rightarrow u_i - u_{i-1}$ , so  $B((u_i^\lambda)_{i=\overline{1, N}}) \rightarrow B((u_i)_{i=\overline{1, N}})$ . In addition, we have  $J_\lambda u_i^\lambda (= u_i^\lambda - \lambda A_\lambda u_i^\lambda) \rightarrow u_i$  as  $\lambda \rightarrow 0$ ,  $i = \overline{1, N}$ .

Since  $A$  is demiclosed, this enables us to pass to the limit as  $\lambda \rightarrow 0$  in (3.20) written under the form

$$-B((u_i^\lambda)_{i=\overline{1, N}}) - \lambda(u_i^\lambda)_{i=\overline{1, N}} - (f_i)_{i=\overline{1, N}} \in \mathcal{A}((J_\lambda u_i^\lambda)_{i=\overline{1, N}}), \quad (3.44)$$

and one obtains that  $(u_i)_{i=\overline{1, N}}$  verifies problem (2.3). The uniqueness follows like in Theorem 3.1. The proof is complete.  $\square$

We now replace the boundedness of  $D(\beta)$  by the conditions

$$\inf \left\{ -\frac{(y, x)}{\|x\|}, y \in -\beta(x) \right\} \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty, \quad (3.45)$$

$$\inf \left\{ \frac{(y, x)}{\|x\|}, y \in \alpha(x) \right\} \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty. \quad (3.46)$$

We get the following result.

**THEOREM 3.3.** *If  $A$ ,  $\alpha$ , and  $\beta$  are maximal monotone operators in  $H$  satisfying hypotheses (3.1), (3.2), (3.3), (3.45), and (3.46), then for given  $a, b \in H$ ,  $g_i \in H$ , and  $p_i, r_i, k_i > 0, i = \overline{1, N}$ , problem (1.3), (1.4) has at least one solution  $(u_i)_{i=\overline{1, N}} \in D(A)^N$ . The solution is unique up to an additive constant.*

*Proof.* One uses again the form (2.3) of problem (1.3), (1.4) and approximates it by (3.20). In order to prove the boundedness of  $u_0^\lambda$  and  $u_{N+1}^\lambda$  with respect to  $\lambda$ , consider the auxiliary problem

$$\begin{aligned} v_{i+1}^\lambda - (1 + \theta_i) v_i^\lambda + \theta_i v_{i-1}^\lambda &= c_i A_\lambda v_i^\lambda + \lambda v_i^\lambda + f_i, \quad i = \overline{1, N}, \\ v_0^\lambda &= a, \quad v_{N+1}^\lambda = b. \end{aligned} \quad (3.47)$$

This problem is a particular case of problem (2.3), where the operator  $A_\lambda + \lambda I$  is maximal monotone and strongly monotone and  $\alpha, \beta$  are the subdifferential  $\partial j$  of the lower-semicontinuous, convex, and proper function  $j : H \rightarrow \mathbb{R}$  as presented in (1.5). Then Theorem 2.3 implies the existence of a unique solution  $(v_i^\lambda)_{i=\overline{1, N}}$  of (3.47).

A multiplication of the difference between (3.20) and (3.47) by  $a_i(u_i^\lambda - v_i^\lambda)$  followed by a summation with respect to  $i$  leads to

$$\begin{aligned} \sum_{i=1}^N a_{i-1} \|u_i^\lambda - v_i^\lambda - u_{i-1}^\lambda + v_{i-1}^\lambda\|^2 \\ \leq a_N (u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda, u_N^\lambda - v_N^\lambda) - (u_1^\lambda - v_1^\lambda - u_0^\lambda + a, u_0^\lambda - a) \end{aligned} \quad (3.48)$$

or, equivalently,

$$\begin{aligned} a_N \|u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda\|^2 + \sum_{i=1}^N a_{i-1} \|u_i^\lambda - v_i^\lambda - u_{i-1}^\lambda + v_{i-1}^\lambda\|^2 \\ \leq a_N (u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda, u_{N+1}^\lambda - b) - (u_1^\lambda - v_1^\lambda - u_0^\lambda + a, u_0^\lambda - a). \end{aligned} \quad (3.49)$$

From this inequality and the boundary conditions in (3.20), we can easily get

$$\begin{aligned} 0 &\leq (u_1^\lambda - u_0^\lambda, u_0^\lambda - a) - a_N (u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b) \\ &\leq (v_1^\lambda - a, u_0^\lambda - a) + a_N (v_N^\lambda - b, u_{N+1}^\lambda - b). \end{aligned} \quad (3.50)$$

Since problem (3.47) is a particular case of (3.20), where  $D(\beta)$  is bounded, we can use the proof of Theorem 3.2 to deduce the boundedness of  $v_1^\lambda$  and  $v_N^\lambda$  in  $H$  with respect to  $\lambda$ . Hence, there exist two constants  $C_1$  and  $C_2$  independent of  $\lambda$  such that

$$0 \leq (u_1^\lambda - u_0^\lambda, u_0^\lambda - a) - a_N (u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b) \leq C_1 \|u_0^\lambda - a\| + C_2 \|u_{N+1}^\lambda - b\|. \quad (3.51)$$

From (3.45), (3.46), and (3.51), we obtain that  $u_0^\lambda$  and  $u_{N+1}^\lambda$  are bounded. Indeed, if not, say  $\|u_0^\lambda - a\| \rightarrow \infty$  on a subsequence denoted again by  $\lambda$ . By (3.46), it follows that

$$\frac{(u_1^\lambda - u_0^\lambda, u_0^\lambda - a)}{\|u_0^\lambda - a\|} \longrightarrow \infty \quad \text{as } \lambda \longrightarrow 0. \quad (3.52)$$

If  $R_\lambda = \|u_{N+1}^\lambda - b\|/\|u_0^\lambda - a\|$  is bounded, then we get a contradiction in (3.51). If  $R_\lambda$  is unbounded, then dividing (3.51) by  $\|u_{N+1}^\lambda - b\|$  and using condition (3.45) for  $x = u_{N+1}^\lambda - b$  and  $y = u_{N+1}^\lambda - u_N^\lambda$ , we arrive again at a contradiction. This demonstrates the boundedness of  $u_0^\lambda$  and  $u_{N+1}^\lambda$ .

From now on, the proof follows that of Theorem 3.2.  $\square$

## References

- [1] A. R. Aftabizadeh and N. H. Pavel, *Boundary value problems for second order differential equations and a convex problem of Bolza*, Differential Integral Equations **2** (1989), no. 4, 495–509.
- [2] ———, *Nonlinear boundary value problems for some ordinary and partial differential equations associated with monotone operators*, J. Math. Anal. Appl. **156** (1991), no. 2, 535–557.
- [3] R. P. Agarwal and D. O'Regan, *Difference equations in abstract spaces*, J. Austral. Math. Soc. Ser. A **64** (1998), no. 2, 277–284.
- [4] ———, *Existence principles for continuous and discrete equations on infinite intervals in Banach spaces*, Math. Nachr. **207** (1999), 5–19.
- [5] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, *Discrete second order inclusions*, J. Difference Equ. Appl. **9** (2003), no. 10, 879–885.
- [6] N. C. Apreutesei, *Existence and asymptotic behavior for a class of second order difference equations*, J. Difference Equ. Appl. **9** (2003), no. 9, 751–763.
- [7] ———, *A finite difference scheme in Hilbert spaces*, An. Univ. Craiova Ser. Mat. Inform. **30** (2003), 21–29.
- [8] ———, *On a class of difference equations of monotone type*, J. Math. Anal. Appl. **288** (2003), no. 2, 833–851.
- [9] V. Barbu, *A class of boundary problems for second order abstract differential equations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **19** (1972), 295–319.
- [10] ———, *Sur un problème aux limites pour une classe d'équations différentielles non linéaires abstraits du deuxième ordre en  $t$* , C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A459–A462 (French).
- [11] ———, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leiden, 1976.
- [12] H. Brézis, *Équations d'évolution du second ordre associées à des opérateurs monotones*, Israel J. Math. **12** (1972), 51–60 (French).
- [13] E. Mitidieri and G. Moroşanu, *Asymptotic behaviour of the solutions of second order difference equations associated to monotone operators*, Numer. Funct. Anal. Optim. **8** (1985/1986), no. 3-4, 419–434.
- [14] G. Moroşanu, *Second-order difference equations of monotone type*, Numer. Funct. Anal. Optim. **1** (1979), no. 4, 441–450.
- [15] L. Véron, *Problèmes d'évolution du second ordre associés à des opérateurs monotones*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 1099–1101 (French).
- [16] ———, *Équations non linéaires avec conditions aux limites de type Sturm-Liouville*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat. **24** (1978), no. 2, 277–287 (French).

N. C. Apreutesei: Department of Mathematics, Technical University “Gh. Asachi” of Iasi, 11, Bd. Carol I, 700506 Iasi, Romania

E-mail address: [napreut@math.tuiasi.ro](mailto:napreut@math.tuiasi.ro)

## Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

### Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

### Guest Editors

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; [piqueira@lac.usp.br](mailto:piqueira@lac.usp.br)

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; [elbert@lac.inpe.br](mailto:elbert@lac.inpe.br)

**Celso Grebogi**, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; [grebogi@abdn.ac.uk](mailto:grebogi@abdn.ac.uk)