EXISTENCE AND GLOBAL STABILITY OF POSITIVE PERIODIC SOLUTIONS OF A DISCRETE PREDATOR-PREY SYSTEM WITH DELAYS

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We study the existence and global stability of positive periodic solutions of a periodic discrete predator-prey system with delay and Holling type III functional response. By using the continuation theorem of coincidence degree theory and the method of Lyapunov functional, some sufficient conditions are obtained.

1. Introduction

Many realistic problems could be solved on the basis of constructing suitable mathematical models, but it is obvious that a perfect model cannot be achieved because even if we could put all possible factors in a model, the model could never predict ecological catastrophes or mother nature caprice. Therefore, the best we can do is to look for analyzable models that describe as well as possible the reality on populations. From a mathematical point of view, the art of good modelling relies on the following: (i) a sound understanding and appreciation of the biological problem; (ii) a realistic mathematical representation of the important biological phenomena; (iii) finding useful solutions, preferably quantitative; (iv) a biological interpretation of the mathematical results in terms of insights and predictions.

Usually a mathematical model could be described by two types of systems: a continuous system or a discrete one. When the size of the population is rarely small or the population has nonoverlapping generations, we may prefer the discrete models. Among all the mathematical models, the predator-prey systems play a fundamental and crucial role (for more details, we refer to [3, 6]). In general, a predator-prey system may have the form

$$\begin{aligned} x' &= rx\left(1-\frac{x}{K}\right)-\varphi(x)y, \\ y' &= y\left(\mu\varphi(x)-D\right), \end{aligned} \tag{1.1}$$

where $\varphi(x)$ is the functional response function. Massive work has been done on this issue. We refer to the monographs [4, 10, 18, 20] for general delayed biological systems and to

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[2, 8, 9, 11, 21, 24] for investigation on predator-prey systems. Here, $\varphi(x)$ may be different response functions: standard type II and type III response functions (Holling [12]), Ivlev's functional response (Ivlev [17]), and Rosenzweig functional response (Rosenzweig [22]). Systems with Holling-type functional response have been investigated by many authors, see, for example, Hsu and Huang [13], Rosenzweig and MacArthur [22, 23]. They studied the stability of the equilibria, existence of Hopf bifurcation, limit cycles, homoclinic loops, and even catastrophe.

On the other hand, in view of the periodic variation of the environment (e.g., food supplies, mating habits, seasonal affects of weather, etc.), it would be of interest to study the global existence and global stability of positive solutions for periodic systems [18]. Recently, some excellent existence results have been obtained by using the coincidence degree method (see, e.g., [5, 14, 15, 16, 19, 27]).

Motivated by the above considerations, we will consider the discrete predator-prey system with Holling type III functional response. The corresponding continuous system which has been investigated in our previous articles [25, 26] with discrete delays takes the form

$$\begin{aligned} N_1'(t) &= N_1(t)\left[b_1(t) - a_1(t)N_1(t-\tau_1)\right] - \frac{\alpha_1(t)N_1^2(t)N_2(t-\sigma)}{1+mN_1^2(t)}, \\ N_2'(t) &= N_2(t)\left[-b_2(t) - a_2(t)N_2(t) + \frac{\alpha_2(t)N_1^2(t-\tau_2)}{1+mN_1^2(t-\tau_2)}\right], \end{aligned} \tag{1.2}$$

where $N_1(t)$ and $N_2(t)$ represent the densities of the prey population and predator population at time t, respectively; m, τ_1, τ_2, and σ are nonnegative constants; $a_1(t)$, $b_1(t)$, $\alpha_1(t)$, $a_2(t)$, $b_2(t)$, and $\alpha_2(t)$ are all continuous functions; $b_1(t)$ stands for prey intrinsic growth rate, $b_2(t)$ stands for the death rate of the predator, m stands for half capturing saturation; the function $N_1(t)[b_1(t) - a_1(t)N_1(t-\tau_1)]$ represents the specific growth rate of the prey in the absence of predator; and $N_1^2(t)/(1+mN_1^2(t))$ denotes the predator response function, which reflects the capture ability of the predator.

We assume that the average growth rates in (1.2) change at regular intervals of time, then we can incorporate this aspect in (1.2) and obtain the following modified system:

$$\begin{aligned} \frac{1}{N_1(t)}\frac{dN_1(t)}{dt} &= \left[b_1([t]) - a_1([t])N_1([t]-[\tau_1])\right] - \frac{\alpha_1([t])N_1([t])N_2([t]-[\sigma])}{1+mN_1^2([t])}, \\ \frac{1}{N_2(t)}\frac{dN_2(t)}{dt} &= -b_2([t]) - a_2([t])N_2([t]) + \frac{\alpha_2([t])N_1^2([t]-[\tau_2])}{1+mN_1^2([t]-[\tau_2])}, \quad t \neq 0,1,2,\ldots, \end{aligned} \tag{1.3}$$

where $[t]$ denotes the integer part of t, $t \in (0,+\infty)$. By a solution of (1.3) we mean a function $N = (N_1, N_2)^T$, which is defined for $t \in (0,+\infty)$, and possesses the following properties:

(1) N is continuous on $[0,+\infty)$;
(2) the derivatives $dN_1(t)/dt$, $dN_2(t)/dt$ exist at each point $t \in [0,+\infty)$ with the possible exception of the points $t \in \{0,1,2,\ldots\}$, where left-sided derivatives exist;
(3) the equations in (1.3) are satisfied on each interval $[k, k+1)$ with $k = 0,1,2,\ldots$.

On any interval of the form $[k,k+1)$, $k=0,1,2,\ldots$, we can integrate (1.3) and obtain for $k\le t<k+1$, $k=0,1,2,\ldots$,

$$\begin{aligned} N_1(t) &= N_1(k)\exp\left\{\left[b_1(k)-a_1(k)N_1(k-[\tau_1])-\frac{\alpha_1(k)N_1(k)N_2(k-[\sigma])}{1+mN_1^2(k)}\right](t-k)\right\},\\ N_2(t) &= N_2(k)\exp\left\{\left[-b_2(k)-a_2(k)N_2(k)+\frac{\alpha_2(k)N_1^2(k-[\tau_2])}{1+mN_1^2(k-[\tau_2])}\right](t-k)\right\}. \end{aligned} \tag{1.4}$$

Let $t\to k+1$; we obtain from (1.4) that

$$\begin{aligned} N_1(k+1) &= N_1(k)\exp\left\{b_1(k)-a_1(k)N_1(k-[\tau_1])-\frac{\alpha_1(k)N_1(k)N_2(k-[\sigma])}{1+mN_1^2(k)}\right\},\\ N_2(k+1) &= N_2(k)\exp\left\{-b_2(k)-a_2(k)N_2(k)+\frac{\alpha_2(k)N_1^2(k-[\tau_2])}{1+mN_1^2(k-[\tau_2])}\right\}, \end{aligned} \tag{1.5}$$

which is a discrete time analogue of system (1.2), where $N_1(t)$, $N_2(t)$ are the densities of the prey population and predator population at time t.

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^2$ denote the sets of all integers, nonnegative integers, real numbers, nonnegative real numbers, and two-dimensional Euclidean vector space, respectively. Throughout this paper, we always assume that $b_i:\mathbb{Z}\to\mathbb{R}$ and $a_i,\alpha_i:\mathbb{Z}\to\mathbb{R}^+$ $(i=1,2)$ are periodic functions such that

$$b_i(k+\omega)=b_i(k),\quad a_i(k+\omega)=a_i(k),\quad \alpha_i(k+\omega)=\alpha_i(k),\quad i=1,2, \tag{1.6}$$

for any $k\in\mathbb{Z}$ and $\overline{b_i}>0$ $(i=1,2)$, where ω is a positive integer and $\overline{b_i}$ is defined as below.

For convenience, we denote

$$I_\omega=\{0,1,\ldots,\omega-1\},\qquad \overline{g}=\frac{1}{\omega}\sum_{k=0}^{\omega-1}g(k),\qquad \overline{G}=\frac{1}{\omega}\sum_{k=0}^{\omega-1}|g(k)|, \tag{1.7}$$

where $\{g(k)\}$ is an ω-periodic sequence of real numbers defined for $k\in\mathbb{Z}$.

The exponential form of (1.5) assures that for any initial condition $N(0)>0$, $N(k)$ remains positive. In the rest of this paper, for biological reasons, we only consider solutions $N(k)$ with

$$N_i(-k)\ge 0,\quad k=1,2,\ldots,\max\{[\tau_1],[\tau_2],[\sigma]\},\qquad N_i(0)>0,\quad i=1,2. \tag{1.8}$$

2. Existence of positive periodic solution

In order to obtain the existence of positive periodic solution of (1.5), for the reader's convenience, we will summarize in the following a few concepts and results from [7] that will be basic for this section.

Let $X, \mathbb{Z}$ be normed vector spaces, $L : \operatorname{Dom} L \subset X \to \mathbb{Z}$ a linear mapping, and $N : X \to \mathbb{Z}$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker} L = \operatorname{Codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Z}$. If L is a Fredholm mapping of index zero, there exist continuous projections $P : X \to X$ and $Q : \mathbb{Z} \to \mathbb{Z}$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of the map L by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof of our main theorem, we will use the following result from Gaines and Mawhin [7].

Lemma 2.1 (continuation theorem). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose that*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ satisfies $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \tag{2.1}$$

Then the operator equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Now we state two lemmas which are useful to prove the main theorem for the existence of a positive ω-periodic solution.

Lemma 2.2 (see [5]). *Let $g : \mathbb{Z} \to \mathbb{R}$ be a function satisfying $g(k + \omega) = g(k)$, $k \in \mathbb{Z}$. Then for any fixed $k_1, k_2 \in I_\omega$ and $k \in \mathbb{Z}$,*

$$\begin{aligned} g(k) &\leq g(k_1) + \sum_{k=0}^{\omega-1} |g(k+1) - g(k)|, \\ g(k) &\geq g(k_2) - \sum_{k=0}^{\omega-1} |g(k+1) - g(k)|. \end{aligned} \tag{2.2}$$

Lemma 2.3. *If* (h_1) $(\overline{\alpha}_2 - m\overline{b}_2)^{-1/2}(\overline{b}_2)^{1/2} < \overline{b}_1/\overline{a}_1 \leq 27/m^2$ *and* (h_2) $\overline{\alpha}_2 > m\overline{b}_2$ *hold, then the system of algebraic equations*

$$\begin{aligned} \overline{b}_1 - \overline{a}_1 u_1 - \overline{\alpha}_1 \frac{u_1 u_2}{1 + m u_1^2} &= 0, \\ \overline{b}_2 + \overline{a}_2 u_2 - \overline{\alpha}_2 \frac{u_1^2}{1 + m u_1^2} &= 0 \end{aligned} \tag{2.3}$$

has a unique solution $(u_1^*, u_2^*)^T \in \mathbb{R}^2$ with $u_i^* > 0$, $i = 1, 2$.

Proof. Consider the functions

$$f(u_1) = \frac{(1+mu_1^2)(\overline{b}_1 - \overline{a}_1 u_1)}{\overline{\alpha}_1 u_1}, \quad u_1 > 0,$$
$$g(u_1) = \frac{-\overline{b}_2 + (\overline{\alpha}_2 - m\overline{b}_2)u_1^2}{\overline{a}_2(1+mu_1^2)}, \quad u_1 > 0. \tag{2.4}$$

It is easy to see that

$$f'(u_1) = \frac{1}{\overline{\alpha}_1}\left(\frac{-\overline{b}_1}{u_1^2} + m\overline{b}_1 - 2m\overline{a}_1 u_1\right),$$
$$f''(u_1) = \frac{1}{\overline{\alpha}_1}\left(\frac{2\overline{b}_1}{u_1^3} - 2m\overline{a}_1\right). \tag{2.5}$$

From (h_1) we know that

$$f'(u_1) \leq 0. \tag{2.6}$$

Notice that

$$f(0) = +\infty, \qquad f(+\infty) = -\infty,$$
$$g(0) = \frac{-\overline{b}_2}{\overline{a}_2} < 0, \qquad g(+\infty) = \frac{(\overline{\alpha}_2 - m\overline{b}_2)}{\overline{a}_2(1+mu_1^2)}, \tag{2.7}$$

and in view of (h_2), we have

$$g'(u_1) > 0 \quad \text{for } u_1 > 0. \tag{2.8}$$

From the above discussion we may conclude that the curve $f(u_1) = g(u_1)$ has only a unique zero point. It follows that the algebraic equations (2.3) have a unique solution. The proof is complete. □

Define

$$l_2 = \{y = y(k) : y(k) \in \mathbb{R}^2, \ k \in \mathbb{Z}\}. \tag{2.9}$$

For $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$, define $|\theta| = \max\{\theta_1, \theta_2\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω-periodic sequences equipped with the norm

$$\|y\| = \max_{k \in I_\omega} |y(k)|, \tag{2.10}$$

that is,

$$l^{\omega} = \{y = y(k) : y(k+\omega) = y(k),\ y(k) \in \mathbb{R}^2,\ k \in \mathbb{Z}\}. \tag{2.11}$$

It is not difficult to show that l^{ω} is a finite-dimensional Banach space.

Set

$$\begin{aligned} l_0^{\omega} &= \left\{ y = y(k) \in l^{\omega} : \sum_{k=0}^{\omega-1} y(k) = 0 \right\}, \\ l_c^{\omega} &= \{y = y(k) \in l^{\omega} : y(k) = h \in \mathbb{R}^2,\ k \in \mathbb{Z}\}. \end{aligned} \tag{2.12}$$

Then it follows that l_0^{ω} and l_c^{ω} are both closed linear subspaces of l^{ω} and

$$l^{\omega} = l_0^{\omega} \oplus l_c^{\omega}, \qquad \dim l_c^{\omega} = 2. \tag{2.13}$$

Now we state our main result of this section.

Theorem 2.4. *Assume that* (h_1), (h_3) $2\sqrt{m}\,\overline{b}_1 > \overline{\alpha}_1 \exp\{H_{21}\}$, *and* ($h_4$)

$$\frac{\exp\{2H_{12}\}}{1 + m\exp\{2H_{12}\}}\overline{\alpha}_2 > \overline{b}_2 \tag{2.14}$$

hold, where

$$\begin{aligned} H_{21} &= \ln\left\{\frac{\overline{\alpha}_2 - m\overline{b}_2}{m\overline{a}_2}\right\} + (\overline{B}_2 + \overline{b}_2)\omega, \\ H_{12} &= \ln\left\{\frac{2\sqrt{m}\,\overline{b}_1 - \overline{\alpha}_1 \exp\{H_{21}\}}{2\sqrt{m}\,\overline{a}_1}\right\} - (\overline{B}_1 + \overline{b}_1)\omega. \end{aligned} \tag{2.15}$$

Then (1.5) has at least one positive ω-periodic solution.

Proof. Make the change of variables

$$N_1(t) = \exp\{x_1(t)\}, \qquad N_2(t) = \exp\{x_2(t)\}; \tag{2.16}$$

then (1.5) can be reformulated as

$$\begin{aligned} x_1(k+1)-x_1(k) &= b_1(k)-a_1(k)\exp\{x_1(k-[\tau_1])\} \\ &\quad -\frac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}, \\ x_2(k+1)-x_2(k) &= -b_2(k)-a_2(k)\exp\{x_2(k)\} \\ &\quad +\frac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}. \end{aligned} \tag{2.17}$$

Define

$$\begin{aligned} X = Y = l^{\omega}, &\qquad (Lx)(k) = x(k+1)-x(k), \\ (Nx)(k) &= \begin{bmatrix} b_1(k)-a_1(k)\exp\{x_1(k-[\tau_1])\}-\dfrac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}} \\ -b_2(k)-a_2(k)\exp\{x_2(k)\}+\dfrac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}} \end{bmatrix} \\ &\equiv \begin{bmatrix} \triangle_1(k) \\ \triangle_2(k) \end{bmatrix} \end{aligned} \tag{2.18}$$

for any $x \in X$ and $k \in \mathbb{Z}$. It is easy to see that L is a bounded linear operator,

$$\begin{gathered} \operatorname{Ker} L = l_c^{\omega}, \qquad \operatorname{Im} L = l_0^{\omega}, \\ \dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L; \end{gathered} \tag{2.19}$$

then it follows that L is a Fredholm mapping of index zero.

Set

$$\begin{aligned} Px &= \frac{1}{\omega}\sum_{k=0}^{\omega-1} x(s), \quad x \in X, \\ Qz &= \frac{1}{\omega}\sum_{k=0}^{\omega-1} z(s), \quad z \in Y, \end{aligned} \tag{2.20}$$

and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q). \tag{2.21}$$

Furthermore, the generalized inverse to L,

$$K_P : \operatorname{Im} L \longrightarrow \operatorname{Ker} P \cap \operatorname{Dom} L, \tag{2.22}$$

exists and can be read as

$$K_P(z) = \sum_{s=0}^{k-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)z(s). \tag{2.23}$$

Thus,

$$QNx = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\left[b_1(k) - a_1(k)\exp\{x_1(k-[\tau_1])\} - \dfrac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}\right] \\ \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1}\left[-b_2(k) - a_2(k)\exp\{x_2(k)\} + \dfrac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}\right] \end{bmatrix},$$

$$K_P(I-Q)Nx = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}\triangle_1(s) \\ \dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}\triangle_2(s)\end{bmatrix} - \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}(\omega-s)\triangle_1(s) \\ \dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}(\omega-s)\triangle_2(s)\end{bmatrix} - \begin{bmatrix} \left(k-\dfrac{\omega+1}{2}\right)\dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}\triangle_1(s) \\ \left(k-\dfrac{\omega+1}{2}\right)\dfrac{1}{\omega}\displaystyle\sum_{s=0}^{\omega-1}\triangle_2(s)\end{bmatrix}. \tag{2.24}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$ by using the Arzelà-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open bounded set Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$,

$$\begin{aligned} x_1(k+1)-x_1(k) &= \lambda\left[b_1(k) - a_1(k)\exp\{x_1(k-[\tau_1])\} - \frac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}\right], \\ x_2(k+1)-x_2(k) &= \lambda\left[-b_2(k) - a_2(k)\exp\{x_2(k)\} + \frac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}\right]. \end{aligned} \tag{2.25}$$

Assume that $x(t) \in X$ is an ω-periodic solution of (2.25) for a certain $\lambda \in (0,1)$. Summing on both sides of (2.25) from 0 to $\omega - 1$ with respect to k, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\omega-1}[x_1(k+1)-x_1(k)] \\
&\quad = \lambda \sum_{k=0}^{\omega-1}\left[b_1(k)-a_1(k)\exp\{x_1(k-[\tau_1])\}-\frac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}\right], \\
&\sum_{k=0}^{\omega-1}[x_2(k+1)-x_2(k)] \\
&\quad = \lambda \sum_{k=0}^{\omega-1}\left[-b_2(k)-a_2(k)\exp\{x_2(k)\}+\frac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}\right].
\end{aligned} \tag{2.26}$$

Notice that

$$\sum_{k=0}^{\omega-1}[x_1(k+1)-x_1(k)] = \sum_{k=0}^{\omega-1}[x_2(k+1)-x_2(k)] = 0. \tag{2.27}$$

Thus

$$\bar{b}_1\omega = \sum_{k=0}^{\omega-1}\left[a_1(k)\exp\{x_1(k-[\tau_1])\}+\frac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}\right], \tag{2.28}$$

$$\bar{b}_2\omega = \sum_{k=0}^{\omega-1}\left[-a_2(k)\exp\{x_2(k)\}+\frac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}\right]. \tag{2.29}$$

From (2.25), (2.28), and (2.29), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\omega-1}\left|x_1(k+1)-x_1(k)\right| \\
&\quad \le \sum_{k=0}^{\omega-1}\left[\left|b_1(k)\right|+a_1(k)\exp\{x_1(k-[\tau_1])\}+\frac{\alpha_1(k)\exp\{x_1(k)+x_2(k-[\sigma])\}}{1+m\exp\{2x_1(k)\}}\right] \\
&\quad = (\overline{B}_1+\bar{b}_1)\omega,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
&\sum_{k=0}^{\omega-1}\left|x_2(k+1)-x_2(k)\right| \\
&\quad \le \sum_{k=0}^{\omega-1}\left|b_2(k)\right|+\sum_{k=0}^{\omega-1}\left[-a_2(k)\exp\{x_2(k)\}+\frac{\alpha_2(k)\exp\{2x_1(k-[\tau_2])\}}{1+m\exp\{2x_1(k-[\tau_2])\}}\right] \\
&\quad = (\overline{B}_2+\bar{b}_2)\omega.
\end{aligned} \tag{2.31}$$

Note that $x(t) = (x_1(t), x_2(t))^T \in X$; then there exist $\xi_i, \eta_i \in I_\omega$ $(i = 1,2)$ such that

$$x_i(\xi_i) = \min_{k \in I_\omega} x_i(k), \quad x_i(\eta_i) = \max_{k \in I_\omega} x_i(k), \quad i = 1,2. \tag{2.32}$$

In view of (2.29), we get

$$\bar{b}_2 + \bar{a}_2 \exp\{x_2(\xi_2)\} \le \bar{\alpha}_2 \frac{\exp\{2x_1(k-[\tau_2])\}}{1 + m\exp\{2x_1(k-[\tau_2])\}} \le \frac{\bar{\alpha}_2}{m}, \tag{2.33}$$

thus

$$x_2(\xi_2) \le \ln\left\{\frac{\bar{\alpha}_2/m - \bar{b}_2}{\bar{a}_2}\right\}. \tag{2.34}$$

Therefore, by Lemma 2.2, we obtain

$$\begin{aligned} x_2(k) &\le x_2(\xi_2) + \sum_{k=0}^{\omega-1} |x_2(s+1) - x_2(s)| \\ &\le \ln\left\{\frac{\bar{\alpha}_2/m - \bar{b}_2}{\bar{a}_2}\right\} + (\bar{B}_2 + \bar{b}_2)\omega = H_{21}. \end{aligned} \tag{2.35}$$

From (2.28), we know that

$$\bar{a}_1 \omega \exp\{x_1(\xi_1)\} \le \sum_{k=0}^{\omega-1} [a_1(k)\exp\{x_1(k-[\tau_1])\}] \le \bar{b}_1 \omega, \tag{2.36}$$

so we get

$$x_1(\xi_1) \le \ln\left\{\frac{\bar{b}_1}{\bar{a}_1}\right\}. \tag{2.37}$$

Combine (2.37) with (2.30); also, in view of Lemma 2.2, we conclude that

$$x_1(k) \le x_1(\xi_1) + \sum_{k=0}^{\omega-1} |x_1(s+1) - x_1(s)| \le \ln\left\{\frac{\bar{b}_1}{\bar{a}_1}\right\} + (\bar{B}_1 + \bar{b}_1)\omega := H_{11}. \tag{2.38}$$

Formulas (2.35) and (2.28) imply that

$$\begin{aligned} \bar{b}_1 \omega &\le \sum_{k=0}^{\omega-1} \left[a_1(k)\exp\{x_1(\eta_1)\} + \frac{\alpha_1(k)\exp\{x_1(k)\}\exp\{H_{21}\}}{1 + m\exp\{2x_1(k)\}}\right] \\ &\le \bar{a}_1 \omega \exp\{x_1(\eta_1)\} + \frac{\bar{\alpha}_1 \omega \exp\{H_{21}\}}{2\sqrt{m}}. \end{aligned} \tag{2.39}$$

Direct calculation yields

$$x_1(\eta_1) \ge \ln\left\{\frac{2\sqrt{m}\,\bar{b}_1 - \bar{\alpha}_1 \exp\{H_{21}\}}{2\sqrt{m}\,\bar{a}_1}\right\}, \tag{2.40}$$

thus, by Lemma 2.2,

$$\begin{aligned} x_1(k) &\geq x_1(\eta_1) - \sum_{k=0}^{\omega-1} |x_1(s+1) - x_1(s)| \\ &\geq \ln\left\{\frac{2\sqrt{m}\,\overline{b}_1 - \overline{\alpha}_1 \exp\{H_{21}\}}{2\sqrt{m}\,\overline{a}_1}\right\} - (\overline{B}_1 + \overline{b}_1)\omega = H_{12}. \end{aligned} \tag{2.41}$$

From (2.29), (2.41), and the monotonicity of the function

$$\frac{\exp\{2u\}}{1+m\exp\{2u\}} \quad (m>0), \tag{2.42}$$

we have

$$\overline{b}_2\omega + \overline{a}_2\omega\exp\{x_2(\eta_2)\} \geq \sum_{k=0}^{\omega-1} \frac{\alpha_2(k)\exp\{2x_1(\xi_1)\}}{1+m\exp\{2x_1(\xi_1)\}} \geq \frac{\exp\{2H_{12}\}}{1+m\exp\{2H_{12}\}}\overline{\alpha}_2\omega; \tag{2.43}$$

this means that

$$x_2(\eta_2) \geq \ln\left\{\frac{(\exp\{2H_{12}\}/(1+m\exp\{2H_{12}\}))\overline{\alpha}_2 - \overline{b}_2}{\overline{a}_2}\right\}. \tag{2.44}$$

From (2.44), (2.31), and Lemma 2.2, we know that

$$\begin{aligned} x_2(k) &\geq x_2(\eta_2) - \sum_{k=0}^{\omega-1} |x_2(s+1) - x_2(s)| \\ &\geq \ln\left\{\frac{(\exp\{2H_{12}\}/(1+m\exp\{2H_{12}\}))\overline{\alpha}_2 - \overline{b}_2}{\overline{a}_2}\right\} - (\overline{B}_2 + \overline{b}_2)\omega := H_{22}. \end{aligned} \tag{2.45}$$

Inequalities (2.38) and (2.41) imply that

$$|x_1(k)| \leq \max\{|H_{11}|, |H_{12}|\} := H_1. \tag{2.46}$$

On the other hand, (2.35) and (2.45) lead to

$$|x_2(k)| \leq \max\{|H_{21}|, |H_{22}|\} := H_2. \tag{2.47}$$

Obviously, H_1 and H_2 are independent of the choice of λ. Under the assumptions in Theorem 2.4, by Lemma 2.3, we can easily know that the algebraic equations

$$\begin{aligned} \overline{b}_1 - \overline{a}_1 u_1 - \overline{\alpha}_1 \frac{u_1 u_2}{1+mu_1^2} &= 0, \\ \overline{b}_2 + \overline{a}_2 u_2 - \overline{\alpha}_2 \frac{u_1^2}{1+mu_1^2} &= 0 \end{aligned} \tag{2.48}$$

have a unique solution $(u_1^*, u_2^*)^T$ with $u_i^* > 0$ $(i = 1,2)$.

Let $H = H_1 + H_2 + H_3$, where $H_3 > 0$ is large enough such that

$$\left\|\left(\ln\{u_1^*\},\ln\{u_2^*\}\right)^T\right\| = \max\{|\ln\{u_1^*\}|,|\ln\{u_2^*\}|\} < H_3, \tag{2.49}$$

and define

$$\Omega = \left\{x(t) = \left(x_1(t),x_2(t)\right)^T \in X : \|x\| < H\right\}. \tag{2.50}$$

It is easy to see that Ω satisfies Lemma 2.1(a). When $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2$, x is a constant vector in $\mathbb{R}^2$ with $\|x\| = H$. Then

$$QNx = \begin{bmatrix} \bar{b}_1 - \bar{a}_1 \exp\{x_1\} - \bar{\alpha}_1 \dfrac{\exp\{x_1 + x_2\}}{1 + m\exp\{2x_1\}} \\ -\bar{b}_2 - \bar{a}_2 \exp\{x_2\} + \bar{\alpha}_2 \dfrac{\exp\{2x_1\}}{1 + m\exp\{2x_1\}} \end{bmatrix} \neq 0. \tag{2.51}$$

Since $\operatorname{Im} P = \operatorname{Ker} L$, J can be chosen as the identity mapping. In view of the assumptions in Theorem 2.4, direct calculation yields

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \tag{2.52}$$

By now, we have proved that Ω satisfies all the conditions in Lemma 2.1. Hence (2.17) has at least one solution $(x_1^*(t),x_2^*(t))^T$ in $\operatorname{Dom} L \cap \overline{\Omega}$. Set $N_1^*(t) = \exp\{x_1^*(t)\}$, $N_2^*(t) = \exp\{x_2^*(t)\}$; then $N^*(t) = (N_1^*(t),N_2^*(t))^T$ is a positive ω-periodic solution of (1.5). This completes the proof. □

3. Global asymptotic stability

The purpose of this section is to present sufficient conditions for the global asymptotic stability of system (1.5) when the delays are all zero. The method we use here is to construct a suitable Lyapunov function.

THEOREM 3.1. *Assume that* (h_1), (h_2), *and* (h_3) *hold and, furthermore, suppose that there exist positive numbers* ν, c_1, *and* c_2 *such that*

$$c_1 a_1(k) + \frac{c_1\alpha_1(k)\exp\{H_{22}\}}{1 + m\exp\{2H_{11}\}} - \frac{c_1\alpha_1(k)\exp\{H_{21}\}}{4} - \frac{c_2\alpha_2(k)}{\sqrt{m}\left(1 + m\exp\{2H_{12}\}\right)} \geq \nu, \tag{3.1}$$

$$c_2 a_2(k) - c_1 \frac{\alpha_1(k)}{2\sqrt{m}} \geq \nu, \tag{3.2}$$

$$a_2(k)\exp\{H_{21}\} \leq 1, \tag{3.3}$$

$$a_1(k)\exp\{H_{11}\} + \frac{\alpha_1(k)\exp\{H_{21}\}}{2\sqrt{m}\left(1 + m\exp\{2H_{12}\}\right)} - \frac{m\alpha_1(k)\exp\{3H_{12} + H_{22}\}}{\left(1 + m\exp\{2H_{11}\}\right)^2} \leq 1. \tag{3.4}$$

Then the positive solution of system (1.5) is globally asymptotically stable.

Proof. Let $\{N_i^*(k)\}$ $(i = 1,2)$ be a positive solution of system (1.5). Introduce the change of variables

$$u_1(k) = N_1(k) - N_1^*(k), \qquad u_2(k) = N_2(k) - N_2^*(k). \tag{3.5}$$

Then, from system (1.5), we can obtain

$$\begin{aligned}
u_1(k+1) &= N_1(k)\exp\left\{b_1(k) - a_1(k)N_1(k) - \frac{\alpha_1(k)N_1(k)N_2(k)}{1+mN_1^2(k)}\right\} \\
&\quad - N_1^*(k)\exp\left\{b_1(k) - a_1(k)N_1^*(k) - \frac{\alpha_1(k)N_1^*(k)N_2^*(k)}{1+mN_1^{*2}(k)}\right\} \\
&= \left[N_1(k)\exp\left\{-a_1(k)u_1(k) - \alpha_1(k)\left[\frac{N_1(k)N_2(k)}{1+mN_1^2(k)} - \frac{N_1^*(k)N_2^*(k)}{1+mN_1^{*2}(k)}\right]\right\} - N_1^*(k)\right] \\
&\quad \times \exp\left\{b_1(k) - a_1(k)N_1^*(k) - \frac{\alpha_1(k)N_1^*(k)N_2^*(k)}{1+mN_1^{*2}(k)}\right\} \\
&= \left\{\left[1 - a_1(k)N_1^*(k) - \frac{\alpha_1(k)N_1^*(k)N_2^*(k)(1-mN_1^{*2}(k))}{(1+mN_1^{*2}(k))^2}\right]\frac{u_1(k)}{N_1^*(k)}\right. \\
&\quad \left. - \frac{\alpha_1(k)N_1^*(k)}{1+mN_1^{*2}(k)}u_2(k) + f_1\right\}N_1^*(k+1), \\
u_2(k+1) &= N_2(k)\exp\left\{-b_2(k) - a_2(k)N_2(k) + \frac{\alpha_2(k)N_1^2(k)}{1+mN_1^2(k)}\right\} \\
&\quad - N_2^*(k)\exp\left\{-b_2(k) - a_2(k)N_2^*(k) + \frac{\alpha_2(k)N_1^{*2}(k)}{1+mN_1^{*2}(k)}\right\} \\
&= \left[N_2(k)\exp\left\{-a_2(k)u_2(k) + \frac{\alpha_2(k)N_1^2(k)}{1+mN_1^2(k)} - \frac{\alpha_2(k)N_1^{*2}(k)}{1+mN_1^{*2}(k)}\right\} - N_2^*(k)\right] \\
&\quad \times \exp\left\{-b_2(k) - a_2(k)N_2^*(k) + \frac{\alpha_2(k)N_1^{*2}(k)}{1+mN_1^{*2}(k)}\right\} \\
&= \left[(1 - a_2(k)N_2^*(k))\frac{u_2(k)}{N_2^*(k)} + \frac{2\alpha_2(k)N_1^*(k)}{(1+mN_1^{*2}(k))^2}u_1(k) + f_2\right] \\
&\quad \times N_2^*(k+1),
\end{aligned} \tag{3.6}$$

where $|f_i|/\|u\|$ converges, uniformly with respect to $k \in N$, to zero as $\|u\| \to 0$.

Define a function V by

$$V(u(k)) = c_1\left|\frac{u_1(k)}{N_1^*(k)}\right| + c_2\left|\frac{u_2(k)}{N_2^*(k)}\right|, \tag{3.7}$$

where c_1, c_2 are positive constants given in (3.1). Calculating the difference of V along the solution of the system, we obtain

$$\begin{aligned}
\Delta V &= c_1\left(\left|\frac{u_1(k+1)}{N_1^*(k+1)}\right| - \left|\frac{u_1(k)}{N_1^*(k)}\right|\right) + c_2\left(\left|\frac{u_2(k+1)}{N_2^*(k+1)}\right| - \left|\frac{u_2(k)}{N_2^*(k)}\right|\right)\\
&\le -\left\{c_1a_1(k) + \frac{c_1\alpha_1(k)N_2^*(k)\left(1-mN_1^{*2}(k)\right)}{\left(1+mN_1^{*2}(k)\right)^2}\right\}\left|u_1(k)\right|\\
&\quad + c_1\frac{\alpha_1(k)N_1^*(k)}{1+mN_1^{*2}(k)}\left|u_2(k)\right| - c_2a_2(k)\left|u_2(k)\right|\\
&\quad + c_2\frac{2\alpha_2(k)N_1^*(k)}{\left(1+mN_1^{*2}(k)\right)^2}\left|u_1(k)\right| + \sum_{i=1}^{2}c_i\left|f_i\right|\\
&= -\left\{c_1a_1(k) + \frac{c_1\alpha_1(k)N_2^*(k)\left(1-mN_1^{*2}(k)\right)}{\left(1+mN_1^{*2}(k)\right)^2} - c_2\frac{2\alpha_2(k)N_1^*(k)}{\left(1+mN_1^{*2}(k)\right)^2}\right\}\left|u_1(k)\right|\\
&\quad - \left\{c_2a_2(k) - c_1\frac{\alpha_1(k)N_1^*(k)}{1+mN_1^{*2}(k)}\right\}\left|u_2(k)\right| + \sum_{i=1}^{2}c_i\left|f_i\right|\\
&\le -\left\{c_1a_1(k) + \frac{c_1\alpha_1(k)\exp\{H_{22}\}}{1+m\exp\{2H_{11}\}} - \frac{c_1\alpha_1(k)\exp\{H_{21}\}}{4}\right.\\
&\qquad\left. - \frac{c_2\alpha_2(k)}{\sqrt{m}\left(1+m\exp\{2H_{12}\}\right)}\right\}\left|u_1(k)\right|\\
&\quad - \left\{c_2a_2(k) - c_1\frac{\alpha_1(k)}{2\sqrt{m}}\right\}\left|u_2(k)\right| + \sum_{i=1}^{2}c_i\left|f_i\right|.
\end{aligned} \tag{3.8}$$

Since $|f_i|/\|u\|$ converges uniformly to zero as $\|u\| \to 0$, it follows from conditions (3.1) and (3.2) that there is a positive σ such that if k is sufficiently large and $\|u\| < \sigma$, then

$$\Delta V \le -\frac{\nu}{2}\left\{\left|u_1(k)\right| + \left|u_2(k)\right|\right\} < -\frac{\nu}{4}\|u\|. \tag{3.9}$$

This means that the trivial solution of (3.6) is uniformly asymptotically stable, and so is the solution $N^*(k) = (N_1^*(k), N_2^*(k))$ of (1.5).

Notice that

$$\max\{p(x), q(x)\} = \frac{\left(\left|p(x)-q(x)\right| + p(x) + q(x)\right)}{2} \le \left|p(x)\right| + \left|q(x)\right|. \tag{3.10}$$

Define

$$\begin{aligned}
\Phi(x) &= \frac{2x}{\min\{\exp\{H_{12}\}, \exp\{H_{22}\}\}},\\
\Psi(x) &= \frac{x}{\max\{\exp\{H_{11}\}, \exp\{H_{21}\}\}}.
\end{aligned} \tag{3.11}$$

Then

$$\Psi(\|u\|) \leq V(u(k)) \leq \Phi(\|u\|). \tag{3.12}$$

From the Lyapunov asymptotic stability theorem [1], also in view of the positive definition of V and (3.9), we obtain that the trivial solution of (3.6) is globally asymptotically stable. By the medium of (3.5), we reach the conclusion that the solution $N^*(k) = (N_1^*(k), N_2^*(k))$ of (1.5) is globally asymptotically stable. The proof is complete. □

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