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Research Article

Global Asymptotic Stability in a Class of Difference Equations

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We study the difference equation $x_n = [(f \times g_1 + g_2 + h)/(g_1 + f \times g_2 + h)](x_{n-1},\dots,x_{n-r})$, $n = 1,2,\dots,$ $x_{1-r},\dots,x_0 > 0$, where $f, g_1, g_2 : (R_+)^r \to R_+$ and $h : (R_+)^r \to [0,+\infty)$ are all continuous functions, and $\min_{1\le i\le r}\{u_i, 1/u_i\} \le f(u_1,\dots,u_r) \le \max_{1\le i\le r}\{u_i, 1/u_i\}$, $(u_1,\dots,u_r)^T \in (R_+)^r$. We prove that this difference equation admits $c = 1$ as the globally asymptotically stable equilibrium. This result extends and generalizes some previously known results.

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1. Introduction

Ladas [1] suggested investigating the nonlinear difference equation

$$x_n = \frac{x_{n-1} + x_{n-2}x_{n-3}}{x_{n-1}x_{n-2} + x_{n-3}}, \quad n = 1,2,\dots, \; x_{-2}, x_{-1}, x_0 > 0. \tag{1.1}$$

Since then, it has been proved that $c = 1$ is the common globally asymptotically stable equilibrium of this difference equation and all of the following difference equations (where a and b are nonnegative constants):

$$x_n = \frac{x_{n-2} + x_{n-1}x_{n-3}}{x_{n-1}x_{n-2} + x_{n-3}}, \quad n = 1,2,\dots, \; x_{-2}, x_{-1}, x_0 > 0 \quad (\text{see [1]}), \tag{1.2}$$

$$x_n = \frac{x_{n-1}x_{n-2} + x_{n-3} + a}{x_{n-1} + x_{n-2}x_{n-3} + a}, \quad n = 1,2,\dots, \; x_{-2}, x_{-1}, x_0 > 0 \quad (\text{see [6,12]}), \tag{1.3}$$

$$x_n = \frac{x_{n-2} + x_{n-1}x_{n-3} + a}{x_{n-1}x_{n-2} + x_{n-3} + a}, \quad n = 1,2,\dots, \; x_{-2}, x_{-1}, x_0 > 0 \quad (\text{see [6]}), \tag{1.4}$$

$$x_n = \frac{x_{n-1}+x_{n-2}x_{n-3}+a}{x_{n-1}x_{n-2}+x_{n-3}+a}, \quad n=1,2,\ldots,\ x_{-2},x_{-1},x_0>0 \quad \text{(see [14])}, \tag{1.5}$$

$$x_n = \frac{x_{n-1}x_{n-2}+x_{n-3}+a}{x_{n-2}+x_{n-1}x_{n-3}+a}, \quad n=1,2,\ldots,\ x_{-2},x_{-1},x_0>0 \quad \text{(see [14])}, \tag{1.6}$$

$$x_n = \frac{x_{n-1}^b x_{n-3}+x_{n-4}^b+a}{x_{n-1}^b+x_{n-3}x_{n-4}^b+a}, \quad n=1,2,\ldots,\ x_{-3},x_{-2},x_{-1},x_0>0 \quad \text{(see [7])}, \tag{1.7}$$

$$x_n = \frac{x_{n-k}^b x_{n-m}+x_{n-l}^b+a}{x_{n-k}^b+x_{n-m}x_{n-l}^b+a}, \quad n=1,2,\ldots,\ x_{1-\max\{k,m,l\}},\ldots,x_0>0 \quad \text{(see [8])}. \tag{1.8}$$

Motivated by the above work and the work by Sun and Xi [2], this article addresses the difference equation

$$x_n = \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n-1},\ldots,x_{n-r}), \quad n=1,2,\ldots,\ x_{1-r},\ldots,x_0>0, \tag{1.9}$$

where $f,g_1,g_2:(R_+)^r\to R_+$ and $h:(R_+)^r\to[0,+\infty)$ are all continuous functions, and

$$\min_{1\le i\le r}\{u_i,1/u_i\} \le f(u_1,\ldots,u_r) \le \max_{1\le i\le r}\{u_i,1/u_i\}, \quad (u_1,\ldots,u_r)^T\in(R_+)^r. \tag{1.10}$$

It can be seen that (1.9) subsumes (1.1) and (1.8). For example, if we let $r=\max\{k,l,m\}$, $f(x_{n-1},\ldots,x_{n-r})=x_{n-m}$, $g_1(x_{n-1},\ldots,x_{n-r})=x_{n-k}^b$, $g_2(x_{n-1},\ldots,x_{n-r})=x_{n-l}^b$, and $h(x_1,\ldots,x_r)\equiv a$, then (1.9) reduces to (1.8).

We prove that (1.9) admits $c=1$ as the globally asymptotically stable equilibrium. As a consequence, our result includes all of the above-mentioned results.

2. Preliminary knowledge

For two functions, $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$, we adopt the following notations:

$$\begin{aligned}
[f+g](x_1,\ldots,x_n) &:= f(x_1,\ldots,x_n)+g(x_1,\ldots,x_n),\\
[f\times g](x_1,\ldots,x_n) &:= f(x_1,\ldots,x_n)\times g(x_1,\ldots,x_n),\\
\left[\frac{f}{g}\right](x_1,\ldots,x_n) &:= \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)} \quad \text{if } g(x_1,\ldots,x_n)\neq 0.
\end{aligned} \tag{2.1}$$

Let R_+ denote the whole set of positive real numbers. The part metric (or Thompson's metric) [3, 4] is a metric defined on $(R_+)^r$ in the following way: for any $X=(x_1,\ldots,x_r)^T\in(R_+)^r$ and

$$Y=(y_1,\ldots,y_r)^T\in(R_+)^r, \qquad p(X,Y):=-\log_2\min_{1\le i\le r}\{x_i/y_i,y_i/x_i\}. \tag{2.2}$$

Theorem 2.1 (see [5, Theorem 2.2], see also [3]). *Let $T:(R_+)^r \to (R_+)^r$ be a continuous mapping with an equilibrium $C \in (R_+)^r$. Consider the following difference equation:*

$$X_n = T(X_{n-1}), \quad n = 1,2,\ldots,\ X_0 \in (R_+)^r. \tag{2.3}$$

Suppose there is a positive integer k such that $p(T^k(X),C) < p(X,C)$ holds for all $X \neq C$. Then C is globally asymptotically stable.

Theorem 2.2 (see [6, page 1]). *Let $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, $c_1,\ldots,c_n$ be positive numbers. Then*

$$\min\left\{\frac{a_i}{b_i} : 1 \le i \le n\right\} \le \frac{\sum_{i=1}^n c_i a_i}{\sum_{i=1}^n c_i b_i} \le \max\left\{\frac{a_i}{b_i} : 1 \le i \le n\right\}. \tag{2.4}$$

Moreover, one of the two equalities holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

3. Main result

The main result of this article is the following.

Theorem 3.1. *Consider the difference equation*

$$x_n = \left[\frac{f \times g_1 + g_2 + h}{g_1 + f \times g_2 + h}\right](x_{n-1},\ldots,x_{n-r}), \quad n = 1,2,\ldots,\ x_{1-r},\ldots,x_0 > 0, \tag{3.1}$$

where $f,g_1,g_2:(R_+)^r \to R_+$ and $h:(R_+)^r \to [0,+\infty)$ are all continuous functions, and

$$\min_{1\le i\le r}\{u_i, 1/u_i\} \le f(u_1,\ldots,u_r) \le \max_{1\le i\le r}\{u_i, 1/u_i\}, \quad (u_1,\ldots,u_r)^T \in (R_+)^r. \tag{3.2}$$

Let $\{x_n\}$ be a solution of (3.1). Then the following assertions hold:

(i) *for all $n \ge 1$ and $j \ge 0$, one has*

$$\min_{1\le i\le r}\{x_{n-i}, 1/x_{n-i}\} \le x_{n+j} \le \max_{1\le i\le r}\{x_{n-i}, 1/x_{n-i}\}; \tag{3.3}$$

(ii) *there exist $n \ge 1$ and $j \ge 0$ such that one of the two equalities in chain (3.3) holds if and only if $(x_{n-1},\ldots,x_{n-r}) = (1,\ldots,1)$;*

(iii) *$c = 1$ is the globally asymptotically stable equilibrium of (3.1).*

Proof. (i) For any given $n \geq 1$, we prove the assertion by induction on j. By Theorem 2.2 and chain (3.3), we have

$$\begin{aligned}
x_n &= \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n-1},\dots,x_{n-r}) \geq \min\left\{f(x_{n-1},\dots,x_{n-r}),\frac{1}{f(x_{n-1},\dots,x_{n-r})}\right\}\\
&\geq \min\left\{\min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\},\frac{1}{\max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}}\right\} = \min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\},\\
x_n &= \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n-1},\dots,x_{n-r}) \leq \max\left\{f(x_{n-1},\dots,x_{n-r}),\frac{1}{f(x_{n-1},\dots,x_{n-r})}\right\}\\
&\leq \max\left\{\max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\},\frac{1}{\max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}}\right\} = \max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}.
\end{aligned} \tag{3.4}$$

So the assertion is true for $j=0$.

Suppose the assertion is true for all integer k $(0\leq k\leq j-1)$, that is,

$$\min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\} \leq x_{n+k} \leq \max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}, \quad 0\leq k\leq j-1. \tag{3.5}$$

By Theorem 2.2, chain (3.2), and the inductive hypothesis, we get

$$\begin{aligned}
x_{n+j} &= \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n+j-1},\dots,x_{n+j-r})\\
&\geq \min\left\{f(x_{n+j-1},\dots,x_{n+j-r}),\frac{1}{f(x_{n+j-1},\dots,x_{n+j-r})}\right\}\\
&\geq \min\left\{\min_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\},\frac{1}{\max_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\}}\right\} = \min_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\}\\
&\geq \min\left\{\min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\},\frac{1}{\max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}}\right\} = \min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\};
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
x_{n+j} &= \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n+j-1},\dots,x_{n+j-r})\\
&\leq \max\left\{f(x_{n+j-1},\dots,x_{n+j-r}),\frac{1}{f(x_{n+j-1},\dots,x_{n+j-r})}\right\}\\
&\leq \max\left\{\max_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\},\frac{1}{\min_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\}}\right\} = \max_{1\leq i\leq r}\{x_{n+j-i},1/x_{n+j-i}\}\\
&\leq \max\left\{\max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\},\frac{1}{\min_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}}\right\} = \max_{1\leq i\leq r}\{x_{n-i},1/x_{n-i}\}.
\end{aligned} \tag{3.7}$$

Thus the assertion is true for j. The inductive proof of this assertion is complete.

(ii) The sufficiency follows immediately from the first assertion of this theorem. *Necessity.* Suppose there exist $n \geq 1$ and $j \geq 0$ such that $x_{n+j} = \min_{1\leq i\leq r}\{x_{n-i}, 1/x_{n-i}\}$. Then all of the equalities in chain (3.6) hold. This chain of equalities plus Theorem 2.2 yield $f(x_{n+j-1},\dots,x_{n+j-r}) = 1$ and, hence, $(x_{n-1},\dots,x_{n-r}) = (1,\dots,1)$. Likewise, one can show that $(x_{n-1},\dots,x_{n-r}) = (1,\dots,1)$ if $x_{n+j} = \max_{1\leq i\leq r}\{x_{n-i}, 1/x_{n-i}\}$.

(iii) The system of first-order difference equations associated with (3.1) is

$$Y_n = T(Y_{n-1}), \quad n = 1,2,\dots, \tag{3.8}$$

where $T : (R_+)^r \to (R_+)^r$ is a mapping defined by

$$T\big((y_1,\dots,y_r)^T\big) = \left(y_2,\dots,y_r,\left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](y_r,\dots,y_1)\right)^T. \tag{3.9}$$

By chain (3.2), we have $f(1,\dots,1) = 1$. Hence, $C = (1,\dots,1)^T$ is an equilibrium of system (3.8). Consider an arbitrary $X = (x_1,\dots,x_r)^T \in (R_+)^r$, $X \neq C$. Then

$$T^r\big((x_1,\dots,x_r)^T\big) = (x_{r+1},\dots,x_{2r})^T, \tag{3.10}$$

where $x_j = [(f\times g_1+g_2+h)/(g_1+f\times g_2+h)](x_{j-1},\dots,x_{j-r})$, $r+1 \leq j \leq 2r$. By the first two assertions of this theorem, we induce

$$\min_{r+1\leq i\leq 2r}\{x_i, 1/x_i\} > \min\left\{\min_{1\leq i\leq r}\{x_i, 1/x_i\}, \frac{1}{\max_{1\leq i\leq r}\{x_i, 1/x_i\}}\right\} = \min_{1\leq i\leq r}\{x_i, 1/x_i\}. \tag{3.11}$$

Hence,

$$p(T^r(X), C) = -\log_2 \min_{r+1\leq i\leq 2r}\{x_i, 1/x_i\} < -\log_2 \min_{1\leq i\leq r}\{x_i, 1/x_i\} = p(X,C). \tag{3.12}$$

By Theorem 2.1, we conclude that C is the globally asymptotically stable equilibrium of system (3.8). This implies that $c = 1$ is the globally asymptotically stable equilibrium of (3.1). □

4. Applications

Example 4.1. Consider the difference equation

$$x_n = \left[\frac{f\times g_1+g_2+h}{g_1+f\times g_2+h}\right](x_{n-1},\dots,x_{n-r}), \quad n = 1,2,\dots,\ x_{1-r},\dots,x_0 > 0, \tag{4.1}$$

where $g_1, g_2 : (R_+)^r \to R_+$ and $h : (R_+)^r \to [0,+\infty)$ are all continuous functions, $1 \leq p \leq r$, $1 \leq q \leq r$, $1 \leq s \leq r$, $f(u_1,\dots,u_r) = (u_p+u_q+u_s)/3$, and $(u_1,\dots,u_r)^T \in (R_+)^r$.

As $f(u_1,\dots,u_r)$ is the arithmetic mean of u_p, u_q, and u_s, we get

$$\begin{aligned} f(u_1,\dots,u_r) &\leq \max\{u_p,u_q,u_s\} \leq \max_{1\leq i\leq r}\{u_i, 1/u_i\}, \\ f(u_1,\dots,u_r) &\geq \min\{u_p,u_q,u_s\} \geq \min_{1\leq i\leq r}\{u_i, 1/u_i\}. \end{aligned} \tag{4.2}$$

By Theorem 3.1, $c = 1$ is the globally asymptotically stable equilibrium of (4.1).

Example 4.2. Consider the difference equation

$$x_n = \left[\frac{f \times g_1 + g_2 + h}{g_1 + f \times g_2 + h}\right](x_{n-1},\ldots,x_{n-r}), \quad n = 1,2,\ldots,\ x_{1-r},\ldots,x_0 > 0, \tag{4.3}$$

where $g_1,g_2 : (R_+)^r \to R_+$ and $h : (R_+)^r \to [0,+\infty)$ are all continuous functions, $1 \le p \le r$, $1 \le q \le r$, $1 \le s \le r$, $f(u_1,\ldots,u_r) = (u_p + u_q + 1/u_s)/3$, and $(u_1,\ldots,u_r)^T \in (R_+)^r$.

As $f(u_1,\ldots,u_r)$ is the arithmetic mean of u_p, u_q, and $1/\ u_s$, we get

$$\begin{aligned} f(u_1,\ldots,u_r) &\le \max\{u_p,u_q,1/u_s\} \le \max_{1\le i\le r}\{u_i,1/u_i\}, \\ f(u_1,\ldots,u_r) &\ge \min\{u_p,u_q,1/u_s\} \ge \min_{1\le i\le r}\{u_i,1/u_i\}, \end{aligned} \tag{4.4}$$

By Theorem 3.1, $c = 1$ is the globally asymptotically stable equilibrium of (4.3).

Example 4.3. Consider the difference equation

$$x_n = \left[\frac{f \times g_1 + g_2 + h}{g_1 + f \times g_2 + h}\right](x_{n-1},\ldots,x_{n-r}), \quad n = 1,2,\ldots,\ x_{1-r},\ldots,x_0 > 0, \tag{4.5}$$

where $g_1,g_2 : (R_+)^r \to R_+$ and $h : (R_+)^r \to [0,+\infty)$ are all continuous functions, $1 \le p \le r$, $1 \le q \le r$, $1 \le s \le r$, $f(u_1,\ldots,u_r) = \sqrt[3]{u_p u_q / u_s}$, and $(u_1,\ldots,u_r)^T \in (R_+)^r$

As $f(u_1,\ldots,u_r)$ is the geometric mean of u_p, u_q, and $1/u_s$, we get

$$\begin{aligned} f(u_1,\ldots,u_r) &\le \max\{u_p,u_q,1/u_s\} \le \max_{1\le i\le r}\{u_i,1/u_i\}, \\ f(u_1,\ldots,u_r) &\ge \min\{u_p,u_q,1/u_s\} \ge \min_{1\le i\le r}\{u_i,1/u_i\}. \end{aligned} \tag{4.6}$$

By Theorem 3.1, $c = 1$ is the globally asymptotically stable equilibrium of (4.5).

5. Conclusions

This article has studied the global asymptotic stability of a class of difference equations. The result obtained extends and generalizes some previous results. We are attempting to apply the technique used in this article to deal with other generic difference equations which include some well-studied difference equations such as those in [7, 8].

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