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Research Article

Triple Positive Solutions of Fourth-Order Four-Point Boundary Value Problems for p-Laplacian Dynamic Equations on Time Scales

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A new triple fixed-point theorem is applied to investigate the existence of at least triple positive solutions of fourth-order four-point boundary value problems for p-Laplacian dynamic equations on a time scale. The interesting point is that we choose an inversion technique employed by Avery and Peterson in 1998.

1. Introduction

Recently, there have been many papers working on the existence of positive solutions to boundary value problems for differential equations on time scales (see, e.g., [1–4] and the references therein). This has been mainly due to their unification of the theory of differential and difference equations. An introduction to this unification is given in [5]. Now, this study is still a new area of fairly theoretical exploration in mathematics. However, it has led to several important applications, for example, in the study of insect population models, neural networks, heat transfer, and epidemic models (see, e.g., [1, 5]). We let $\mathbb{T}$ be any time scale (nonempty closed subset of $\mathbb{R}$) and let $[a,b]$ be subset of $\mathbb{T}$ such that $[a,b]=\{t\in\mathbb{T}: a\leq t\leq b\}$. Thus, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_o$, that is, the real numbers, the integers, the natural numbers, and the nonnegative integers, are examples of time scales.

In this paper, we study the existence of multiple positive solutions for the fourth-order four-point nonlinear dynamic equation on time scales with p-Laplacian:

$$\left(\phi_p\left(x^{\Delta\nabla}(t)\right)\right)^{\nabla\Delta}-w(t)f\left(x(t)\right)=0,\quad t\in(0,1),\tag{1.1}$$

subject to the following boundary conditions:

$$x(0) - \lambda x^{\Delta}(\eta) = x^{\Delta}(1) = 0, \qquad x^{\Delta\nabla}(0) = \alpha_1 x^{\Delta\nabla}(\xi), \qquad x^{\Delta\nabla}(1) = \beta_1 x^{\Delta\nabla}(\xi), \tag{1.2}$$

where $\lambda \geq 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $0 < \xi, \eta < 1$, $\phi_p(s)$ is p-Laplacian operator, that is to say, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, and

(H_1) the function $f : R \to [0, +\infty)$ is continuous,

(H_2) $w(t) \in C_{\mathrm{rd}}([0,1], [0, +\infty))$, where $C_{\mathrm{rd}}([0,1], [0, +\infty))$ denotes the set of all right dense continuous functions from $\mathbb{T}$ to $[0, +\infty)$.

We remark that by a solution x of (1.1)-(1.2) we mean $x : \mathbb{T} \to \mathbb{R}$ which is delta/nabla differentiable; $x^{\Delta\nabla}$ and $(|x^{\Delta\nabla}|^{p-2}x^{\Delta\nabla})^{\nabla\Delta}$ are both continuous on $\mathbb{T}_{k^2} \cap \mathbb{T}^{k^2}$, and x satisfies (1.1)-(1.2). If $x^{\Delta\nabla}(t) \leq 0$ on $[0,1]_{\mathbb{T}_k \cap \mathbb{T}^k}$, then we say that x is concave on $[0,1]$.

p-Laplacian problems with two-, three-, m-point boundary conditions for ordinary differential equations and finite difference equations have been studied extensively (see [6, 7] and the references therein). However, there are not many concerning the p-Laplacian problems on time scales. In this paper, by using a new triple fixed-point theorem due to Avery [8] in a cone, we prove that there exist at least triple positive solutions of problem (1.1)-(1.2). Our results generalize the recent paper by Bai et al. [7] to some degree. Meanwhile, we choose an inversion technique to simplify our arguments, which is a variation of the technique employed by Avery and Peterson in [9].

The time-scale-related notations adopted in this paper can be found in almost all literatures related to time scales. The readers who are unfamiliar with this area can consult, for example, [5] for details. Throughout this paper, we assume that $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with $0 \in \mathbb{T}_{k^2}$, $1 \in \mathbb{T}^{k^2}$.

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces, and we then state the triple fixed-point theorem for a cone-preserving operator. The following definitions can be found in the book by Guo and Lakshmikantham [10].

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$,

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\bigl(tx + (1-t)y\bigr) \geq t\alpha(x) + (1-t)\alpha(y) \tag{2.1}$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, one says that the map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y) \tag{2.2}$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, and θ be nonnegative continuous convex functionals on P, α and ψ nonnegative continuous concave functionals on P. Then, for positive real numbers h, a, b, d, and c, one defines the following convex sets:

$$\begin{gathered} P(\gamma, c) = \{x \in P \mid \gamma(x) < c\}, \qquad P(\gamma, \theta, \alpha, a, b, c) = \{x \in P \mid a \leq \alpha(x),\ \theta(x) \leq b,\ \gamma(x) \leq c\}, \\ P(\gamma, \alpha, a, c) = \{x \in P \mid a \leq \alpha(x),\ \gamma(x) \leq c\}, \qquad Q(\gamma, \beta, d, c) = \{x \in P \mid \beta(x) \leq d,\ \gamma(x) \leq c\}, \\ Q(\gamma, \beta, \psi, h, d, c) = \{x \in P \mid h \leq \psi(x),\ \beta(x) \leq d,\ \gamma(x) \leq c\}. \end{gathered} \tag{2.3}$$

The following fixed-point theorem due to Avery [8] which is a generalization of the Leggett-Williams fixed-point theorem will be fundamental.

Theorem 2.3 (see [8]). *Let P be a cone in a real Banach space E. Let γ, β, and θ be nonnegative continuous convex functionals on P, α, ψ nonnegative continuous concave functionals on P, and there exist two positive numbers c and M such that*

$$\alpha(x) \leq \beta(x), \qquad \|x\| \leq M\gamma(x) \tag{2.4}$$

for all $x \in \overline{P(\gamma, c)}$. Suppose $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist positive numbers $h, d, a, b \geq 0$, with $0 < d < a$ such that

(S$_1$) *$\{x \in P(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$ and $\alpha(Tx) > a$ for $x \in P(\gamma, \theta, \alpha, a, b, c)$;*

(S$_2$) *$\{x \in Q(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$ and $\beta(Tx) < d$ for $x \in Q(\gamma, \beta, \psi, h, d, c)$;*

(S$_3$) *$\alpha(Tx) > a$ for $x \in P(\gamma, \alpha, a, c)$ with $\theta(Tx) > b$;*

(S$_4$) *$\beta(Tx) < d$ for $x \in Q(\gamma, \beta, d, c)$ with $\psi(Tx) < h$.*

Then, T has at least three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad d < \beta(x_3), \quad \text{with } \alpha(x_3) < a. \tag{2.5}$$

3. Main results

Let E denote the real Banach space $E = C([\rho^2(0), \sigma(1)], R)$ with the norm

$$\|x\|_{0,T} := \sup\{|x(t)| : t \in [\rho^2(0), \sigma(1)]\}, \quad x \in E. \tag{3.1}$$

Define a cone $P \subset E$ by

$$P = \{x \in E : x \geq 0,\ x(t) \text{ is concave and nondecreasing on } [\rho^2(0), \sigma(1)]\}. \tag{3.2}$$

In our main results, we will make use of the following lemmas.

Lemma 3.1. *If $y \in E$, then BVP*

$$\begin{gathered} -x^{\Delta\nabla}(t) = -\phi_q(y(t)), \quad t \in (0,1), \\ x(0) - \lambda x^{\Delta}(\eta) = 0, \quad x^{\Delta}(1) = 0, \end{gathered} \tag{3.3}$$

has a unique solution

$$x(t) = -\int_0^1 h(t,s)\phi_q(y(s))\nabla s, \tag{3.4}$$

where

$$h(t,s) = \begin{cases} s, & s \leq t < \eta \text{ or } s \leq \eta \leq t, \\ t, & t \leq s \leq \eta, \\ s+\lambda, & \eta \leq s \leq t, \\ t+\lambda, & \eta \leq t \leq s \text{ or } t < \eta \leq s. \end{cases} \tag{3.5}$$

Proof. In fact, if x is a solution of (3.3), then

$$x(t) = \int_0^t (t-s)\phi_q(y(s))\nabla s + \overline{A}t + \overline{B}, \quad t \in (0,1). \tag{3.6}$$

By the boundary condition of (3.3), we have

$$\overline{A} = -\int_0^1 \phi_q(y(s))\nabla s, \qquad \overline{B} = x(0) = \lambda x^{\Delta}(\eta) = -\lambda\int_\eta^1 \phi_q(y(s))\nabla s. \tag{3.7}$$

Therefore,

$$\begin{aligned} x(t) &= \int_0^t (t-s)\phi_q(y(s))\nabla s - t\int_0^1 \phi_q(y(s))\nabla s - \lambda\int_\eta^1 \phi_q(y(s))\nabla s \\ &= -\int_0^1 h(t,s)\phi_q(y(s))\nabla s. \end{aligned} \tag{3.8}$$

□

Lemma 3.2. *If $y \in E$, then BVP*

$$\begin{gathered} -y^{\nabla\Delta}(t) = -w(t)f(x(t)), \quad t \in (0,1), \\ y(0) = \phi_p(\alpha_1)y(\xi), \qquad y(1) = \phi_p(\beta_1)y(\xi) \end{gathered} \tag{3.9}$$

has a unique solution

$$y(t) = -\frac{1}{M}\int_0^1 g(t,s)w(s)f(x(s))\Delta s, \tag{3.10}$$

where

$$g(t,s) = \begin{cases} s(1-t) + \phi_p(\beta_1)s(t-\xi), & s \leq t < \xi \text{ or } s \leq \xi \leq t, \\ t(1-s) + \phi_p(\beta_1)t(s-\xi) + \phi_p(\alpha_1)(1-\xi)(s-t), & t \leq s \leq \xi, \\ s(1-t) + \phi_p(\beta_1)\xi(t-s) + \phi_p(\alpha_1)(1-t)(\xi-s), & \xi \leq s \leq t, \\ (1-s)(t - \phi_p(\alpha_1)t + \phi_p(\alpha_1)\xi), & \xi \leq t \leq s \text{ or } t < \xi \leq s, \end{cases} \tag{3.11}$$

and $M = 1 - \phi_p(\alpha_1) - (\phi_p(\beta_1) - \phi_p(\alpha_1))\xi \neq 0$.

Proof. In fact, if y is a solution of (3.9), then

$$y(t) = \int_0^t (t-s)w(s)f(x(s))\Delta s + A^*t + B^*, \quad t \in (0,1). \tag{3.12}$$

By the boundary condition of (3.9), we have

$$\begin{aligned} B^* &= \phi_p(\alpha_1)\int_0^\xi (\xi - s)w(s)f(s,x(s))\Delta s + \phi_p(\alpha_1)\xi A^* + \phi_p(\alpha_1)B^*, \\ &\int_0^1 (1-s)w(s)f(s,x(s))\Delta s + A^* + B^* \\ &\quad = \phi_p(\beta_1)\int_0^\xi (\xi - s)w(s)f(s,x(s))\Delta s + \phi_p(\beta_1)\xi A^* + \phi_p(\beta_1)B^*. \end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned} y(t) &= \int_0^t (t-s)w(s)f(s,x(s))\Delta s - \frac{(1-\phi_p(\alpha_1))t}{M}\int_0^1 (1-s)f(s)\Delta s \\ &\quad + \frac{(\phi_p(\beta_1)+\phi_p(\alpha_1))t}{M}\int_0^\xi (\xi - s)f(s)\Delta s - \frac{\phi_p(\alpha_1)\xi}{M}\int_0^1 (1-s)f(s)\Delta s \\ &\quad + \frac{\phi_p(\alpha_1)}{M}\int_0^\xi (\xi - s)w(s)f(s,x(s))\Delta s \\ &= -\frac{1}{M}\int_0^1 g(t,s)w(s)f(s,x(s))\Delta s. \end{aligned} \tag{3.14}$$

Obviously, if $M > 0$, then $g(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$.

Suppose that x is a solution of (1.1)-(1.2). Then, from Lemma 3.1, we have

$$x(t) = -\int_0^1 h(t,s)\phi_q(y(s))\nabla s. \tag{3.15}$$

Substituting (3.10) into (3.15), we have

$$x(t) = \frac{1}{\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s. \tag{3.16}$$

□

Lemma 3.3. *Assume that* $0 < t_1 < t_2 < 1$ *and* $\eta, \xi \in (0,1)$. *Then, one has for* $s \in [\rho^2(0), \sigma(1)]$,

$$\begin{gathered} \frac{h(t_1,s)}{h(t_2,s)} \geq \frac{t_1}{t_2}, \\ \frac{h(1,s)}{h(\xi,s)} \leq \frac{1}{\xi}. \end{gathered} \tag{3.17}$$

The proof follows by routine calculations.

For the sake of applying Theorem 2.3, we let the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ be defined on the cone P by

$$\gamma(x) = \max_{t\in[0,t_3]} x(t) = x(t_3), \quad x \in P, \qquad \psi(x) = \min_{t\in[\xi,\sigma(1)]} x(t) = x(\xi), \quad x \in P,$$
$$\beta(x) = \max_{t\in[\xi,\sigma(1)]} x(t) = x(\sigma(1)), \quad x \in P, \qquad \alpha(x) = \min_{t\in[t_1,t_2]} x(t) = x(t_1), \quad x \in P, \tag{3.18}$$
$$\theta(x) = \max_{t\in[t_1,t_2]} x(t) = x(t_2), \quad x \in P,$$

where t_1, t_2, and $t_3 \in (0,1)$ and $t_1 < t_2$.

It is clear that

$$\alpha(x) = x(t_1) \le x(\sigma(1)) = \beta(x), \quad \|x\|_{0,T} = x(\sigma(1)) \le \frac{\sigma(1)}{t_3}x(t_3) = \frac{\sigma(1)}{t_3}\gamma(x), \quad \text{for } x \in P. \tag{3.19}$$

Theorem 3.4. *Assume that* (H_1) *and* (H_2) *hold. Let positive numbers* $0 < a < b < c$ *satisfy* $0 < a < b < (t_1/t_2)b \le c$ *and suppose that* $f(x)$ *satisfies the following conditions:*

(H_3) $f(x) \le \phi_q(a/c)$ *for* $0 \le x \le a$,

(H_4) $f(x) \ge \phi_q(b/B)$ *for* $b \le x \le (t_1/t_2)b$,

(H_5) $f(x) \le \phi_q(c/A)$ *for* $0 \le x \le (\sigma(1)/t_3)c$,

where

$$A = \frac{1}{\phi_q(M)}\int_0^1 h(t_3,s)\left[\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\Delta\tau\right)\right]\nabla s,$$
$$B = \frac{1}{\phi_q(M)}\int_0^1 h(t_1,s)\left[\phi_q\left(\int_{t_1}^{t_2} g(s,\tau)w(\tau)\Delta\tau\right)\right]\nabla s,$$
$$C = \frac{1}{\phi_q(M)}\int_0^1 h(\sigma(1),s)\left[\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\Delta\tau\right)\right]\nabla s, \tag{3.20}$$
$$M = 1 - \phi_p(\alpha_1) - (\phi_p(\beta_1) - \phi_p(\alpha_1))\xi > 0.$$

Then, boundary value problem (1.1)-(1.2) *has at least three positive solutions* $x_1, x_2, x_3 \in \overline{P(\gamma,c)}$ *satisfying*

$$x_i(t_3) \le c, \quad i = 1,2,3, \qquad x_1(t_1) > b, \qquad x_2(\sigma(1)) < a, \qquad x_3(t_1) < b, \qquad x_3(\sigma(1)) > a. \tag{3.21}$$

Proof. Define a completely continuous operator $\Psi : P_1 \to E$ by

$$(\Psi x)(t) = \frac{1}{\phi_q(M)}\int_0^1 h(t,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s. \tag{3.22}$$

It is not difficult to prove that x is a positive solution of (1.1)-(1.2) if and only if x is a fixed point of Ψ in P.

First, we prove that $\Psi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

By $M > 0$, we obtain $\Psi x \geq 0$ for $x \in P$. On the other hand, by (3.22), we have

$$\begin{aligned} (\Psi x)^{\Delta}(t) &= \int_t^1 \phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \geq 0, \\ (\Psi x)^{\Delta\nabla}(t) &= -\phi_q\left(\int_0^1 g(t,s)w(s)f(x(s))\right)\Delta s \leq 0. \end{aligned} \tag{3.23}$$

Therefore, $\Psi : P \to P_1$.

For $x \in \overline{P(\gamma, c)}$, notice that $\alpha(x) \leq \beta(x)$, $\|x\| \leq (\sigma(1)/t_3)\gamma(x) \leq (\sigma(1)/t_3)c$; so $0 \leq x(t) \leq (\sigma(1)/t_3)\gamma(x) \leq (\sigma(1)/t_3)c$. By ($H_5$), we have

$$\begin{aligned} \gamma(\Psi x) &= \max_{t\in[0,t_3]}\left|(\Psi x)(t)\right| = (\Psi x)(t_3) \\ &= \frac{1}{\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\ &= \frac{1}{\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\phi_p\frac{c}{A}\Delta\tau\right)\nabla s \\ &\leq \frac{C}{A}\frac{1}{\phi_q(M)}\int_0^1 h(t_3,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\Delta\tau\right)\nabla s \\ &= c. \end{aligned} \tag{3.24}$$

Therefore, $\Psi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

To check condition (S_1) of Theorem 2.3, we choose

$$x_1(t) = b + \varepsilon_1, \quad \text{for } 0 < \varepsilon_1 < \frac{t_2}{t_1}b - b, \qquad x_2(t) = a - \varepsilon_2, \quad \text{for } 0 < \varepsilon_2 < a - t_3 a. \tag{3.25}$$

It is easy to see that $x_1 \in P(\gamma, \theta, \alpha, b, (t_2/t_1)b, c)$, $x_2 \in Q(\gamma, \beta, \psi, \xi a, a, c)$, and $\alpha(x_1) > b$, $\beta(x_2) < a$. Therefore,

$$\left\{x \in P\left(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c\right) \mid \alpha(x) > b\right\} \neq \varnothing, \qquad \left\{x \in Q(\gamma, \beta, \psi, \xi a, a, c) \mid \beta(x) < a\right\} \neq \varnothing. \tag{3.26}$$

Hence, for $t \in [t_1, t_2]$, $x \in P(\gamma, \theta, \alpha, b, (t_2/t_1)b, c)$, there are

$$x(t) \geq x(t_1) = \alpha(x) \geq b, \qquad x(t) \leq x(t_2) = \theta(x) \leq \frac{t_2}{t_1}b. \tag{3.27}$$

Thus, for $t \in [t_1, t_2]$, $x \in P(\gamma, \theta, \alpha, b, (t_2/t_1)b, c)(t_1)$, by condition ($H_4$), we have

$$\begin{aligned} \alpha(\Psi x) &= \min_{t\in[t_1,t_2]}\left|(\Psi x)(t)\right| = (\Psi x)(t_1) \\ &= \frac{1}{\phi_q(M)}\int_0^1 h(t_1,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\ &> \frac{1}{\phi_q(M)}\int_0^1 h(t_1,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\phi_p\frac{b}{B}\Delta\tau\right)\nabla s \\ &= \frac{b}{B\phi_q(M)}\int_0^1 h(t_1,s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\Delta\tau\right)\nabla s = b. \end{aligned} \tag{3.28}$$

This shows that condition (S_1) of Theorem 2.3 is satisfied.

Secondly, we prove that $\beta(\Psi x) < a$ for all $x \in Q(\gamma, \beta, \psi, \xi a, a, c)$.

In fact, for $t \in [0, \sigma(1)]$, $x \in Q(\gamma, \beta, \psi, \xi a, a, c)$, there is

$$0 \leq x(t) \leq x(\sigma(1)) = \beta(x) \leq a. \tag{3.29}$$

Thus, for $t \in [0, \sigma(1)]$, $x \in Q(\gamma, \beta, \psi, \xi a, a, c)$, by condition (H_3), we have

$$\begin{aligned}
\beta(\Psi x) &= \max_{t\in[\xi,\sigma(1)]} \left|(\Psi x)(t)\right| = (\Psi x)\,(\sigma(1)) \\
&= \frac{1}{\phi_q(M)} \int_0^1 h(\sigma(1), s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\
&< \frac{1}{\phi_q(M)} \int_0^1 h(\sigma(1), s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\phi_p\frac{a}{C}\Delta\tau\right)\nabla s \\
&= \frac{a}{C\phi_q(M)} \int_0^1 h(\sigma(1), s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)\Delta\tau\right)\nabla s = a.
\end{aligned} \tag{3.30}$$

Thus, condition (S_2) of Theorem 2.3 is satisfied.

Next, we show that $\alpha(\Psi x) > b$ for $x \in P(\gamma, \alpha, b, c)$ and $\theta(\Psi x) > (t_2/t_1)b$. In fact,

$$\begin{aligned}
\alpha(\Psi x) &= \min_{t\in[t_1,t_2]} \left|(\Psi x)(t)\right| = (\Psi x)(t_1) \\
&= \frac{1}{\phi_q(M)} \int_0^1 h(t_1, s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\
&= \frac{1}{\phi_q(M)} \int_0^1 \frac{h(t_1, s)}{h(t_2, s)} h(t_2, s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\
&\geq \frac{t_1}{t_2}(\Psi x)(t_2) = \frac{t_1}{t_2}\theta(\Psi x) > b.
\end{aligned} \tag{3.31}$$

Finally, we show that condition (S_4) of Theorem 2.3 also holds. In fact, for $x \in Q(\gamma, \beta, a, c)$ and $\psi(\Psi x) < \xi a$, we have

$$\begin{aligned}
\beta(\Psi x) &= \max_{t\in[\xi,\sigma(1)]} \left|(\Psi x)(t)\right| = (\Psi x)\,(\sigma(1)) \\
&= \frac{1}{\phi_q(M)} \int_0^1 h(\sigma(1), s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\
&= \frac{1}{\phi_q(M)} \int_0^1 \frac{h(\sigma(1), s)}{h(\xi, s)} h(\xi, s)\phi_q\left(\int_0^1 g(s,\tau)w(\tau)f(x(\tau))\Delta\tau\right)\nabla s \\
&\leq \frac{1}{\xi}(\Psi x)\,(\xi) = \frac{1}{\xi}\psi(\Psi x) < a.
\end{aligned} \tag{3.32}$$

So, condition (S_4) of Theorem 2.3 is satisfied. Therefore, an application of Theorem 2.3 implies that problem (1.1)-(1.2) has at least three positive solutions x_1, x_2, and x_3 such that

$$x_i(t_3) \leq c, \quad i = 1,2,3, \qquad x_1(t_1) > b, \qquad x_2(\sigma(1)) < a, \qquad x_3(t_1) < b, \qquad x_3(\sigma(1)) > a. \tag{3.33}$$

The proof is complete. □

Remark 3.5. By the same method of this paper, we can also consider the following BVP:

$$\left(\phi_p\left(x^{\Delta\nabla}(t)\right)\right)^{\nabla\Delta} - w(t)f\left(x(t)\right) = 0, \quad t \in (0,1), \tag{3.34}$$

subject to the following boundary conditions:

$$x(1) + \lambda x^{\Delta}(\eta) = x^{\Delta}(0) = 0, \qquad x^{\Delta\nabla}(0) = \alpha_1 x^{\Delta\nabla}(1), \qquad x^{\Delta\nabla}(1) = \beta_1 x^{\Delta\nabla}(\xi), \tag{3.35}$$

where $\mathbb{T}$ is a time scale, $\lambda \geq 0,\ \alpha_1 \geq 0,\ \beta_1 \geq 0,\ 0 < \xi,\ \eta < 1,\ \phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$, and (H_1)-(H_2) hold.

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