Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2009, Article ID 360871, 17 pages
doi:10.1155/2009/360871

Research Article

Nonlinear Discrete Periodic Boundary Value Problems at Resonance

Ruyun Ma[1] and Huili Ma[2]

[1] *Department of Mathematics, Northwest Normal University, Lanzhou 730070, China*
[2] *College of Economics and Management, Northwest Normal University, Lanzhou 730070, China*

Correspondence should be addressed to Ruyun Ma, ruyun_ma@126.com

Received 25 June 2009; Revised 4 October 2009; Accepted 6 December 2009

Recommended by Kanishka Perera

Let $T \in \mathbb{N}$ be an integer with $T > 2$, and let $\mathbb{T} := \{1, \dots, T\}$. We study the existence of solutions of nonlinear discrete problems $\Delta^2 u(t-1) + \lambda_k a(t)u(t) + g(t, u(t)) = h(t)$, $t \in \mathbb{T}$, $u(0) = u(T)$, $u(1) = u(T+1)$, where $a, h : \mathbb{T} \to \mathbb{R}$ with $a > 0$, λ_k is the kth eigenvalue of the corresponding linear eigenvalue problem.

1. Introduction

Initialed by Lazer and Leach [1], much work has been devoted to the study of existence result for nonlinear periodic boundary value problem

$$\begin{gathered} y''(x) + m^2 y(x) + \widehat{g}(x, y(x)) = e(x), \quad x \in (0, 2\pi), \\ y(0) = y(2\pi), \qquad y'(0) = y'(2\pi), \end{gathered} \tag{1.1}$$

where $m \geq 0$ is an integer. Results from the paper have been extended to partial differential equations by several authors. The reader is referred, for detail, to Landesman and Lazer [2], Amann et al. [3], Brézis and Nirenberg [4], Fučík and Hess [5], and Iannacci and Nkashama [6] for some reference along this line. Concerning (1.1), results have been carried out by many authors also. Let us mention articles by Mawhin and Ward [7], Conti et al. [8], Omari and Zanolin [9], Ding and Zanolin [10], Capietto and Liu [11], Iannacci and Nkashama [12], Chu et al. [13], and the references therein.

However, relatively little is known about the discrete analog of (1.1) of the form

$$\Delta^2 u(t-1) + \lambda_k a(t)u(t) + g(t,u(t)) = h(t), \quad t \in \mathbb{T},$$
$$u(0) = u(T), \qquad u(1) = u(T+1), \tag{1.2}$$

where $\mathbb{T} := \{1,\ldots,T\}$, $a,h : \mathbb{T} \to \mathbb{R}$ with $a > 0$, $g(t,s) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is continuous in s. The likely reason is that the spectrum theory of the corresponding linear problem

$$\Delta^2 u(t-1) + \lambda_k a(t)u(t) = 0, \quad t \in \mathbb{T},$$
$$u(0) = u(T), \qquad u(1) = u(T+1) \tag{1.3}$$

was not established until [14]. In [14], Wang and Shi showed that the linear eigenvalue problem (1.3) has exactly T real eigenvalues

$$\mu_0 < \mu_1 \le \mu_2 < \cdots < \mu_{T-2} \le \mu_{T-1}, \quad \text{when } T \text{ is odd},$$
$$\mu_0 < \mu_1 \le \mu_2 < \cdots \le \mu_{T-2} < \mu_{T-1}, \quad \text{when } T \text{ is even}. \tag{1.4}$$

Suppose that these above eigenvalues have $N+1$ different values λ_k, $(k = 0,1,\ldots,N)$. Then (1.4) can be rewritten as

$$\lambda_0 < \lambda_1 < \cdots < \lambda_N. \tag{1.5}$$

For each λ_k, we denote its eigenspace by M_k. If $\dim M_k = 1$, then we assume that $M_k := \operatorname{span}\{\varphi_k\}$ in which φ_k is the eigenfunction of λ_k. If $\dim M_k = 2$, then we assume that $M_k := \operatorname{span}\{\varphi_k, \varphi_k\}$ in which φ_k and φ_k are two linearly independent eigenfunctions of λ_k.

It is the purpose of this paper to prove the existence results for problem (1.2) when there occurs resonance at the eigenvalue λ_k and the nonlinear function g may "touching" the eigenvalue λ_{k+1}. To have the wit, we have what follows.

Theorem 1.1. *Let $a,h : \mathbb{T} \to \mathbb{R}$ with $a > 0$, $g(t,s) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is continuous in s, and for some $r^* < 0 < R^*$,*

$$g(t,x) \ge A(t), \quad \forall x \ge R^*,$$
$$g(t,x) \le B(t), \quad \forall x \le r^*, \tag{1.6}$$

where $A,B : \mathbb{T} \to \mathbb{R}$ are two given functions. Suppose for some $1 \le k \le N-1$,

$$\dim M_{k+1} = 2. \tag{1.7}$$

Assume that for all $\varepsilon > 0$, there exist a constant $R = R(\varepsilon) > 0$ and a function $b : \mathbb{T} \to \mathbb{R}$ such that

$$\left|g(t,u)\right| \le (\Gamma(t) + \varepsilon)a(t)|u| + b(t), \quad t \in \mathbb{T},\ |u| \ge R, \tag{1.8}$$

where $\Gamma : \mathbb{T} \to \mathbb{R}$ *is a given function satisfying*

$$0 \le \Gamma(t) \le \lambda_{k+1} - \lambda_k, \quad t \in \mathbb{T}, \tag{1.9}$$

and for at least $[T/2] + 2$ *points in* $[1, T]$,

$$\Gamma(t) < \lambda_{k+1} - \lambda_k, \tag{1.10}$$

where $[r]$ *denotes the integer part of the real number* r.

Then (1.2) *has at least one solution provided*

$$\sum_{t=1}^{T} h(t)v(t) < \sum_{v(t)>0} g_+(t)v(t) + \sum_{v(t)<0} g_-(t)v(t), \tag{1.11}$$

where $v \in M_k$, $v \neq 0$, *and*

$$g_+(t) = \liminf_{u \to +\infty} g(t, u), \qquad g_-(t) = \limsup_{u \to -\infty} g(t, u). \tag{1.12}$$

In [12], Iannacci and Nkashama proved the analogue of Theorem 1.1 for continuous-time nonlinear periodic boundary value problems (1.1). Our paper is motivated by Iannacci and Nkashama [12]. However, as we will see below, there are big differences between the continuous case and the discrete case. The main tool we use is the Leray-Schauder continuation theorem (see Mawhin [15, Theorem IV.5]).

Finally, we note that when $a(t) \equiv 1$ in (1.2), the existence of odd solutions or even solutions was investigated by R. Ma and H. Ma [16] under some parity conditions on the nonlinearities. The existence of solutions of second-order discrete problem at resonance was studied by Rodriguez in [17], in which the nonlinearity is required to be bounded. For other results on discrete boundary value problems, see Kelley and Peterson [18], Agarwal and O'Regan [19], Rachunkova and Tisdell [20], Yu and Guo [21], Atici and Cabada [22], Bai and Xu [23]. However, these papers do not address the problem under "asymptotic nonuniform resonance" conditions.

2. Preliminaries

Let

$$\widehat{\mathbb{T}} = \{0, 1, \ldots, T + 1\}. \tag{2.1}$$

Let

$$D := \left\{ u : \widehat{\mathbb{T}} \longrightarrow \mathbb{R} \mid u(0) = u(T),\ u(1) = u(T + 1) \right\}. \tag{2.2}$$

Then D is a Hilbert space under the inner product

$$\langle u, v\rangle = \sum_{t=1}^{T} a(t)u(t)v(t), \tag{2.3}$$

and the corresponding norm is

$$\|u\| := \sqrt{\langle u, u\rangle} = \left(\sum_{t=1}^{T} a(t)u(t)u(t)\right)^{1/2}. \tag{2.4}$$

Thus,

$$\begin{gathered}
\langle \varphi_k, \varphi_k\rangle = 0 \quad \text{if } \dim M_k = 2, \\
\langle \psi_j, \psi_k\rangle = 0, \quad \text{for } j, k \in \{0, 1, \ldots, N\},\ j \neq k, \\
\langle \varphi_j, \varphi_k\rangle = 0, \quad \text{for } j, k \in \{0, 1, \ldots, N\},\ j \neq k.
\end{gathered} \tag{2.5}$$

In the rest of the paper, we always assume that

$$\begin{gathered}
\|\psi_k\| = 1, \quad \text{for } k \in \{0, 1, \ldots, N\}, \\
\|\varphi_k\| = 1 \quad \text{if } \dim M_k = 2.
\end{gathered} \tag{2.6}$$

Define a linear operator $L : D \to D$ by

$$\begin{gathered}
(Lu)(t) = -\Delta^2 u(t-1), \quad t \in \mathbb{T}, \\
(Lu)(0) := (Lu)(1), \\
(Lu)(T+1) := (Lu)(T).
\end{gathered} \tag{2.7}$$

Lemma 2.1 (see [16]). *Let $u, w \in D$. Then*

$$\sum_{k=1}^{T} w(k)\Delta^2 u(k-1) = -\sum_{k=1}^{T} \Delta u(k)\Delta w(k). \tag{2.8}$$

Similar to [12, Lemma 3], we can prove the following.

Lemma 2.2 (see [12]). *Suppose that*

(i) *there exist $A, B : \mathbb{T} \to \mathbb{R}$ and real numbers $r < 0 < R$, such that*

$$\begin{gathered}
g(t, x) \geq A(t), \quad \forall x \geq R, \\
g(t, x) \leq B(t), \quad \forall x \leq r,
\end{gathered} \tag{2.9}$$

(ii) *there exist* $\alpha, \beta : \mathbb{T} \to [0, \infty)$ *and a constant* $B_0 > 0$ *such that*

$$\left|g(t,u)\right| \leq \alpha(t)|u| + \beta(t), \quad t \in \mathbb{T},\ |u| \geq B_0. \tag{2.10}$$

Then for each real number $\kappa > 0$, *there is a decomposition*

$$g(t,x) = q_\kappa(t,x) + e_\kappa(t,x) \tag{2.11}$$

of g *satisfying*

$$0 \leq x q_\kappa(t,x), \quad t \in \mathbb{T},\ x \in \mathbb{R}, \tag{2.12}$$

$$\left|q_\kappa(t,u)\right| \leq \alpha(t)|u| + \beta(t) + \kappa, \quad t \in \mathbb{T},\ |u| \geq \max\{1, B_0\}, \tag{2.13}$$

and there exists a function $\sigma_\kappa : \mathbb{T} \to [0, \infty)$ *depending on* r, R, *and* g *such that*

$$|e_\kappa(t,x)| \leq \sigma_\kappa(t), \quad t \in \mathbb{T},\ x \in \mathbb{R}. \tag{2.14}$$

3. Existence of Periodic Solutions

In this section, we need to give some lemmas first, which have vital importance to prove Theorem 1.1.

For convenience, we set

$$\varphi_k := 0, \quad \text{as } \dim M_k = 1. \tag{3.1}$$

Thus, for any $u \in D$, we have the following Fourier expansion:

$$u(t) = a_0 + \sum_{i=1}^{N} \left[a_i \psi_i(t) + b_i \varphi_i(t)\right], \quad t \in \mathbb{T}. \tag{3.2}$$

Let us write

$$u(t) = \overline{u}(t) + u^0(t) + \tilde{u}(t), \qquad u^{\perp}(t) = u(t) - u^0(t), \tag{3.3}$$

where

$$\begin{aligned} \overline{u}(t) &= a_0 + \sum_{i=1}^{k-1} \left[a_i \varphi_i(t) + b_i \psi_i(t)\right], \\ u^0(t) &= a_k \varphi_k(t) + b_k \psi_k(t), \\ \tilde{u}(t) &= \sum_{i=k+1}^{N} \left[a_i \varphi_i(t) + b_i \psi_i(t)\right]. \end{aligned} \tag{3.4}$$

Lemma 3.1. *Suppose that for* $1 \leq k \leq N-1$, λ_{k+1} *is an eigenvalue of* (1.3) *of multiplicity 2. Let* $\Gamma : \mathbb{T} \to \mathbb{R}$ *be a given function satisfying*

$$0 \leq \Gamma(t) \leq \lambda_{k+1} - \lambda_k, \quad t \in \mathbb{T}, \tag{3.5}$$

and for at least $[T/2]+2$ *points in* $[1,T]$,

$$\Gamma(t) < \lambda_{k+1} - \lambda_k. \tag{3.6}$$

Then there exists a constant $\delta = \delta(\Gamma) > 0$ such that for all $u \in D$, one has

$$\sum_{t=1}^{T}\left[\Delta^2 u(t-1) + \lambda_k a(t)u(t) + \Gamma(t)a(t)u(t)\right]\left[\overline{u}(t) + u^0(t) - \widetilde{u}(t)\right] \geq \delta \left\| u^{\perp} \right\|^2. \tag{3.7}$$

Proof. For $u \in D$,

$$\Delta^2 u(t-1) = -a(t)\sum_{i=1}^{N}\left[a_i\lambda_i\varphi_i(t) + b_i\lambda_i\varphi_i(t)\right]. \tag{3.8}$$

Taking into account the orthogonality of $\overline{u}$, u^0, and $\widetilde{u}$ in D, we have

$$\begin{aligned}
&\sum_{t=1}^{T}\left[\Delta^2 u(t-1) + \lambda_k a(t)u(t) + \Gamma(t)a(t)u(t)\right]\left[\overline{u}(t) + u^0(t) - \widetilde{u}(t)\right] \\
&\quad = \sum_{t=1}^{T}\left[\Delta^2 \overline{u}(t-1) + \lambda_k a(t)\overline{u}(t)\right]\overline{u}(t) + \sum_{t=1}^{T}\Gamma(t)a(t)\left[\overline{u}(t) + u^0(t)\right]^2 \\
&\qquad + \sum_{t=1}^{T}\left[\Delta^2 \widetilde{u}(t-1) + \lambda_k a(t)\widetilde{u}(t) + \Gamma(t)a(t)\widetilde{u}(t)\right]\left[-\widetilde{u}(t)\right] \\
&\qquad + \sum_{t=1}^{T}\left[\Delta^2 u^0(t-1) + \lambda_k a(t)u^0(t)\right]u^0(t) \\
&\quad = \sum_{t=1}^{T}\left[-(\Delta\overline{u}(t))^2 + \lambda_k a(t)\overline{u}^2(t)\right] + \sum_{t=1}^{T}\Gamma(t)a(t)\left[\overline{u}(t) + u^0(t)\right]^2 \\
&\qquad + \sum_{t=1}^{T}\left[(\Delta\widetilde{u}(t))^2 - \lambda_k a(t)\widetilde{u}^2(t) - \Gamma(t)a(t)\widetilde{u}^2(t)\right] \\
&\quad \geq (\lambda_k - \lambda_{k-1})\sum_{t=1}^{T}a(t)\overline{u}^2(t) + \sum_{t=1}^{T}\left[\Delta\widetilde{u}(t)\right]^2 - \sum_{t=1}^{T}(\lambda_k a(t) + \Gamma(t)a(t))\widetilde{u}^2(t).
\end{aligned} \tag{3.9}$$

Set

$$\Lambda(\overline{u}) = (\lambda_k - \lambda_{k-1})\sum_{t=1}^{T} a(t)\overline{u}^2(t). \tag{3.10}$$

Then,

$$\Lambda(\overline{u}) \geq \delta_1 \|\overline{u}\|^2, \tag{3.11}$$

where δ_1 is a positive constant less than $\lambda_k - \lambda_{k-1}$.

Let

$$\Lambda_\Gamma(\widetilde{u}) = \sum_{t=1}^{T} [\Delta\widetilde{u}(t)]^2 - \sum_{t=1}^{T} (\lambda_k a(t) + \Gamma(t)a(t))\widetilde{u}^2(t). \tag{3.12}$$

We claim that $\Lambda_\Gamma(\widetilde{u}) \geq 0$ with the equality holding only if $\widetilde{u} = A_0\psi_{k+1} + B_0\varphi_{k+1}$, where $A_0, B_0 \in \mathbb{R}$ are constants.

In fact, we have from Lemma 2.1 that

$$\begin{aligned}
\Lambda_\Gamma(\widetilde{u}) &= \sum_{t=1}^{T} [\Delta\widetilde{u}(t)]^2 - \sum_{t=1}^{T} (\lambda_k a(t) + \Gamma(t)a(t))\widetilde{u}^2(t) \\
&= -\sum_{t=1}^{T} \widetilde{u}(t)\Delta^2\widetilde{u}(t-1) - \sum_{t=1}^{T} (\lambda_k a(t) + \Gamma(t)a(t))\widetilde{u}^2(t) \\
&= \sum_{t=1}^{T} \sum_{i=k+1}^{N} \left[a_i\psi_i(t) + b_i\varphi_i(t)\right] \sum_{i=k+1}^{N} \lambda_i a(t)\left[a_i\psi_i(t) + b_i\varphi_i(t)\right] \\
&\quad - \sum_{t=1}^{T} (\lambda_k a(t) + \Gamma(t)a(t))\left(\sum_{i=k+1}^{N} \left[a_i\psi_i(t) + b_i\varphi_i(t)\right]\right)^2 \\
&\geq \sum_{t=1}^{T} \sum_{i=k+1}^{N} \left[a_i\psi_i(t) + b_i\varphi_i(t)\right] \sum_{j=k+1}^{N} \lambda_j a(t)\left[a_j\psi_j(t) + b_j\varphi_j(t)\right] \\
&\quad - \sum_{t=1}^{T} \lambda_{k+1} a(t)\left(\sum_{i=k+1}^{N} \left[a_i\psi_i(t) + b_i\varphi_i(t)\right]\right)\left(\sum_{j=k+1}^{N} \left[a_j\psi_j(t) + b_j\varphi_j(t)\right]\right) \\
&= \sum_{i=k+1}^{N} \sum_{j=k+1}^{N} a_i a_j \lambda_j \sum_{t=1}^{T} a(t)\psi_i(t)\psi_j(t) + \sum_{i=k+1}^{N} \sum_{j=k+1}^{N} b_i b_j \lambda_j \sum_{t=1}^{T} a(t)\varphi_i(t)\varphi_j(t) \\
&\quad - \sum_{i=k+1}^{N} \sum_{j=k+1}^{N} a_i a_j \lambda_{k+1} \sum_{t=1}^{T} a(t)\psi_i(t)\psi_j(t)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=k+1}^{N} \sum_{j=k+1}^{N} b_i b_j \lambda_{k+1} \sum_{t=1}^{T} a(t) \varphi_i(t) \varphi_j(t) \\
&= \sum_{j=k+1}^{N} a_j^2 (\lambda_j - \lambda_{k+1}) + \sum_{j=k+1}^{N} b_j^2 (\lambda_j - \lambda_{k+1}) \\
&= \sum_{j=k+1}^{N} \left(a_j^2 + b_j^2\right) (\lambda_j - \lambda_{k+1}) \geq 0.
\end{aligned}
\tag{3.13}
$$

Obviously, $\Lambda_\Gamma(\widetilde{u}) = 0$ implies that $a_{k+2} = \cdots = a_N = b_{k+2} = \cdots b_N = 0$, and accordingly $\widetilde{u}(t) = A_0 \varphi_{k+1}(t) + B_0 \varphi_{k+1}(t)$ for some $A_0, B_0 \in \mathbb{R}$.

Next we prove that $\Lambda_\Gamma(\widetilde{u}) = 0$ implies $\widetilde{u} = 0$. Suppose to the contrary that $\widetilde{u} \neq 0$.

We note that $\widetilde{u}$ has at most $[T/2]+1$ zeros in $\mathbb{T}$. Otherwise, $\widetilde{u}$ must have two consecutive zeros in $\mathbb{T}$, and subsequently, $\widetilde{u} \equiv 0$ in $[0, T+1]$ by (1.3). This is a contradiction.

Using (3.6) and the fact that $\widetilde{u}$ has at most $[T/2] + 1$ zeros in $\mathbb{T}$, it follows that

$$
\begin{aligned}
\Lambda_\Gamma(\widetilde{u}) &= \sum_{t=1}^{T} (\lambda_{k+1} a(t) - \lambda_k a(t) - \Gamma(t) a(t)) [\widetilde{u}(t)]^2 \\
&= \sum_{t \in \mathbb{T}, \widetilde{u}(t) \neq 0} a(t) [\lambda_{k+1} - \lambda_k - \Gamma(t)] [\widetilde{u}(t)]^2 \\
&> 0,
\end{aligned}
\tag{3.14}
$$

which contradicts $\Lambda_\Gamma(\widetilde{u}) = 0$. Hence, $\widetilde{u} = 0$.

We claim that there is a constant $\delta_2 = \delta_2(\Gamma) > 0$ such that

$$
\Lambda_\Gamma(\widetilde{u}) \geq \delta_2 \|\widetilde{u}\|^2. \tag{3.15}
$$

Assume that the claim is not true. Then we can find a sequence $\{\widetilde{u}_n\} \subset D$ and $\widetilde{u} \in D$, such that, by passing to a subsequence if necessary,

$$
0 \leq \Lambda_\Gamma(\widetilde{u}_n) \leq \frac{1}{n}, \quad \|\widetilde{u}_n\| = 1, \tag{3.16}
$$

$$
\|\widetilde{u}_n - \widetilde{u}\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.17}
$$

From (3.17), it follows that

$$\begin{aligned}\left|\sum_{t=1}^{T}[\Delta\widetilde{u}_n(t)]^2-\sum_{t=1}^{T}[\Delta\widetilde{u}(t)]^2\right| &= \left|\sum_{t=1}^{T}[\widetilde{u}_n(t+1)-\widetilde{u}_n(t)]^2-\sum_{t=1}^{T}[\widetilde{u}(t+1)-\widetilde{u}(t)]^2\right| \\ &\le \sum_{t=1}^{T}\left|\widetilde{u}_n^2(t+1)-\widetilde{u}^2(t+1)\right|+\sum_{t=1}^{T}\left|\widetilde{u}_n^2(t)-\widetilde{u}^2(t)\right| \\ &\quad +2\sum_{t=1}^{T}(|\widetilde{u}_n(t)|\,|\widetilde{u}_n(t+1)-\widetilde{u}(t+1)|+|\widetilde{u}(t+1)|\,|\widetilde{u}_n(t)-\widetilde{u}(t)|) \\ &\longrightarrow 0.\end{aligned} \tag{3.18}$$

By (3.12), (3.16), and (3.17), we obtain, for $n \to \infty$,

$$\sum_{t=1}^{T}[\Delta\widetilde{u}_n(t)]^2 \longrightarrow \sum_{t=1}^{T}(\lambda_k a(t)+\Gamma(t)a(t))[\widetilde{u}(t)]^2, \tag{3.19}$$

and hence

$$\sum_{t=1}^{T}[\Delta\widetilde{u}(t)]^2 \le \sum_{t=1}^{T}(\lambda_k a(t)+\Gamma(t)a(t))[\widetilde{u}(t)]^2, \tag{3.20}$$

that is,

$$\Lambda_\Gamma(\widetilde{u}) \le 0. \tag{3.21}$$

By the first part of the proof, $\widetilde{u}=0$, so that, by (3.19), $\sum_{t=1}^{T}[\Delta\widetilde{u}_n(t)]^2 \to 0$, a contradiction with the second equality in (3.16).

Set $\delta=\min\{\delta_1,\delta_2\}>0$ and observing that $\|u^\perp\|^2=\|\widetilde{u}\|^2+\|\overline{u}\|^2$ the proof is complete. □

Lemma 3.2. *Let Γ be as in Lemma 3.1 and let $\delta>0$ be associated with Γ by that lemma. Let $\varepsilon>0$. Let $p:\mathbb{T}\to\mathbb{R}$ be a function satisfying*

$$0\le p(t)\le \Gamma(t)+\varepsilon. \tag{3.22}$$

Then for all $u\in D$, one has

$$\sum_{t=1}^{T}\left[\Delta^2u(t-1)+\lambda_k a(t)u(t)+p(t)a(t)u(t)\right]\left[\overline{u}(t)+u^0(t)-\widetilde{u}(t)\right]\ge(\delta-\varepsilon)\left\|u^\perp\right\|^2. \tag{3.23}$$

Proof. Using the computations in the proof of Lemma 3.1 and (3.22), we obtain

$$
\begin{aligned}
&\sum_{t=1}^{T}\left[\Delta^2 u(t-1)+\lambda_k a(t)u(t)+p(t)a(t)u(t)\right]\left[\overline{u}(t)+u^0(t)-\widetilde{u}(t)\right]\\
&\quad=\sum_{t=1}^{T}\left[\Delta^2\overline{u}(t-1)+\lambda_k a(t)\overline{u}(t)\right]\overline{u}(t)+\sum_{t=1}^{T}p(t)a(t)\left[\overline{u}(t)+u^0(t)\right]^2\\
&\qquad+\sum_{t=1}^{T}\left[\Delta^2\widetilde{u}(t-1)+\lambda_k a(t)\widetilde{u}(t)+p(t)a(t)\widetilde{u}(t)\right](-\widetilde{u}(t))\\
&\qquad+\sum_{t=1}^{T}\left[\Delta^2 u^0(t-1)+\lambda_k a(t)u^0(t)\right]u^0(t)\\
&\quad\geq\sum_{t=1}^{T}\left[(\Delta\widetilde{u}(t))^2-\left(\lambda_k a(t)+p(t)a(t)\right)(\widetilde{u}(t))^2\right]\\
&\qquad+\sum_{t=1}^{T}\left[-(\Delta\overline{u}(t))^2+\lambda_k a(t)(\overline{u}(t))^2\right]\\
&\quad\geq\sum_{t=1}^{T}\left[(\Delta\widetilde{u}(t))^2-(\lambda_k a(t)+\Gamma(t)a(t))(\widetilde{u}(t))^2\right]-\sum_{t=1}^{T}\varepsilon a(t)(\widetilde{u}(t))^2\\
&\qquad+\sum_{t=1}^{T}\left[-(\Delta\overline{u}(t))^2+\lambda_k a(t)(\overline{u}(t))^2\right]\\
&\quad\geq\delta\left\|u^{\perp}\right\|^2-\varepsilon\|\widetilde{u}\|^2.
\end{aligned}
\tag{3.24}
$$

So that, using (3.7), (3.8), the relation $\widetilde{u}(t)=\sum_{i=k+1}^{N}[a_i\varphi_i(t)+b_i\varphi_i(t)]$, and Lemma 2.1, it follows that

$$
\sum_{t=1}^{T}\left[\Delta^2 u(t-1)+\lambda_k a(t)u(t)+p(t)a(t)u(t)\right]\left[\overline{u}(t)+u^0(t)-\widetilde{u}(t)\right]\geq(\delta-\varepsilon)\left\|u^{\perp}\right\|^2. \tag{3.25}
$$

□

Proof of Theorem 1.1. The proof is motivated by Iannacci and Nkashama [12].

Let $\delta>0$ be associated to the function Γ by Lemma 3.1. Then, by assumption (1.8), there exist $R(\delta)>0$ and $b:\mathbb{T}\to\mathbb{R}$, such that

$$
\left|g(t,u)\right|\leq\left(\Gamma(t)+\left(\frac{\delta}{4}\right)\right)a(t)|u|+b(t), \tag{3.26}
$$

for all $t\in\mathbb{T}$ and all $u\in\mathbb{R}$ with $|u|\geq R$. Hence, (1.2) is equivalent to

$$
\begin{gathered}
\Delta^2 u(t-1)+\lambda_k a(t)u(t)+q_1(t,u(t))+e_1(t,u(t))=h(t),\\
u(0)=u(T),\qquad u(1)=u(T+1),
\end{gathered}
\tag{3.27}
$$

where q_1 and e_1 satisfy (2.12) and (2.14) with $\kappa = 1$. Moreover, by (2.13)

$$\left|q_1(t,u)\right| \le \left(\Gamma(t) + \left(\frac{\delta}{4}\right)\right)a(t)|u| + b(t) + 1, \quad t \in \mathbb{T},\ |u| > \max\{1, R\}. \tag{3.28}$$

Let $\overline{B} > \max\{1, R\}$, so that

$$\frac{b(t)+1}{|u|} < \frac{\delta}{4}a(t), \quad t \in \mathbb{T},\ |u| > \overline{B}. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$0 \le u^{-1}q_1(t,u) \le \left(\Gamma(t) + \frac{\delta}{2}\right)a(t), \quad t \in \mathbb{T},\ |u| \ge \overline{B}. \tag{3.30}$$

Define $\gamma : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t,u) = \begin{cases} u^{-1}q_1(t,u), & |u| \ge \overline{B}, \\ \overline{B}^{-1}q_1\left(t,\overline{B}\right)\left(\dfrac{u}{\overline{B}}\right) + \left(1 - \dfrac{u}{\overline{B}}\right)\Gamma(t)a(t), & 0 \le u < \overline{B}, \\ \overline{B}^{-1}q_1\left(t,-\overline{B}\right)\left(\dfrac{u}{\overline{B}}\right) + \left(1 + \dfrac{u}{\overline{B}}\right)\Gamma(t)a(t), & -\overline{B} < u \le 0. \end{cases} \tag{3.31}$$

So we have

$$0 \le \gamma(t,u) \le \left(\Gamma(t) + \frac{\delta}{2}\right)a(t), \quad t \in \mathbb{T},\ u \in \mathbb{R}. \tag{3.32}$$

Define $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$f(t,u) = e_1(t,u) + q_1(t,u) - \gamma(t,u)u. \tag{3.33}$$

Then there exists $\nu : \mathbb{T} \to [0,\infty)$ such that

$$\left|f(t,u)\right| \le \nu(t), \quad t \in \mathbb{T},\ u \in \mathbb{R}. \tag{3.34}$$

Therefore, (1.2) is equivalent to

$$\begin{gathered} \Delta^2 u(t-1) + \lambda_k a(t)u(t) + \gamma(t,u(t))u(t) + f(t,u(t)) = h(t), \\ u(0) = u(T), \qquad u(1) = u(T+1). \end{gathered} \tag{3.35}$$

To prove that (1.2) has at least one solution in D, it suffices, according to the Leray-Schauder continuation method [15], to show that all of the possible solutions of the family of equations

$$\Delta^2 u(t-1) + \lambda_k a(t)u(t) + (1-\eta)\tau a(t)u(t) + \eta\gamma(t,u(t))u(t) + \eta f(t,u(t)) = \eta h(t), \quad t \in \mathbb{T},$$
$$u(0) = u(T), \qquad u(1) = u(T+1) \tag{3.36}$$

(in which $\eta \in [0,1]$, $\tau \in (0, \lambda_{k+1} - \lambda_k)$ with $\tau < \delta/4$, τ fixed) are bounded by a constant K_0 which is independent of η and u.

Notice that, by (3.32), we have

$$0 \le (1-\eta)\tau a(t) + \eta\gamma(t,u) \le \left(\Gamma(t) + \frac{\delta}{2}\right)a(t), \quad t \in \mathbb{T},\ u \in \mathbb{R}. \tag{3.37}$$

It is clear that for $\eta = 0$, (3.36) has only the trivial solution. Now if $u \in D$ is a solution of (3.36) for some $\eta \in (0,1)$, using Lemma 3.2 and Cauchy's inequality, we obtain

$$\begin{aligned} 0 &= \sum_{t=1}^{T}\left(\overline{u}(t) + u^0(t) - \widetilde{u}(t)\right)\left(\Delta^2 u(t-1) + \lambda_k a(t)u(t) + \left[(1-\eta)\tau a(t) + \eta\gamma(t,u(t))\right]u(t)\right) \\ &\quad + \sum_{t=1}^{T}\left(\overline{u}(t) + u^0(t) - \widetilde{u}(t)\right)(\eta f(t,u(t)) - \eta h(t)) \\ &\ge \left(\frac{\delta}{2}\right)\left\|u^{\perp}\right\|^2 - \zeta\left(\|\overline{u}\| + \|\widetilde{u}\| + \left\|u^0\right\|\right)(\|\nu\| + \|h\|), \end{aligned} \tag{3.38}$$

where

$$\zeta = \left(\frac{\sqrt{T}}{\min_{t\in\mathbb{T}}\sqrt{a(t)}}\right)^2. \tag{3.39}$$

So we conclude that

$$0 \ge \left(\frac{\delta}{2}\right)\left\|u^{\perp}\right\|^2 - \beta\left(\left\|u^{\perp}\right\| + \left\|u^0\right\|\right), \tag{3.40}$$

for some constant $\beta > 0$, depending only on a, ν and h (but not on u or η). Taking $\alpha = \beta\delta^{-1}$, we get

$$\left\|u^{\perp}\right\| \le \alpha + \left(\alpha^2 + 2\alpha\left\|u^0\right\|\right)^{1/2}. \tag{3.41}$$

We claim that there exists $\rho > 0$, independent of u and η, such that for all possible solutions of (3.36)

$$\|u\| < \rho. \tag{3.42}$$

Suppose on the contrary that the claim is false. Then there exists $\{(\eta_n, u_n)\} \subset (0,1) \times D$ with $\|u_n\| \geq n$ and for all $n \in \mathbb{N}$,

$$\begin{gathered}\Delta^2 u_n(t-1) + \lambda_k a(t) u_n(t) + (1-\eta_n)\tau a(t) u_n(t) + \eta_n g(t, u_n(t)) = \eta_n h(t), \\ u_n(0) = u_n(T), \qquad u_n(1) = u_n(T+1).\end{gathered} \tag{3.43}$$

From (3.41), it can be shown that

$$\left\|u_n^0\right\| \longrightarrow \infty, \qquad \left\|u_n^{\perp}\right\|\left(\left\|u_n^0\right\|\right)^{-1} \longrightarrow 0, \tag{3.44}$$

and accordingly, $u_n^{\perp}(\|u_n^0\|)^{-1}$ is bounded in D.

Setting $v_n = (u_n/\|u_n\|)$, we have

$$\begin{aligned}\Delta^2 v_n(t-1) &+ \lambda_k a(t) v_n(t) + \tau a(t) v_n(t) \\ &= \eta_n\left(\frac{h(t)}{\|u_n\|}\right) + \eta_n \tau a(t) v_n(t) - \eta_n\left(\frac{g(t, u_n(t))}{\|u_n\|}\right), \quad t \in \mathbb{T}, \\ &v_n(0) = v_n(T), \qquad v_n(1) = v_n(T+1).\end{aligned} \tag{3.45}$$

Define an operator $A : D \to D$ by

$$\begin{gathered}(Aw)(t) := \Delta^2 w(t-1) + \lambda_k a(t) w(t) + \tau a(t) w(t), \quad t \in \mathbb{T}, \\ (Aw)(0) := (Aw)(T), \qquad (Aw)(1) := (Aw)(T+1).\end{gathered} \tag{3.46}$$

Then $A^{-1} : D \to D$ is completely continuous since D is finite dimensional. Now, (3.45) is equivalent to

$$v_n(t) = A^{-1}\left[\eta_n\left(\frac{h(\cdot)}{\|u_n\|}\right) + \eta_n \tau a(\cdot) v_n(\cdot) - \eta_n\left(\frac{g(\cdot, u_n(\cdot))}{\|u_n\|}\right)\right](t), \quad t \in \mathbb{T}. \tag{3.47}$$

By (3.26), it follows that $\{(g(\cdot, u_n(\cdot))/\|u_n\|\}$ is bounded. Using (3.47), we may assume that (taking a subsequence and relabeling if necessary) $v_n \to v$ in $(D, \|\cdot\|)$, $\|v\| = 1$ and $v(0) = v(T)$, $v(1) = v(T+1)$.

On the other hand, using (3.41), we deduce immediately that

$$\left\|v_n^{\perp}\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.48}$$

Therefore,

$$v(t) = a_k\varphi_k(t) + b_k\psi_k(t), \quad t \in \widehat{\mathbb{T}}. \tag{3.49}$$

Rewrite $v_n = v_n^0 + v_n^{\perp}$, and let, taking a subsequence and relabeling if necessary,

$$v_n^0 \longrightarrow v^*, \quad \text{in } D. \tag{3.50}$$

Set

$$I_+ = \{t \in \mathbb{T} : v^*(t) > 0\}, \qquad I_- = \{t \in \mathbb{T} : v^*(t) < 0\}. \tag{3.51}$$

Since $u(t) \not\equiv 0$ in $\mathbb{T}$, $I_+ \neq \emptyset$ or $I_- \neq \emptyset$.

We claim that

$$\lim_{n\to\infty} u_n(t) = \infty, \quad \forall t \in I_+, \tag{3.52}$$

$$\lim_{n\to\infty} u_n(t) = -\infty, \quad \forall t \in I_-. \tag{3.53}$$

We may assume that $I_+ \neq \emptyset$, and only deal with the case $t \in I_+$. The other case can be treated by similar method.

It follows from (3.50) that

$$\left\|v_n^0 - v^*\right\|_\infty := \max\left\{\left|v_n^0(t) - v^*(t)\right| \mid t \in \mathbb{T}\right\} \longrightarrow 0, \quad n \longrightarrow \infty, \tag{3.54}$$

which implies that for all n sufficiently large,

$$v_n^0(t) \geq \frac{1}{2}v^*(t) > 0, \quad \forall t \in I_+. \tag{3.55}$$

On the other hand, we have from (3.44), (3.55), and the fact $\|u_n\| \geq \|u_n^0\|$ that there exists $\overline{N} > 0$ such that for $n > \overline{N}$ and $t \in I_+$,

$$u_n(t) = u_n^0(t) + u_n^{\perp}(t) = \|u_n\|\left(v_n^0(t) + \frac{u_n^{\perp}(t)}{\|u_n\|}\right) \geq \frac{1}{2}\|u_n\|v_n^0(t). \tag{3.56}$$

This together with (3.55) implies that for $n \geq \overline{N}$,

$$u_n(t) \geq \frac{1}{4}\|u_n\|v^*(t), \quad t \in T_+. \tag{3.57}$$

Therefore, (3.52) holds.

Now let us come back to (3.43). Multiplying both sides of (3.43) by v_n^0 and summing from 1 to T, we get that

$$0 \le \eta_n^{-1}(1-\eta_n)\tau\left\|v_n^0\right\|^2\|u_n\| = \sum_{t=1}^{T}\left[h(t) - g(t,u_n(t))\right]v_n^0(t). \tag{3.58}$$

Combining this with (3.52) and (3.53), it follows that

$$\sum_{t=1}^{T} h(t)v^*(t) \ge \sum_{v(t)>0} g_+(t)v^*(t) + \sum_{v(t)<0} g_-(t)v^*(t). \tag{3.59}$$

However, this contradicts (1.11). □

Example 3.3. By [16], the eigenvalues and eigenfunctions of

$$\begin{gathered}\Delta^2 y(t-1) + \lambda y(t) = 0,\\ y(0) = y(7), \qquad y(1) = y(8)\end{gathered} \tag{3.60}$$

can be listed as follows:

$$\begin{gathered}\lambda_0 = 0, \qquad \varphi_0 = 1,\\ \lambda_1 = 2 - 2\cos\frac{2\pi}{7}, \qquad \varphi_1(t) = \sin\frac{2\pi t}{7}, \qquad \varphi_1(t) = \cos\frac{2\pi t}{7},\\ \lambda_2 = 2 - 2\cos\frac{4\pi}{7}, \qquad \varphi_2(t) = \sin\frac{4\pi t}{7}, \qquad \varphi_2(t) = \cos\frac{4\pi t}{7},\\ \lambda_3 = 2 - 2\cos\frac{6\pi}{7}, \qquad \varphi_2(t) = \sin\frac{6\pi t}{7}, \qquad \varphi_2(t) = \cos\frac{6\pi t}{7}.\end{gathered} \tag{3.61}$$

Let us consider the nonlinear discrete periodic boundary value problem

$$\begin{gathered}\Delta^2 y(t-1) + \lambda_1 y(t) + g(t,y(t)) = h(t),\\ y(0) = y(7), \qquad y(1) = y(8),\end{gathered} \tag{3.62}$$

where

$$g(t,s) = (\lambda_2 - \lambda_1)\cdot\left|\sin\left[\frac{\pi}{7}\left(t+\frac{5}{2}\right)\right]\right|\cdot\left(s + \frac{s}{1+s^2}\right), \quad (t,s) \in \mathbb{T}\times\mathbb{R}. \tag{3.63}$$

Obviously, $g_+(t) = +\infty$, $g_-(t) = -\infty$, and $\dim M_2 = 2$. If we take that

$$\Gamma(t) = (\lambda_2 - \lambda_1)\cdot\left|\sin\left[\frac{\pi}{7}\left(t+\frac{5}{2}\right)\right]\right|, \tag{3.64}$$

then

$$\Gamma(1) = \lambda_2 - \lambda_1; \qquad \Gamma(j) < \lambda_2 - \lambda_1, \quad \text{for } j = 2, \ldots, 7. \tag{3.65}$$

Now, it is easy to verify that g satisfies all conditions of Theorem 1.1. Consequently, for any 7-periodic function $h : \mathbb{Z} \to \mathbb{R}$, (3.62) has at least one solution.

Acknowledgments

This work was supported by the NSFC (no. 10671158), the NSF of Gansu Province (no. 3ZS051-A25-016), NWNU-KJCXGC-03-17, NWNU-KJCXGC-03-18, the Spring-Sun program (no. Z2004-1-62033), SRFDP (no. 20060736001), and the SRF for ROCS, SEM (2006 [311]).

References

[1] A. C. Lazer and D. E. Leach, “Bounded perturbations of forced harmonic oscillators at resonance,” *Annali di Matematica Pura ed Applicata*, vol. 82, pp. 49–68, 1969.

[2] E. M. Landesman and A. C. Lazer, “Nonlinear perturbations of linear elliptic boundary value problems at resonance,” *Journal of Applied Mathematics and Mechanics*, vol. 19, pp. 609–623, 1970.

[3] H. Amann, A. Ambrosetti, and G. Mancini, “Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities,” *Mathematische Zeitschrift*, vol. 158, no. 2, pp. 179–194, 1978.

[4] H. Brézis and L. Nirenberg, “Characterizations of the ranges of some nonlinear operators and applications to boundary value problems,” *Annali della Scuola Normale Superiore di Pisa*, vol. 5, no. 2, pp. 225–326, 1978.

[5] S. Fučík and P. Hess, “Nonlinear perturbations of linear operators having nullspace with strong unique continuation property,” *Nonlinear Analysis*, vol. 3, no. 2, pp. 271–277, 1979.

[6] R. Iannacci and M. N. Nkashama, “Nonlinear boundary value problems at resonance,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, no. 4, pp. 455–473, 1987.

[7] J. Mawhin and J. R. Ward Jr., “Periodic solutions of some forced Liénard differential equations at resonance,” *Archiv der Mathematik*, vol. 41, no. 4, pp. 337–351, 1983.

[8] G. Conti, R. Iannacci, and M. N. Nkashama, “Periodic solutions of Liénard systems at resonance,” *Annali di Matematica Pura ed Applicata*, vol. 139, pp. 313–327, 1985.

[9] P. Omari and F. Zanolin, “Existence results for forced nonlinear periodic BVPs at resonance,” *Annali di Matematica Pura ed Applicata*, vol. 141, pp. 127–157, 1985.

[10] T. R. Ding and F. Zanolin, “Time-maps for the solvability of periodically perturbed nonlinear Duffing equations,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 17, no. 7, pp. 635–653, 1991.

[11] A. Capietto and B. Liu, “Quasi-periodic solutions of a forced asymmetric oscillator at resonance,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 56, no. 1, pp. 105–117, 2004.

[12] R. Iannacci and M. N. Nkashama, “Unbounded perturbations of forced second order ordinary differential equations at resonance,” *Journal of Differential Equations*, vol. 69, no. 3, pp. 289–309, 1987.

[13] J. Chu, P. J. Torres, and M. Zhang, “Periodic solutions of second order non-autonomous singular dynamical systems,” *Journal of Differential Equations*, vol. 239, no. 1, pp. 196–212, 2007.

[14] Y. Wang and Y. Shi, “Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions,” *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 56–69, 2005.

[15] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, vol. 40 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1979.

[16] R. Ma and H. Ma, “Unbounded perturbations of nonlinear discrete periodic problem at resonance,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2602–2613, 2009.

[17] J. Rodriguez, “Nonlinear discrete Sturm-Liouville problems,” *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 380–391, 2005.

[18] W. G. Kelley and A. C. Peterson, *Difference Equations*, Academic Press, Boston, Mass, USA, 1991.

[19] R. P. Agarwal and D. O’Regan, “Boundary value problems for discrete equations,” *Applied Mathematics Letters*, vol. 10, no. 4, pp. 83–89, 1997.

[20] I. Rachunkova and C. C. Tisdell, "Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 4, pp. 1236–1245, 2007.

[21] J. Yu and Z. Guo, "On boundary value problems for a discrete generalized Emden-Fowler equation," *Journal of Differential Equations*, vol. 231, no. 1, pp. 18–31, 2006.

[22] F. M. Atici and A. Cabada, "Existence and uniqueness results for discrete second-order periodic boundary value problems," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 1417–1427, 2003.

[23] D. Bai and Y. Xu, "Nontrivial solutions of boundary value problems of second-order difference equations," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 297–302, 2007.