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Research Article

Sturm-Picone Comparison Theorem of Second-Order Linear Equations on Time Scales

Chao Zhang and Shurong Sun

School of Science, University of Jinan, Jinan, Shandong 250022, China

Correspondence should be addressed to Chao Zhang, ss_zhangc@ujn.edu.cn

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This paper studies Sturm-Picone comparison theorem of second-order linear equations on time scales. We first establish Picone identity on time scales and obtain our main result by using it. Also, our result unifies the existing ones of second-order differential and difference equations.

1. Introduction

In this paper, we consider the following second-order linear equations:

$$\left(p_1(t)x^{\Delta}(t)\right)^{\Delta} + q_1(t)x^{\sigma}(t) = 0, \tag{1.1}$$

$$\left(p_2(t)y^{\Delta}(t)\right)^{\Delta} + q_2(t)y^{\sigma}(t) = 0, \tag{1.2}$$

where $t \in [\alpha,\beta] \cap \mathbb{T}$, $p_1^{\Delta}(t)$, $p_2^{\Delta}(t)$, $q_1(t)$, and $q_2(t)$ are real and rd-continuous functions in $[\alpha,\beta] \cap \mathbb{T}$. Let $\mathbb{T}$ be a time scale, $\sigma(t)$ be the forward jump operator in $\mathbb{T}$, y^{Δ} be the delta derivative, and $y^{\sigma}(t) := y(\sigma(t))$.

First we briefly recall some existing results about differential and difference equations. As we well know, in 1909, Picone [1] established the following identity.

Picone Identity

If $x(t)$ and $y(t)$ are the nontrivial solutions of

$$\begin{aligned} &(p_1(t)x'(t))' + q_1(t)x(t) = 0, \\ &(p_2(t)y'(t))' + q_2(t)y(t) = 0, \end{aligned} \tag{1.3}$$

where $t \in [\alpha, \beta]$, $p_1'(t)$, $p_2'(t)$, $q_1(t)$, and $q_2(t)$ are real and continuous functions in $[\alpha, \beta]$. If $y(t) \neq 0$ for $t \in [\alpha, \beta]$, then

$$\begin{aligned}&\left(\frac{x(t)}{y(t)}\left(p_1(t)x'(t)y(t) - p_2(t)y'(t)x(t)\right)\right)' \\ &\quad = \left(p_1(t) - p_2(t)\right)x'^2(t) + \left(q_2(t) - q_1(t)\right)x^2(t) + p_2(t)\left(\frac{x(t)y'(t)}{y(t)} - x'(t)\right)^2.\end{aligned} \tag{1.4}$$

By (1.4), one can easily obtain the Sturm comparison theorem of second-order linear differential equations (1.3).

Sturm-Picone Comparison Theorem

Assume that $x(t)$ and $y(t)$ are the nontrivial solutions of (1.3) and a, b are two consecutive zeros of $x(t)$, if

$$p_1(t) \geq p_2(t) > 0, \quad q_2(t) \geq q_1(t), \quad t \in [a, b], \tag{1.5}$$

then $y(t)$ has at least one zero on $[a, b]$.

Later, many mathematicians, such as Kamke, Leighton, and Reid [2–5] developed thier work. The investigation of Sturm comparison theorem has involved much interest in the new century [6, 7]. The Sturm comparison theorem of second-order difference equations

$$\begin{aligned}\Delta\left[p_1(t-1)\Delta x(t-1)\right] + q_1(t)x(t) = 0, \\ \Delta\left[p_2(t-1)\Delta y(t-1)\right] + q_2(t)y(t) = 0,\end{aligned} \tag{1.6}$$

has been investigated in [8, Chapter 8], where $p_1(t) \geq p_2(t) > 0$ on $[\alpha, \beta+1]$, $q_2(t) \geq q_1(t)$ on $[\alpha+1, \beta+1]$, α, β are integers, and Δ is the forward difference operator: $\Delta x(t) = x(t+1) - x(t)$. In 1995, Zhang [9] extended this result. But we will remark that in [8, Chapter 8] the authors employed the Riccati equation and a positive definite quadratic functional in their proof. Recently, the Sturm comparison theorem on time scales has received a lot of attentions. In [10, Chapter 4], the mathematicians studied

$$\begin{aligned}\left(p_1(t)x^{\Delta}(t)\right)^{\nabla} + q_1(t)x(t) = 0, \\ \left(p_2(t)y^{\Delta}(t)\right)^{\nabla} + q_2(t)y(t) = 0,\end{aligned} \tag{1.7}$$

where $p_1(t) \geq p_2(t) > 0$ and $q_2(t) \geq q_1(t)$ for $t \in [\rho(\alpha), \sigma(\beta)] \cap \mathbb{T}$, y^{∇} is the nabla derivative, and they get the Sturm comparison theorem. We will make use of Picone identity on time scales to prove the Sturm-Picone comparison theorem of (1.1) and (1.2).

This paper is organized as follows. Section 2 introduces some basic concepts and fundamental results about time scales, which will be used in Section 3. In Section 3 we first give the Picone identity on time scales, then we will employ this to prove our main result: Sturm-Picone comparison theorem of (1.1) and (1.2) on time scales.

2. Preliminaries

In this section, some basic concepts and some fundamental results on time scales are introduced.

Let $\mathbb{T} \subset \mathbb{R}$ be a nonempty closed subset. Define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \tag{2.1}$$

where $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-scattered, right-dense, left-scattered, and left-dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, and $\rho(t) = t$, respectively. We put $\mathbb{T}^k = \mathbb{T}$ if $\mathbb{T}$ is unbounded above and $\mathbb{T}^k = \mathbb{T} \setminus (\rho(\max \mathbb{T}), \max \mathbb{T}]$ otherwise. The graininess functions $\nu, \mu : \mathbb{T} \to [0, \infty)$ are defined by

$$\mu(t) = \sigma(t) - t, \qquad \nu(t) = t - \rho(t). \tag{2.2}$$

Let f be a function defined on $\mathbb{T}$. f is said to be (delta) differentiable at $t \in \mathbb{T}^k$ provided there exists a constant a such that for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) with

$$\left| f(\sigma(t)) - f(s) - a(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \tag{2.3}$$

In this case, denote $f^{\Delta}(t) := a$. If f is (delta) differentiable for every $t \in \mathbb{T}^k$, then f is said to be (delta) differentiable on $\mathbb{T}$. If f is differentiable at $t \in \mathbb{T}^k$, then

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{\substack{s \to t \\ s \in \mathbb{T}}} \dfrac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0, \\[2ex] \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0. \end{cases} \tag{2.4}$$

If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F(t)$ is called an antiderivative of f on $\mathbb{T}$. In this case, define the delta integral by

$$\int_s^t f(\tau) \Delta\tau = F(t) - F(s) \quad \forall s, t \in \mathbb{T}. \tag{2.5}$$

Moreover, a function f defined on $\mathbb{T}$ is said to be rd-continuous if it is continuous at every right-dense point in $\mathbb{T}$ and its left-sided limit exists at every left-dense point in $\mathbb{T}$.

For convenience, we introduce the following results ([11, Chapter 1], [12, Chapter 1], and [13, Lemma 1]), which are useful in the paper.

Lemma 2.1. *Let $f, g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$.*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f and g are differentiable at t, then fg is differentiable at t and*

$$(fg)^{\Delta}(t) = f^{\sigma}(t)g^{\Delta}(t) + f^{\Delta}(t)g(t) = f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t). \tag{2.6}$$

(iii) *If f and g are differentiable at t, and $f(t)f^{\sigma}(t) \neq 0$, then $f^{-1}g$ is differentiable at t and*

$$\left(gf^{-1}\right)^{\Delta}(t) = \left(g^{\Delta}(t)f(t) - g(t)f^{\Delta}(t)\right)\left(f^{\sigma}(t)f(t)\right)^{-1}. \tag{2.7}$$

(iv) *If f is rd-continuous on $\mathbb{T}$, then it has an antiderivative on $\mathbb{T}$.*

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-increasing at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ provided

(i) $f(\sigma(t_0)) > f(t_0)$ in the case that t_0 is right-scattered;

(ii) there is a neighborhood U of t_0 such that $f(t) > f(t_0)$ for all $t \in U$ with $t > t_0$ in the case that t_0 is right-dense.

If the inequalities for f are reversed in (i) and (ii), f is said to be right-decreasing at t_0.

The following result can be directly derived from (2.4).

Lemma 2.3. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$. If $f^{\Delta}(t_0) > 0$, then f is right-increasing at t_0; and if $f^{\Delta}(t_0) < 0$, then f is right-decreasing at t_0.*

Definition 2.4. One says that a solution $x(t)$ of (1.1) has a generalized zero at t if $x(t) = 0$ or, if t is right-scattered and $x(t)x(\sigma(t)) < 0$. Especially, if $x(t)x(\sigma(t)) < 0$, then we say $x(t)$has a node at $(t + \sigma(t))/2$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)p(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{2.8}$$

Hilger [14] showed that for $t_0 \in \mathbb{T}$ and rd-continuous and regressive p, the solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \qquad y(t_0) = 1 \tag{2.9}$$

is given by $e_p(\cdot, t_0)$, where

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \quad \text{with } \xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1+hz)}{h}, & \text{if } h \neq 0 \\ z, & \text{if } h = 0. \end{cases} \tag{2.10}$$

The development of the theory uses similar arguments and the definition of the nabla derivative (see [10, Chapter 3]).

3. Main Results

In this section, we give and prove the main results of this paper.

First, we will show that the following second-order linear equation:

$$x^{\Delta\Delta}(t) + a_1(t)x^{\Delta\sigma}(t) + a_2(t)x^{\sigma}(t) = 0 \tag{3.1}$$

can be rewritten as (1.1).

Theorem 3.1. *If* $1 + \mu(t)a_1(t) \neq 0$ *and* $a_2(t)$ *is continuous, then* (3.1) *can be written in the form of* (1.1), *with*

$$p_1(t) = e_{a_1}(t, t_0), \qquad q_1(t) = e_{a_1}(t, t_0)a_2(t). \tag{3.2}$$

Proof. Multiplying both sides of (3.1) by $e_{a_1}(t, t_0)$, we get

$$\begin{aligned} 0 &= e_{a_1}(t, t_0)x^{\Delta\Delta}(t) + e_{a_1}(t, t_0)a_1(t)x^{\Delta\sigma}(t) + e_{a_1}(t, t_0)a_2(t)x^{\sigma}(t) \\ &= e_{a_1}(t, t_0)x^{\Delta\Delta}(t) + [e_{a_1}(t, t_0)]^{\Delta}x^{\Delta\sigma}(t) + e_{a_1}(t, t_0)a_2(t)x^{\sigma}(t) \\ &= \left[e_{a_1}(t, t_0)x^{\Delta}(t)\right]^{\Delta} + e_{a_1}(t, t_0)a_2(t)x^{\sigma}(t), \end{aligned} \tag{3.3}$$

where we used Lemma 2.1. This equation is in the form of (1.1) with $p_1(t)$ and $q_1(t)$ as desired. □

Lemma 3.2 (Picone Identity). *Let* $x(t)$ *and* $y(t)$ *be the nontrivial solutions of* (1.1) *and* (1.2) *with* $p_1(t) \geq p_2(t) > 0$ *and* $q_2(t) \geq q_1(t)$ *for* $t \in [\alpha, \beta] \cap \mathbb{T}$. *If* $y(t)$ *has no generalized zeros on* $[\alpha, \beta] \cap \mathbb{T}$, *then the following identity holds:*

$$\begin{aligned} &\left(\frac{x(t)}{y(t)}\left(p_1(t)x^{\Delta}(t)y(t) - p_2(t)y^{\Delta}(t)x(t)\right)\right)^{\Delta} \\ &\quad = (p_1(t) - p_2(t))\left(x^{\Delta}(t)\right)^2 + (q_2(t) - q_1(t))x^2(\sigma(t)) \\ &\qquad + \left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^{\Delta}(t)}{y(t)}x(\sigma(t)) - \sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^{\Delta}(t)\right)^2. \end{aligned} \tag{3.4}$$

Proof. We first divide the left part of (3.4) into two parts

$$\begin{aligned} \left(\frac{x(t)}{y(t)}\left(p_1(t)x^{\Delta}(t)y(t) - p_2(t)y^{\Delta}(t)x(t)\right)\right)^{\Delta} &= \left(p_1(t)x^{\Delta}(t)x(t) - \frac{p_2(t)y^{\Delta}(t)}{y(t)}x^2(t)\right)^{\Delta} \\ &= \left(p_1(t)x^{\Delta}(t)x(t)\right)^{\Delta} - \left(\frac{p_2(t)y^{\Delta}(t)}{y(t)}x^2(t)\right)^{\Delta}. \end{aligned} \tag{3.5}$$

From (1.1) and the product rule (Lemma 2.1(ii)), we have

$$\begin{aligned}\left(p_1(t)x^{\Delta}(t)x(t)\right)^{\Delta} &= \left(p_1(t)x^{\Delta}(t)\right)^{\Delta}x(\sigma(t)) + p_1(t)x^{\Delta}(t)x^{\Delta}(t)\\ &= p_1(t)\left(x^{\Delta}(t)\right)^2 - q_1(t)x^2(\sigma(t)) \quad \forall t\in[\alpha,\beta]\cap\mathbb{T}.\end{aligned} \tag{3.6}$$

It follows from (1.2), (2.4), product and quotient rules (Lemma 2.1(ii), (iii)) and the assumption that $y(t)$ has no generalized zeros on $[\alpha,\beta]\cap\mathbb{T}$ that

$$\begin{aligned}&\left(\frac{p_2(t)y^{\Delta}(t)}{y(t)}x^2(t)\right)^{\Delta}\\
&= x^2(\sigma(t))\left(\frac{p_2(t)y^{\Delta}(t)}{y(t)}\right)^{\Delta} + x(\sigma(t))x^{\Delta}(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)} + x^{\Delta}(t)x(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)}\\
&= x^2(\sigma(t))\left(-q_2(t) - p_2(t)\frac{\left(y^{\Delta}(t)\right)^2}{y(t)y(\sigma(t))}\right) + x(\sigma(t))x^{\Delta}(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)}\\
&\quad + x^{\Delta}(t)\left(x(\sigma(t)) - \mu(t)x^{\Delta}(t)\right)\frac{p_2(t)y^{\Delta}(t)}{y(t)}\\
&= p_2(t)\left(x^{\Delta}(t)\right)^2 - q_2(t)x^2(\sigma(t)) - p_2(t)\frac{\left(y^{\Delta}(t)\right)^2x^2(\sigma(t))}{y(t)y(\sigma(t))}\\
&\quad + 2x(\sigma(t))x^{\Delta}(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)} - \left(p_2(t) + \mu(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)}\right)\left(x^{\Delta}(t)\right)^2\\
&= p_2(t)\left(x^{\Delta}(t)\right)^2 - q_2(t)x^2(\sigma(t)) - \frac{y(t)}{p_2(t)y(\sigma(t))}\left(\frac{p_2(t)y^{\Delta}(t)}{y(t)}\right)^2x^2(\sigma(t))\\
&\quad + 2x(\sigma(t))x^{\Delta}(t)\frac{p_2(t)y^{\Delta}(t)}{y(t)} - \frac{p_2(t)y(\sigma(t))}{y(t)}\left(x^{\Delta}(t)\right)^2\\
&= p_2(t)\left(x^{\Delta}(t)\right)^2 - q_2(t)x^2(\sigma(t))\\
&\quad - \left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^{\Delta}(t)}{y(t)}x(\sigma(t)) - \sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^{\Delta}(t)\right)^2 \quad \forall t\in[\alpha,\beta]\cap\mathbb{T}.\end{aligned} \tag{3.7}$$

Combining $(p_1(t)x^{\Delta}(t)x(t))^{\Delta}$ and $-((p_2(t)y^{\Delta}(t)/y(t))x^2(t))^{\Delta}$, we get (3.4). This completes the proof. □

Now, we turn to proving the main result of this paper.

Theorem 3.3 (Sturm-Picone Comparison Theorem). *Suppose that* $x(t)$ *and* $y(t)$ *are the nontrivial solutions of* (1.1) *and* (1.2)*, and* a*,* b *are two consecutive generalized zeros of* $x(t)$*, if*

$$p_1(t) \geq p_2(t) > 0, \quad q_2(t) \geq q_1(t), \quad t \in [a,b] \cap \mathbb{T}, \tag{3.8}$$

then $y(t)$ *has at least one generalized zero on* $[a,b] \cap \mathbb{T}$.

Proof. Suppose to the contrary, $y(t)$ has no generalized zeros on $[a,b] \cap \mathbb{T}$ and $y(t) > 0$ for all $t \in [a,b] \cap \mathbb{T}$.

Case 1. Suppose a, b are two consecutive zeros of $x(t)$. Then by Lemma 3.2, (3.4) holds and integrating it from a to b we get

$$\begin{aligned}
&\int_a^b \left(\frac{x(t)}{y(t)} \left(p_1(t) x^{\Delta}(t) y(t) - p_2(t) y^{\Delta}(t) x(t) \right) \right)^{\Delta} \Delta t \\
&\quad = \int_a^b \Bigg(\left(p_1(t) - p_2(t) \right) \left(x^{\Delta}(t) \right)^2 + \left(q_2(t) - q_1(t) \right) x^2(\sigma(t)) \\
&\qquad + \left(\sqrt{\frac{y(t)}{p_2(t) y(\sigma(t))}} \frac{p_2(t) y^{\Delta}(t)}{y(t)} - \sqrt{\frac{p_2(t) y(\sigma(t))}{y(t)}} x^{\Delta}(t) \right)^2 \Bigg) \Delta t.
\end{aligned} \tag{3.9}$$

Noting that $x(a) = x(b) = 0$, we have

$$\begin{aligned}
&\int_a^b \left(\frac{x(t)}{y(t)} \left(p_1(t) x^{\Delta}(t) y(t) - p_2(t) y^{\Delta}(t) x(t) \right) \right)^{\Delta} \Delta t \\
&\quad = \left(\frac{x(t)}{y(t)} \left(p_1(t) x^{\Delta}(t) y(t) - p_2(t) y^{\Delta}(t) x(t) \right) \right) \Bigg|_a^b \\
&\quad = 0.
\end{aligned} \tag{3.10}$$

Hence, by (3.9) and $p_1(t) \geq p_2(t) > 0$, $q_2(t) \geq q_1(t)$, for all $t \in [a,b] \cap \mathbb{T}$ we have

$$\begin{aligned}
0 &= \int_a^b \Bigg(\left(p_1(t) - p_2(t) \right) \left(x^{\Delta}(t) \right)^2 + \left(q_2(t) - q_1(t) \right) x^2(\sigma(t)) \\
&\qquad + \left(\sqrt{\frac{y(t)}{p_2(t) y(\sigma(t))}} \frac{p_2(t) y^{\Delta}(t)}{y(t)} - \sqrt{\frac{p_2(t) y(\sigma(t))}{y(t)}} x^{\Delta}(t) \right)^2 \Bigg) \Delta t \\
&> 0,
\end{aligned} \tag{3.11}$$

which is a contradiction. Therefore, in Case 1, $y(t)$ has at least one generalized zero on $[a,b]\cap\mathbb{T}$.

Case 2. Suppose a is a zero of $x(t)$, $(b+\sigma(b))/2$ is a node of $x(t), x(b)<0$, and $x(\sigma(b))>0$. It follows from the assumption that $y(t)$ has no generalized zeros on $[a,b]\cap\mathbb{T}$ and that $y(t)>0$ for all $t\in[a,b]\cap\mathbb{T}$ that $y(\sigma(b))>0$. Hence by (2.4) and $p_2(t)\geq p_1(t)>0$ on $[a,b]\cap\mathbb{T}$, we have

$$\begin{aligned}
&\frac{x(b)}{y(b)}\left(p_1(b)x^{\Delta}(b)y(b)-p_2(b)y^{\Delta}(b)x(b)\right)\\
&\quad=\frac{x(b)}{y(b)}\frac{1}{\mu(b)}\left(p_1(b)x(\sigma(b))y(b)-p_2(b)y(\sigma(b))x(b)+\left(p_2(b)-p_1(b)\right)x(b)y(b)\right)\\
&\quad<0.
\end{aligned}\tag{3.12}$$

By integration, it follows from (3.12) and $x(a)=0$ that

$$\begin{aligned}
&\int_a^b\left(\frac{x(t)}{y(t)}\left(p_1(t)x^{\Delta}(t)y(t)-p_2(t)y^{\Delta}(t)x(t)\right)\right)^{\Delta}\Delta t\\
&\quad=\left(\frac{x(t)}{y(t)}\left(p_1(t)x^{\Delta}(t)y(t)-p_2(t)y^{\Delta}(t)x(t)\right)\right)\Bigg|_a^b\\
&\quad=\frac{x(b)}{y(b)}\left(p_1(b)x^{\Delta}(b)y(b)-p_2(b)y^{\Delta}(b)x(b)\right)\\
&\quad<0.
\end{aligned}\tag{3.13}$$

So, from (3.9) and above argument we obtain that

$$\begin{aligned}
0>\int_a^b&\Bigg(\left(p_1(t)-p_2(t)\right)\left(x^{\Delta}(t)\right)^2+\left(q_2(t)-q_1(t)\right)x^2(\sigma(t))\\
&+\left(\sqrt{\frac{y(t)}{p_2(t)y(\sigma(t))}}\frac{p_2(t)y^{\Delta}(t)}{y(t)}-\sqrt{\frac{p_2(t)y(\sigma(t))}{y(t)}}x^{\Delta}(t)\right)^2\Bigg)\Delta t\\
>0,&
\end{aligned}\tag{3.14}$$

which is a contradiction, too. Hence, in Case 2, $y(t)$ has at least one generalized zero on $[a,b]\cap\mathbb{T}$.

Case 3. Suppose $(a+\sigma(a))/2$ is a node of $x(t), x(a) > 0$, $x(\sigma(a)) < 0$, and b is a generalized zero of $x(t)$. Similar to the discussion of (3.12), we have

$$\begin{aligned} &\frac{x(a)}{y(a)}\left(p_1(a)x^{\Delta}(a)y(a) - p_2(a)y^{\Delta}(a)x(a)\right) \\ &\quad = \frac{x(a)}{y(a)}\frac{1}{\mu(a)}\left(p_1(a)x(\sigma(a))y(a) - p_2(a)y(\sigma(a))x(a) + \left(p_2(a)-p_1(a)\right)x(a)y(a)\right) \\ &\quad < 0, \end{aligned} \tag{3.15}$$

which implies

$$\left(p_1(a)x^{\Delta}(a)y(a) - p_2(a)y^{\Delta}(a)x(a)\right) < 0. \tag{3.16}$$

(i) If $(b+\sigma(b))/2$ is a node of $x(t)$, then $x(b) < 0$, $x(\sigma(b)) > 0$. Hence, we have (3.12), that is,

$$\frac{x(b)}{y(b)}\left(p_1(b)x^{\Delta}(b)y(b) - p_2(b)y^{\Delta}(b)x(b)\right) < 0. \tag{3.17}$$

(ii) If b is a zero of $x(t)$, then

$$\frac{x(b)}{y(b)}\left(p_1(b)x^{\Delta}(b)y(b) - p_2(b)y^{\Delta}(b)x(b)\right) = 0. \tag{3.18}$$

It follows from (3.4) and Lemma 2.3 that

$$\frac{x(t)}{y(t)}\left(p_1(t)x^{\Delta}(t)y(t) - p_2(t)y^{\Delta}(t)x(t)\right) \tag{3.19}$$

is right-increasing on $[a,b]\cap\mathbb{T}$. Hence, from (i) and (ii) that

$$\begin{aligned} &\frac{x(a)}{y(a)}\left(p_1(a)x^{\Delta}(a)y(a) - p_2(a)y^{\Delta}(a)x(a)\right) \\ &\quad < \frac{x(\sigma(a))}{y(\sigma(a))}\left(p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a)) - p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a))\right) \\ &\quad < 0, \end{aligned} \tag{3.20}$$

which implies

$$p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a)) - p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a)) > 0. \tag{3.21}$$

From (3.16), (3.21), and (2.4), we have

$$\left(p_1x^{\Delta}y - p_2y^{\Delta}x\right)^{\Delta}(a) = \frac{1}{\mu(a)}\left(\left(p_1x^{\Delta}y - p_2y^{\Delta}x\right)(\sigma(a)) - \left(p_1x^{\Delta}y - p_2y^{\Delta}x\right)(a)\right) > 0. \quad (3.22)$$

Further, it follows from (1.1), (1.2), product rule (Lemma 2.1(ii)), and (3.22) that

$$\left(p_1x^{\Delta}y - p_2y^{\Delta}x\right)^{\Delta}(a) = (q_2(a) - q_1(a))x(\sigma(a))y(\sigma(a)) + (p_1(a) - p_2(a))x^{\Delta}(a)y^{\Delta}(a) > 0. \quad (3.23)$$

If $p_1(a) = p_2(a)$ and from $q_2(a) \ge q_1(a)$, $x(\sigma(a)) < 0$, and $y(\sigma(a)) > 0$ we have

$$(q_2(a) - q_1(a))x(\sigma(a))y(\sigma(a)) < 0. \quad (3.24)$$

This contradicts (3.22). Note that $x^{\Delta}(a) = (1/\mu(a))(x(\sigma(a)) - x(a))$. It follows from $p_1(a) > p_2(a) > 0$, (3.23), and (3.24) that

$$y^{\Delta}(a) < 0. \quad (3.25)$$

On the other hand, it follows from $x(t)$ and $y(t)$ are solutions of (1.1) and (1.2) that

$$\begin{aligned} y(\sigma(a))\left(\left(p_1(a)x^{\Delta}(a)\right)^{\Delta} + q_1(a)x(\sigma(a))\right) &= 0, \\ x(\sigma(a))\left(\left(p_2(a)y^{\Delta}(a)\right)^{\Delta} + q_2(a)y(\sigma(a))\right) &= 0. \end{aligned} \quad (3.26)$$

Combining the above two equations we obtain

$$\left(\left(p_1(a)x^{\Delta}(a)\right)^{\Delta}y(\sigma(a)) - \left(p_2(a)y^{\Delta}(a)\right)^{\Delta}x(\sigma(a))\right) + (q_1(a) - q_2(a))x(\sigma(a))y(\sigma(a)) = 0. \quad (3.27)$$

It follows from (3.27) and (2.4) that

$$\begin{aligned}
&\frac{1}{\mu(a)}\Big\{\Big[p_1(\sigma(a))x^{\Delta}(\sigma(a)) - p_1(a)x^{\Delta}(a)\Big]y(\sigma(a)) - \Big[p_2(\sigma(a))y^{\Delta}(\sigma(a)) - p_2(a)y^{\Delta}(a)\Big]x(\sigma(a))\Big\}\\
&\qquad + \big(q_1(a) - q_2(a)\big)x(\sigma(a))y(\sigma(a))\\
&\quad = \frac{1}{\mu(a)}\Big[p_2(a)y^{\Delta}(a)x(\sigma(a)) - p_1(a)x^{\Delta}(a)y(\sigma(a))\Big]\\
&\qquad + \frac{1}{\mu(a)}\Big[p_1(\sigma(a))x^{\Delta}(\sigma(a))y(\sigma(a)) - p_2(\sigma(a))y^{\Delta}(\sigma(a))x(\sigma(a))\Big]\\
&\qquad + \big(q_1(a) - q_2(a)\big)x(\sigma(a))y(\sigma(a))\\
&\quad = 0.
\end{aligned} \tag{3.28}$$

Hence, from $q_2(a) \geq q_1(a)$, $x(\sigma(a)) < 0$, $y(\sigma(a)) > 0$, and (3.21), we get

$$p_2(a)y^{\Delta}(a)x(\sigma(a)) - p_1(a)x^{\Delta}(a)y(\sigma(a)) < 0. \tag{3.29}$$

By referring to $x^{\Delta}(a) < 0$ and $p_1(a) > p_2(a) > 0$, it follows that

$$y^{\Delta}(a) > 0, \tag{3.30}$$

which contradicts $y^{\Delta}(a) < 0$.

It follows from the above discussion that $y(t)$ has at least one generalized zero on $[a, b] \cap \mathbb{T}$. This completes the proof. □

Remark 3.4. If $p_1(t) \equiv p_2(t) \equiv 1$, then Theorem 3.3 reduces to classical Sturm comparison theorem.

Remark 3.5. In the continuous case: $\mu(t) \equiv 0$. This result is the same as Sturm-Picone comparison theorem of second-order differential equations (see Section 1).

Remark 3.6. In the discrete case: $\mu(t) \equiv 1$. This result is the same as Sturm comparison theorem of second-order difference equations (see [8, Chapter 8]).

Example 3.7. Consider the following three specific cases:

$$\begin{gathered}
[0,1] \cap \mathbb{T} = \left[0, \frac{1}{2}\right] \cup \left[\frac{2}{3}, 1\right],\\
[0,1] \cap \mathbb{T} = \left[0, \frac{1}{2}\right] \cup \left\{\frac{1}{2(N-1)}, \frac{1}{(N-1)}, \frac{3}{2(N-1)}, \ldots, 1\right\}, \quad N > 2,\\
[0,1] \cap \mathbb{T} = \left\{q^k \mid k \geq 0,\ k \in \mathbb{Z}\right\} \cup \{0\}, \quad \text{where } 0 < q < 1.
\end{gathered} \tag{3.31}$$

By Theorem 3.3, we have if $x(t)$ and $y(t)$ are the nontrivial solutions of (1.1) and (1.2), a, b are two consecutive generalized zeros of $x(t)$, and $p_1(t) \geq p_2(t) > 0$, $q_2(t) \geq q_1(t)$, $t \in [a, b] \cap \mathbb{T}$, then $y(t)$ has at least one generalized zero on $[a, b] \cap \mathbb{T}$. Obviously, the above three cases are not continuous and not discrete. So the existing results for the differential and difference equations are not available now.

By Remarks 3.4–3.6 and Example 3.7, the Sturm comparison theorem on time scales not only unifies the results in both the continuous and the discrete cases but also contains more complicated time scales.

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