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Research Article

Uniform Attractor for the Partly Dissipative Nonautonomous Lattice Systems

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The existence of uniform attractor in $l^2 \times l^2$ is proved for the partly dissipative nonautonomous lattice systems with a new class of external terms belonging to $L^2_{\text{loc}}(R, l^2)$, which are locally asymptotic smallness and translation bounded but not translation compact in $L^2_{\text{loc}}(R, l^2)$. It is also showed that the family of processes corresponding to nonautonomous lattice systems with external terms belonging to weak topological space possesses uniform attractor, which is identified with the original one. The upper semicontinuity of uniform attractor is also studied.

1. Introduction

This paper is concerned with the long-time behavior of the following non-autonomous lattice systems:

$$\dot{u}_i + \nu_i (Au)_i + \lambda_i u_i + f_i(u_i, (Bu)_i) + \alpha_i v_i = k_i(t), \quad i \in \mathbb{Z}, \ t > \tau, \tag{1.1}$$

$$\dot{v}_i + \delta_i v_i - \beta_i u_i = g_i(t), \quad i \in \mathbb{Z}, \ t > \tau, \tag{1.2}$$

with initial conditions

$$u_i(\tau) = u_{i,\tau}, \quad v_i(\tau) = v_{i,\tau}, \quad i \in \mathbb{Z}, \ \tau \in \mathbb{R}, \tag{1.3}$$

where $\mathbb{Z}$ is the integer lattice; $\nu_i, \lambda_i, \delta_i > 0$, $\alpha_i \beta_i > 0$, f_i is a nonlinear function satisfying $f_i \in C^1(R \times R, R)$, $i \in \mathbb{Z}$; A is a positive self-adjoint linear operator; $k(t) = (k_i(t))_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ belong to certain metric space, which will be given in the following.

Lattice dynamical systems occur in a wide variety of applications, where the spatial structure has a discrete character, for example, chemical reaction theory, electrical engineering, material science, laser, cellular neural networks with applications to image processing and pattern recognition; see [1–4]. Thus, a great interest in the study of infinite lattice systems has been raising. Lattice differential equations can be considered as a spatial or temporal discrete analogue of corresponding partial differential equations on unbounded domains. It is well known that the long-time behavior of solutions of partial differential equations on unbounded domains raises some difficulty, such as well-posedness and lack of compactness of Sobolev embeddings for obtaining existence of global attractors. Authors in [5–7] consider the autonomous partial equations on unbounded domain in weighted spaces, using the decaying of weights at infinity to get the compactness of solution semigroup. In [8–10], asymptotic compactness of the solutions is used to obtain existence of global compact attractors for autonomous system on unbounded domain. Authors in [11] consider them in locally uniform space. For non-autonomous partial differential equations on bounded domain, many studies on the existence of uniform attractor have been done, for example [12–14].

For lattice dynamical systems, standard theory of ordinary differential equations can be applied to get the well-posedness of it. "Tail ends" estimate method is usually used to get asymptotic compactness of autonomous infinite-dimensional lattice, and by this the existence of global compact attractor is obtained; see [15–17]. Authors in [18, 19] also prove that the uniform smallness of solutions of autonomous infinite lattice systems for large space and time variables is sufficient and necessary conditions for asymptotic compactness of it. Recently, "tail ends" method is extended to non-autonomous infinite lattice systems; see [20–22]. The traveling wave solutions of lattice differential equations are studied in [23–25]. In [18, 26, 27], the existence of global attractors of autonomous infinite lattice systems is obtained in weighted spaces, which do not exclude traveling wave.

In this paper, we investigate the existence of uniform attractor for non-autonomous lattice systems (1.1)–(1.3). The external term in [20] is supposed to belong to $C_b(R, l^2)$ and to be almost periodic function. By Bochner-Amerio criterion, the set of this external term's translation is precompact in $C_b(R, l^2)$. Based on ideas of [28], authors in [14] introduce uniformly ω-limit compactness, and prove that the family of weakly continuous processes with respect to (w.r.t.) certain symbol space possesses compact uniform attractors if the process has a bounded uniform absorbing set and is uniformly ω-limit compact. Motivated by this, we will prove that the process corresponding to problem (1.1)–(1.3) with external terms being locally asymptotic smallness (see Definition 4.5) possesses a compact uniform attractor in $l^2 \times l^2$, which coincides with uniform attractor of the family of processes with external terms belonging to weak closure of translation set of locally asymptotic smallness function in $L^2_{\text{loc}}(R, l^2)$. We also show that locally asymptotic functions are translation bounded in $L^2_{\text{loc}}(R, l^2)$, but not translation compact (tr.c.) in $L^2_{\text{loc}}(R, l^2)$. Since the locally asymptotic smallness functions are not necessary to be translation compact in $C_b(R, l^2)$, compared with [20], the conditions on external terms of (1.1)–(1.3) can be relaxed in this paper.

This paper is organized as follows. In Section 2, we give some preliminaries and present our main result. In Section 3, the existence of a family of processes for (1.1)–(1.3) is obtained. We also show that the family of processes possesses a uniformly (w.r.t $\mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$) absorbing set. In Section 4, we prove the existence of uniform attractor. In Section 5, the upper semicontinuity of uniform attractor will be studied.

2. Main Result

In this section, we describe our main result. Denote by l^2 the Hilbert space defined by

$$l^2 = \left\{ u = (u_i)_{i\in\mathbb{Z}} \mid u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}} u_i^2 < +\infty \right\}, \tag{2.1}$$

with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ given by

$$\langle u, v \rangle = \sum_{i\in\mathbb{Z}} u_i v_i, \qquad \|u\|^2 = \langle u, u \rangle = \sum_{i\in\mathbb{Z}} u_i^2. \tag{2.2}$$

For $l^2 \times l^2$, we endow with the inner and norm as. For $\varphi_j = (u^{(j)}, v^{(j)}) = (u_i^{(j)}, v_i^{(j)})_{i\in\mathbb{Z}} \in l^2 \times l^2,\ j = 1, 2,$

$$\begin{gathered} \langle \varphi_1, \varphi_2 \rangle_{l^2\times l^2} = \left\langle u^{(1)}, u^{(2)} \right\rangle_{l^2} + \left\langle v^{(1)}, v^{(2)} \right\rangle_{l^2} = \sum_{i\in\mathbb{Z}} \left(u_i^{(1)} u_i^{(2)} + v_i^{(1)} v_i^{(2)} \right), \\ \|\varphi\|_{l^2\times l^2}^2 = \langle \varphi, \varphi \rangle_{l^2\times l^2}, \quad \forall \varphi \in l^2 \times l^2. \end{gathered} \tag{2.3}$$

Denote by $L^2_{\text{loc}}(R, l^2)$ the space of function $\phi(s),\ s \in \mathbb{R}$ with values in l^2 that locally 2-power integrable in the Bochner sense, that is,

$$\int_{t_1}^{t_2} \|\phi(s)\|_{l^2}^2 ds < +\infty, \quad \forall [t_1, t_2] \subset \mathbb{R}. \tag{2.4}$$

It is equipped with the local 2-power mean convergence topology. Then, $L^2_{\text{loc}}(R, l^2)$ is a metrizable space. Let $L^2_b(R, l^2)$ be a space of functions $\phi(t)$ from $L^2_{\text{loc}}(R, l^2)$ such that

$$\|\phi(t)\|_{L^2_b(R,l^2)}^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\phi(s)\|_{l^2}^2 ds < \infty. \tag{2.5}$$

Denote by $L^{2,w}_{\text{loc}}(R, l^2)$ the space $L^2_{\text{loc}}(R, l^2)$ endow with the local weak convergence topology.

For each sequence $u = (u_i)_{i\in\mathbb{Z}}$, define linear operators on l^2 by

$$\begin{gathered} (Bu)_i = u_{i+1} - u_i, \quad (B^* u)_i = u_{i-1} - u_i, \quad i \in \mathbb{Z}, \\ (Au)_i = -u_{i+1} + 2u_i - u_{i-1}, \quad i \in \mathbb{Z}. \end{gathered} \tag{2.6}$$

Then

$$\begin{gathered} A = BB^* = B^* B, \\ (B^* u, v) = (u, Bv), \quad \forall u, v \in l^2. \end{gathered} \tag{2.7}$$

For convenience, initial value problem (1.1)–(1.3) can be written as

$$\dot{u} + \nu(Au) + \lambda u + f(u, Bu) + \alpha v = k(t), \quad t > \tau, \tag{2.8}$$

$$\dot{v} + \delta v - \beta u = g(t), \quad t > \tau, \tag{2.9}$$

with initial conditions

$$u(\tau) = u_\tau = (u_{i,\tau})_{i\in\mathbb{Z}}, \quad v(\tau) = v_\tau = (v_{i,\tau})_{i\in\mathbb{Z}}, \quad \tau \in \mathbb{R}, \tag{2.10}$$

where $u=(u_i)_{i\in\mathbb{Z}}, v=(v_i)_{i\in\mathbb{Z}}, \nu(Au)=(\nu_i(Au_i))_{i\in\mathbb{Z}}, f(u,Bu) = (f(u_i,(Bu)_i))_{i\in\mathbb{Z}}, k(t) = (k_i(t))_{i\in\mathbb{Z}}$, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$.

In the following, we give some assumption on nonlinear function $f_i \in C^1(R \times R, R)$, and $\nu_i, \lambda_i, \alpha_i, \beta_i, \delta_i \in \mathbb{R}$:

(H_1)

$$f_i(u_i = 0, (Bu)_i = 0) = 0, \qquad f_i(u_i, (Bu)_i)u_i \geq 0. \tag{2.11}$$

(H_2) There exists a positive-value continuous function $Q : R^+ \mapsto R^+$ such that

$$\sup_{i\in\mathbb{Z}} \max_{u_i,(Bu)_i\in[-r,r]} \left| f'_{i,u_i}(u_i, (Bu)_i) \right| + \sup_{i\in\mathbb{Z}} \max_{u_i,(Bu)_i\in[-r,r]} \left| f'_{i,(Bu)_i}(u_i, (Bu)_i) \right| \leq Q(r). \tag{2.12}$$

(H_3) There exist positive constants $\nu_0, \nu^0, \lambda_0, \lambda^0, \alpha_0, \alpha^0, \beta_0, \beta^0, \sigma_0, \sigma^0$such that

$$\begin{aligned}
&0 < \nu_0 = \min\{\nu_i, : i \in \mathbb{Z}\}, && \nu^0 = \max\{\nu_i, : i \in \mathbb{Z}\} < +\infty,\\
&0 < \lambda_0 = \min\{\lambda_i, : i \in \mathbb{Z}\}, && \lambda^0 = \max\{\lambda_i, : i \in \mathbb{Z}\} < +\infty,\\
&0 < \alpha_0 = \min\{\alpha_i, : i \in \mathbb{Z}\}, && \alpha^0 = \max\{\alpha_i, : i \in \mathbb{Z}\} < +\infty,\\
&0 < \beta_0 = \min\{\beta_i, : i \in \mathbb{Z}\}, && \beta^0 = \max\{\beta_i, : i \in \mathbb{Z}\} < +\infty,\\
&0 < \delta_0 = \min\{\delta_i, : i \in \mathbb{Z}\}, && \delta^0 = \max\{\delta_i, : i \in \mathbb{Z}\} < +\infty.
\end{aligned} \tag{2.13}$$

Let the external term $h(t), g(t)$ belong to $L_b^2(R, l^2)$, it follows from the standard theory of ordinary differential equations that there exists a unique local solution $(u, v) \in C([\tau, t_0), l^2 \times l^2)$ for problem (2.8)–(2.10) if (H_1)–(H_3) hold. For a fixed external term $(k_0(t), g_0(t)) \in L_b^2(R, l^2) \times L_b^2(R, l^2)$, take the symbol space $\mathbf{\Sigma} = \{k_0(s + h) \mid h \in \mathbb{R}\} \times \{g_0(s + h) \mid h \in \mathbb{R}\} = \mathcal{H}(k_0) \times \mathcal{H}(g_0)$, the set contains all translations of $(k_0(s), g_0(s))$ in $L_b^2(R, l^2) \times L_b^2(R, l^2)$. Take the $[\mathbf{\Sigma}]_w = \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$ the closure of $\mathbf{\Sigma}$ in $L_{\rm loc}^{2,w}(R, l^2) \times L_{\rm loc}^{2,w}(R, l^2)$. Denote by $T(h)$ the translation semigroup, $T(h)(k(s), g(s)) = (k(s + h), g(s + h))$ for all $(k, g) \in \mathbf{\Sigma}$ or $[\mathbf{\Sigma}]_w$, $s \in \mathbb{R}$, $h \geq 0$. It is evident that $\{T(h)\}_{h\geq 0}$ is continuous on $\mathbf{\Sigma}$ in the topology of $L_b^2(R, l^2)$ and on $[\mathbf{\Sigma}]_w$ in the topology of $L_{\rm loc}^{2,w}(R, l^2)$, respectively,

$$T(h)\mathbf{\Sigma} = \mathbf{\Sigma} = \mathcal{H}(k_0) \times \mathcal{H}(g_0), \quad T(h)[\mathbf{\Sigma}]_w = [\mathbf{\Sigma}]_w = \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0), \quad \forall h > 0. \tag{2.14}$$

In Section 3, we will show that for every $(k(t), g(t)) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$, and $(u_\tau, v_\tau) = (u_{i,\tau}, v_{i,\tau})_{i\in\mathbb{Z}} \in l^2 \times l^2$, $\tau \in \mathbb{R}$, problem (2.8)–(2.10) has a unique global solution $(u, v)(t) = (u_i, v_i)_{i\in\mathbb{Z}}(t) \in C([\tau, \infty), l^2 \times l^2)$. Thus, there exists a family of processes $\{U_{(k,g)}(t, \tau)\}$ from $l^2 \times l^2$ to $l^2 \times l^2$. In order to obtain the uniform attractor of the family of processes, we suppose the external term is *locally asymptotic smallness* (see Definition 4.5). Let E be a Banach space which the processes acting in, for a given symbol space Ξ, the uniform (w.r.t. $\sigma \in \Xi$) ω-limit set $\omega_{\tau,\Xi}(B)$ of $B \subset E$ is defined by

$$\omega_{\tau,\Xi}(B) = \bigcap_{t\geq\tau} \overline{\bigcup_{\sigma\in\Xi} \bigcup_{s\geq t} U_\sigma(s, \tau)B}^E. \tag{2.15}$$

The first result of this paper is stated in the following, which will be proved in Section 4.

Theorem A. *Assume that $(k_0(s), g_0(s)) \in L^2_{\mathrm{loc}}(R, l^2) \times L^2_{\mathrm{loc}}(R, l^2)$ be locally asymptotic smallness and (H_1)–(H_3) hold. Then the process $\{U_{(k_0,g_0)}\}$ corresponding to problems* (2.8)–(2.10) *with external term $(k_0(s), g_0(s))$ possesses compact uniform (w.r.t. $\tau \in R$) attractor $\mathscr{A}_0$ in $l^2 \times l^2$ which coincides with uniform (w.r.t. $(k(s), g(s)) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$) attractor $\mathscr{A}_{\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)}$ for the family of processes $\{U_{(k,g)}(t, \tau)\}$, $(k, g) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$, that is,*

$$\mathscr{A}_0 = \mathscr{A}_{\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)} = \omega_{0,\mathscr{A}_{\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)}}(B_0) = \bigcup_{(k,g)\in\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)} \mathscr{K}_{(k,g)}(0), \tag{2.16}$$

where B_0 is the uniform (w.r.t. $(k, g) \in \mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)$) absorbing set in $l^2\times l^2$, and $\mathscr{K}_{(k,g)}$ is kernel of the process $\{U_{(k,g)}(t, \tau)\}$. The uniform attractor uniformly (w.r.t. $(k, g) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$) attracts the bounded set in $l^2 \times l^2$.

We also consider finite-dimensional approximation to the infinite-dimensional systems (1.2)-(1.3) on finite lattices. For every positive integer $n > 0$, let $Z_n = Z \cap \{-n \leq i \leq n\}$, consider the following ordinary equations with initial data in $R^{2n+1} \times R^{2n+1}$:

$$\begin{gathered}\dot{u}_i + \nu_i(Au)_i + \lambda_i u_i + f_i(u_i, (Bu)_i) + \alpha_i v_i = k_i(t), \quad i \in Z_n,\ t > \tau,\\ \dot{v}_i + \delta_i v_i - \beta_i u_i = g_i(t), \quad i \in Z_n,\ t > \tau,\\ u(\tau) = (u_i(\tau))_{|i|\leq n} = (u_{i,\tau})_{|i|\leq n}, \quad v(\tau) = (v_i(\tau))_{|i|\leq n} = (v_{i,\tau})_{|i|\leq n}, \quad \tau \in \mathbb{R}.\end{gathered} \tag{2.17}$$

In Section 5, we will show that the finite-dimensional approximation systems possess a uniform attractor $\mathscr{A}_0^n$ in ${}^{2n+1} \times R^{2n+1}$, and these uniform attractors are upper semicontinuous when $n \to \infty$. More precisely, we have the following theorem.

Theorem B. *Assume that $(k_0(s), g_0(s)) \in L^2_b(R, l^2) \times L^2_b(R, l^2)$ and (H_1)–(H_3) hold. Then for every positive integer n, systems* (2.17) *possess compact uniform attractor $\mathscr{A}_0^n$. Further, $\mathscr{A}_0^n$ is upper semicontinuous to $\mathscr{A}_0$ as $n \to \infty$, that is,*

$$\lim_{n\to\infty} d_{l^2\times l^2}\left(\mathscr{A}_0^n, \mathscr{A}_0\right) = 0, \tag{2.18}$$

where

$$d_{l^2\times l^2}\left(\mathscr{A}_0^n,\mathscr{A}_0\right)=\sup_{a\in\mathscr{A}_0^n}\inf_{b\in\mathscr{A}_0}\|a-b\|_{l^2\times l^2}. \tag{2.19}$$

3. Processes and Uniform Absorbing Set

In this section, we show that the process can be defined and there exists a bounded uniform absorbing set for the family of processes.

Lemma 3.1. *Assume that $k_0, g_0 \in L_b^2(R, l^2)$ and (H_1)–(H_3) hold. Let $(k(s), g(s)) \in \mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)$, and $(u_\tau, v_\tau) \in l^2 \times l^2$, $\tau \in R$. Then the solution of* (2.8)–(2.10) *satisfies*

$$\begin{aligned}\|(u,v)(t)\|_{l^2\times l^2}^2 &\le \|(u,v)(\tau)\|_{l^2\times l^2}^2 e^{-(\gamma_0/\eta_0)(t-\tau)}\\&\quad+\frac{1}{\eta_0}\left(\frac{\beta^0}{\lambda_0}\|k_0(s)\|_{L_b^2(R,l^2)}^2+\frac{\alpha^0}{\delta_0}\|g_0(s)\|_{L_b^2(R,l^2)}^2\right)\left(1+\frac{\eta_0}{\gamma_0}\right),\end{aligned} \tag{3.1}$$

where $\eta_0=\min\{\alpha_0,\beta_0\}$, $\gamma_0=\min\{\lambda_0\beta_0,\alpha_0\delta_0\}$.

Proof. Taking the inner product of (2.8) with βu in l^2, by (H_1), we get

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\beta_iu_i^2+\sum_{i\in\mathbb{Z}}\beta_i\nu_i|(Bu)_i|^2+\sum_{i\in\mathbb{Z}}\lambda_i\beta_iu_i^2+\sum_{i\in\mathbb{Z}}\beta_i\alpha_iu_iv_i\le\sum_{i\in\mathbb{Z}}\beta_iu_ik_i(t). \tag{3.2}$$

Similarly, taking the inner product of (2.9) with αv in l^2, we get

$$\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\alpha_iv_i^2+\sum_{i\in\mathbb{Z}}\alpha_i\delta_iu_i^2-\sum_{i\in\mathbb{Z}}\beta_i\alpha_iu_iv_i=\sum_{i\in\mathbb{Z}}\alpha_iv_ig_i(t). \tag{3.3}$$

Note that

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\beta_iu_ik_i(t)&\le\frac{1}{2}\sum_{i\in\mathbb{Z}}\lambda_i\beta_iu_i^2+\frac{1}{2}\sum_{i\in\mathbb{Z}}\frac{\beta_i}{\lambda_i}k_i^2(t),\\\sum_{i\in\mathbb{Z}}\alpha_iv_ig_i(t)&\le\frac{1}{2}\sum_{i\in\mathbb{Z}}\alpha_i\delta_iv_i^2+\frac{1}{2}\sum_{i\in\mathbb{Z}}\frac{\alpha_i}{\delta_i}g_i^2(t).\end{aligned} \tag{3.4}$$

Summing up (3.2) and (3.3), from (3.4), we get

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\left(\beta_iu_i^2+\alpha_iv_i^2\right)+\sum_{i\in\mathbb{Z}}\left(\lambda_i\beta_iu_i^2+\alpha_i\delta_iv_i^2\right)\le\sum_{i\in\mathbb{Z}}\left(\frac{\beta_i}{\lambda_i}k_i^2(t)+\frac{\alpha_i}{\delta_i}g_i^2(t)\right). \tag{3.5}$$

Thus, by (H_3),

$$\eta_0 \frac{d}{dt}\|(u,v)(t)\|^2_{l^2\times l^2} + \gamma_0\|(u,v)(t)\|^2_{l^2\times l^2} \le \left(\frac{\beta^0}{\lambda_0}\|k(t)\|^2_{l^2} + \frac{\alpha^0}{\delta_0}\|g(t)\|^2_{l^2}\right). \tag{3.6}$$

Since $(k(t), g(t)) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$, from [12, Proposition V.4.2.], we have

$$\|k(t)\|^2_{L^2_b(R,l^2)} \le \|k_0(t)\|^2_{L^2_b(R,l^2)}, \qquad \|g(t)\|^2_{L^2_b(R,l^2)} \le \|g_0(t)\|^2_{L^2_b(R,l^2)}. \tag{3.7}$$

From (3.6)-(3.7), applying Gronwall's inequality of generalization (see [12, Lemma II.1.3]), we get (3.1). The proof is completed. □

It follows from Lemma 3.1 that the solution (u, v) of problem (2.8)–(2.10) is defined for all $t \ge \tau$. Therefore, there exists a family processes acting in the space $l^2 \times l^2$: $\{U_{(k,g)}\}$: $U_{(k,g)}(t,\tau)(u_\tau, v_\tau) = (u(t), v(t))$, $U_{(k,g)}(t,\tau) : l^2 \times l^2 \to l^2 \times l^2$, $t \ge \tau$, $\tau \in \mathbb{R}$, where $(u(t), v(t))$ is the solution of (2.8)–(2.10), and the time symbol $(k(s), g(s))$ belongs to $\mathscr{H}(k_0) \times \mathscr{H}(g_0)$ and $\mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$, respectively. The family of processes $\{U_{(k,g)}\}$ satisfies multiplicative properties:

$$\begin{gathered} U_{(k,g)}(t,s) \circ U_{(k,g)}(s,\tau) = U_{(k,g)}(t,\tau), \quad \forall t \ge s \ge \tau,\ \tau \in \mathbb{R}, \\ U_{(k,g)}(\tau,\tau) = \text{Id is the identity operator}, \quad \tau \in \mathbb{R}. \end{gathered} \tag{3.8}$$

Furthermore, the following translation identity holds:

$$U_{(k,g)}(t+h, \tau+h) = U_{T(h)(k,g)}(t,\tau), \quad \forall t \ge \tau,\ \tau \in \mathbb{R},\ h \ge 0. \tag{3.9}$$

The kernel $\mathscr{K}$ of the processes $U_{(k,g)}(t,\tau)$ consists of all bounded complete trajectories of the process $U_{(k,g)}(t,\tau)$, that is,

$$\begin{aligned} \mathscr{K}_{(k,g)} = \Big\{ (u(\cdot), v(\cdot)) \mid \|(u(t), v(t))\|_{l^2\times l^2} \le C_{(u,v)}, \\ U_{(k,g)}(t,\tau)(u(\tau), v(\tau)) = (u(t), v(t)),\ \forall t \ge \tau,\ \tau \in \mathbb{R} \Big\}. \end{aligned} \tag{3.10}$$

$\mathscr{K}(s)$ denotes the kernel section at a times moment $s \in \mathbb{R}$:

$$\mathscr{K}_{(k,g)}(s) = \left\{ (u(s), v(s)) \mid (u(\cdot), v(\cdot)) \in \mathscr{K}_{(k,g)} \right\}. \tag{3.11}$$

Lemma 3.1 also shows that the family of processes possesses a uniform absorbing set in $l^2 \times l^2$.

Lemma 3.2. *Assume that* $k_0, g_0 \in L_b^2(R, l^2)$ *and* (H_1)–(H_3) *hold. Let* $(k(s), g(s)) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$*. Then, there exists a bounded uniform absorbing set* B_0 *in* $l^2 \times l^2$ *for the family of processes* $\{U_{(k,g)}\}_{\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)}$*, that is, for any bounded set* $B \subset l^2 \times l^2$*, there exists* $t_0 = t_0(\tau, B) \geq \tau$*,*

$$\bigcup_{(k,g)\in\mathscr{H}_w(k_0)\times\mathscr{H}_w(g_0)} U_{(k,g)}(t,\tau)B \subset B_0, \quad \forall t \geq t_0. \tag{3.12}$$

Proof. Let $\|(u_\tau, v_\tau)\|_{l^2\times l^2} \leq R$, from (3.1) we have

$$\begin{aligned}\|(u,v)(t)\|^2_{l^2\times l^2} &\leq R^2 e^{-(\gamma_0/\eta_0)(t-\tau)} + \frac{1}{\eta_0}\left(\frac{\beta^0}{\lambda_0}\|k_0(s)\|^2_{L_b^2(R,l^2)} + \frac{\alpha^0}{\delta_0}\|g_0(s)\|^2_{L_b^2(R,l^2)}\right)\left(1+\frac{\eta_0}{\gamma_0}\right) \\ &\leq \frac{2}{\eta_0}\left(\frac{\beta^0}{\lambda_0}\|k_0(s)\|^2_{L_b^2(R,l^2)} + \frac{\alpha^0}{\delta_0}\|g_0(s)\|^2_{L_b^2(R,l^2)}\right)\left(1+\frac{\eta_0}{\gamma_0}\right), \quad \forall t \geq t_0,\end{aligned} \tag{3.13}$$

where

$$t_0 = \frac{\eta_0}{\gamma_0}\ \ln\ \frac{R^2}{X} + \tau, \qquad X = \frac{1}{\eta_0}\left(\frac{\beta^0}{\lambda_0}\|k_0(s)\|^2_{L_b^2(R,l^2)} + \frac{\alpha^0}{\delta_0}\|g_0(s)\|^2_{L_b^2(R,l^2)}\right)\left(1+\frac{\eta_0}{\gamma_0}\right). \tag{3.14}$$

Let $B_0 = \{(u,v)(t) \in l^2 \times l^2 \mid \|(u,v)(t)\|^2_{l^2\times l^2} \leq 2X^2\}$. The proof is completed. □

4. Uniform Attractor

In this section, we establish the existence of uniform attractor for the non-autonomous lattice systems (2.8)–(2.10). Let E be a Banach space, and let Ξ be a subset of some Banach space.

Definition 4.1. $\{U_\sigma(t,\tau)\}$, $\sigma \in \Xi$ is said to be $(E\times\Xi, E)$ weakly continuous, if for any $t \geq \tau$, $\tau \in \mathbb{R}$, the mapping $(u,\sigma) \to \{U_\sigma(t,\tau)u$ is weakly continuous from $E \times \Xi$ to E.

A family of processes $U_\sigma(t,\tau)$, $\sigma \in \Xi$ is said to be *uniformly* (w.r.t. $\sigma \in \Xi$) *ω-limit compact* if for any $\tau \in \mathbb{R}$ and bounded set $B \subset E$, the set $\bigcup_{\sigma\in\Xi}\bigcup_{s\geq t} U_\sigma(s,\tau)B$ is bounded for every t and $\bigcup_{\sigma\in\Xi}\bigcup_{s\geq t} U_\sigma(s,t)B$ is precompact set as $t \to +\infty$. We need the following result in [14].

Theorem 4.2. *Let* Ξ *be the weak closure of* Ξ_0*. Assume that* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Xi$ *is* $(E \times \Xi, E)$ *weakly continuous, and*

(i) *has a bounded uniformly (w.r.t.* $\sigma \in \Xi$*) absorbing set* B_0*,*

(ii) *is uniformly (w.r.t.* $\sigma \in \Xi$*) ω-limit compact.*

Then the families of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Xi_0$, $\sigma \in \Xi$ possess, respectively, compact uniform (w.r.t. $\sigma \in \Xi_0$, $\sigma \in \Xi$, resp.) attractors $\mathcal{A}_{\Xi_0}$ and $\mathcal{A}_{\Xi}$ satisfying

$$\mathcal{A}_{\Xi_0} = \mathcal{A}_{\Xi} = \omega_{0,\Xi}(B_0) = \bigcup_{\sigma\in\Xi} \mathcal{K}_\sigma(0). \tag{4.1}$$

Furthermore, $\mathcal{K}_\sigma(0)$ is nonempty for all $\sigma \in \Xi$.

Let $\mathcal{E}$ be a Banach space and $p \geq 1$, denote the space $L^p_{\text{loc}}(R,\mathcal{E})$ of functions $\rho(s)$, $s \in \mathbb{R}$ with values in $\mathcal{E}$ that are locally p-power integral in the Bochner sense, it is equipped with the local p-power mean convergence topology. Recall the Propositions in [12].

Proposition 4.3. *A set $\Sigma \subset L^p_{\text{loc}}(R,\mathcal{E})$ is precompact in $L^p_{\text{loc}}(R,\mathcal{E})$ if and only if the set $\Sigma_{[t_1,t_2]}$ is precompact in $L^p_{\text{loc}}([t_1,t_2],\mathcal{E})$ for every segment $[t_1,t_2] \subset \mathbb{R}$. Here, $\Sigma_{[t_1,t_2]}$ denotes the restriction of the set Σ to the segment $[t_1,t_2]$.*

Proposition 4.4. *A function $\sigma(s)$ is tr.c. in $L^p_{\text{loc}}(R,\mathcal{E})$ if and only if*

(i) *for any $h \in R$ the set $\{\int_t^{t+h}\sigma(s)ds \mid t \in \mathbb{R}\}$ is precompact in $\mathcal{E}$;*

(ii) *there exists a function $\alpha(s)$, $\alpha(s) \to 0^+ (s \to 0^+)$ such that*

$$\int_t^{t+1} \|\sigma(s) - \sigma(s+l)\|^p_{\mathcal{E}} ds \leq \alpha(|l|), \quad \forall t \in \mathbb{R}. \tag{4.2}$$

Now, one introduces a class of function.

Definition 4.5. A function $\varphi \in L^2_{loc}(R,l^2)$ is said to be locally asymptotic smallness if for any $\epsilon > 0$, there exists positive integer N such that

$$\sup_{t\in R}\int_t^{t+1} \sum_{|i|\geq N} \varphi_i^2(s)ds < \epsilon. \tag{4.3}$$

Denote by $L^2_{\text{las}}(R,l^2)$ the set of all locally asymptotic smallness functions in $L^2_{\text{loc}}(R,l^2)$. It is easy to see that $L^2_{\text{las}}(R,l^2) \subset L^2_b(R,l^2)$. The next examples show that there exist functions in $L^2_b(R,l^2)$ but not in $L^2_{\text{las}}(R,l^2)$, and a function belongs to $L^2_{\text{las}}(R,l^2)$ is not necessary a tr.c. function in $L^2_{\text{loc}}(R,l^2)$.

Example 4.6. Let $\phi(t) = (\phi_i(t))_{i\in\mathbb{Z}}$,

$$\phi_i(t) = \begin{cases} 0, & i \leq 0, \\ \begin{cases} \sqrt{2i} + 4i\sqrt{2i}(t-2i), & 2i - \dfrac{1}{4i} \leq t \leq 2i, \\ \sqrt{2i}, & 2i \leq t \leq 2i + \dfrac{1}{2i}, \\ \sqrt{2i} - 4i\sqrt{2i}\left(t - 2i - \dfrac{1}{2i}\right), & 2i + \dfrac{1}{2i} \leq t \leq 2i + \dfrac{3}{4i}, \\ 0, & \text{otherwise}. \end{cases} & i \geq 1. \end{cases} \tag{4.4}$$

For every $t \in [2i - 1/4i, 2i + 3/4i]$, $i \geq 1$,

$$\begin{aligned}
\int_t^{t+1} \sum_{i\in\mathbb{Z}} |\phi_i(s)|^2 ds &\leq \int_{2i-(1/4i)}^{2i} \left[\sqrt{2i} + 4i\sqrt{2i}(s-2i)\right]^2 ds \\
&\quad + \int_{2i}^{2i+(1/2i)} 2i\, ds + \int_{2i+1/2i}^{2i+3/4i} \left[\sqrt{2i} - 4i\sqrt{2i}\left(s - 2i - \frac{1}{2i}\right)\right]^2 ds \\
&\leq 2i \times \frac{1}{4i} + 2i \times \frac{1}{2i} + 2i \times \frac{1}{4i} \\
&= 2 < +\infty.
\end{aligned} \tag{4.5}$$

Thus,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \sum_{i\in\mathbb{Z}} |\phi_i(s)|^2 ds \leq 2, \tag{4.6}$$

and $\phi(t) \in L_b^2(R, l^2)$. However, for every positive integer N, and for any positive $i \geq N$,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \sum_{|i|\geq N} |\phi_i(s)|^2 ds \geq \int_{2i}^{2i+1/2i} 2i\, ds = 1. \tag{4.7}$$

Therefore, $\phi(t) \notin L_{\text{las}}^2(R, l^2)$.

Example 4.7. $\varphi(t) = (\varphi_i(t))_{i\in\mathbb{Z}}$,

$$\varphi_i(t) = 0, \quad \text{for } i \leq 0,$$

$$\varphi_1(t) = \begin{cases}
\sqrt{2k} + 4(2k)^2\sqrt{2k}\left(t - 2k - \dfrac{j}{2k}\right), & 2k + \dfrac{j}{2k} - \dfrac{1}{4(2k)^2} \leq t \leq 2k + \dfrac{j}{2k}, \\
\sqrt{2k}, & 2k + \dfrac{j}{2k} \leq t \leq 2k + \dfrac{j}{2k} + \dfrac{1}{(2k)^2}, \\
\sqrt{2k} - 4(2k)^2\sqrt{2k}\left(t - 2k - \dfrac{j}{2k} - \dfrac{1}{(2k)^2}\right), & 2k + \dfrac{j}{2k} + \dfrac{1}{(2k)^2} \leq t \leq 2k + \dfrac{j}{2k} + \dfrac{5}{4(2k)^2}, \\
& j = 0, 1, 2, \ldots, 2k-1,\ k \in \mathbb{Z}^+, \\
0, & \text{otherwise}.
\end{cases} \tag{4.8}$$

for $i \geq 2$,

$$\varphi_i(t) = \begin{cases} \sqrt{2i} + (2i)^2\sqrt{2i}(t-2i), & 2i - \dfrac{1}{(2i)^2} \leq t \leq 2i, \\ \sqrt{2i}, & 2i \leq t \leq 2i + \dfrac{1}{(2i)^2}, \\ \sqrt{2i} - (2i)^2\sqrt{2i}\left(t - 2i - \dfrac{1}{(2i)^2}\right), & 2i + \dfrac{1}{(2i)^2} \leq t \leq 2i + \dfrac{2}{(2i)^2}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.9}$$

Here, $\mathbb{Z}^+$ denote the positive integer set.

For every positive integer $N > 1$, $i \geq N$, and for $t \in [2i - (1/(2i)^2), 2i + (2/(2i)^2)]$,

$$\begin{aligned} \int_t^{t+1} \sum_{|i|\geq N} |\varphi_i(s)|^2 ds &\leq \int_{2i-1/(2i)^2}^{2i} \left[\sqrt{2i} + (2i)^2\sqrt{2i}(s-2i)\right]^2 ds \\ &\quad + \int_{2i}^{2i+1/(2i)^2} 2i\, ds + \int_{2i+1/(2i)^2}^{2i+2/(2i)^2} \\ &\quad \times \left[\sqrt{2i} - (2i)^2\sqrt{2i}\left(s - 2i - \frac{1}{(2i)^2}\right)\right]^2 ds \\ &\leq 2i \times \frac{1}{(2i)^2} + 2i \times \frac{1}{(2i)^2} + 2i \times \frac{1}{(2i)^2} \\ &= \frac{3}{2i}, \end{aligned} \tag{4.10}$$

which implies that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \sum_{|i|\geq N} |\varphi_i(s)|^2 ds \leq \frac{3}{2N}. \tag{4.11}$$

Therefore, $\varphi(t) = (\varphi_i(t))_{i\in\mathbb{Z}} \in L^2_{\text{las}}(R, l^2)$. Note that for any $1/(2i)^2 \leq l < 1/2k - (1/(2i)^2)(k > 2)$,

$$\begin{aligned} \int_0^1 \sum_{i\in\mathbb{Z}} |\varphi_i(s+2k) - \varphi_i(s+2k+l)|^2 &\geq \int_0^1 \sum_{i\in\mathbb{Z}} |\varphi_1(s+2k) - \varphi_1(s+2k+l)|^2 \\ &\geq 1. \end{aligned} \tag{4.12}$$

From Proposition 4.4, $\varphi(t) = (\varphi_i(t))_{i\in Z}$ is not translation compact in $L^2_{\text{loc}}(R, l^2)$.

Remark 4.8. Example 4.7 shows that a locally asymptotic function is not necessary translation compact in $C_b(R, l^2)$.

In the following, we give some properties of locally asymptotic smallness function.

Lemma 4.9. *$L^2_{las}(R,l^2)$ is a closed subspace of $L^2_b(R,l^2)$.*

Proof. Let $\{\psi_n\}_{n=1}^{\infty} \subset L^2_{\text{las}}(R,l^2)$ such that

$$\psi_n \longrightarrow \psi \quad \text{in } L^2_b\left(R,l^2\right). \tag{4.13}$$

Then, for any $\epsilon > 0$, there exists positive integer N_1 such that for every $n \geq N_1$,

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} \left\|\psi_n(s) - \psi(s)\right\|^2_{l^2} < \epsilon. \tag{4.14}$$

Since $\psi_n \in L^2_{\text{las}}(R,l^2)$, there exist $N_2 > 0$ such that for all $n \in \mathbb{Z}^+$,

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} \sum_{|i|\geq N_2} \psi^2_{ni}(s)ds < \epsilon. \tag{4.15}$$

Let $n > N_1$, we get that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\int_t^{t+1} \sum_{|i|\geq N_2} \psi^2_i(s)ds \\
&\quad\leq 2\left(\sup_{t\in\mathbb{R}}\int_t^{t+1} \sum_{|i|\geq N_2} \left|\psi_{ni}(s) - \psi_i(s)\right|^2 ds + \sup_{t\in\mathbb{R}}\int_t^{t+1} \sum_{|i|\geq N_2} \psi^2_{ni}(s)ds\right) \\
&\quad< 4\epsilon.
\end{aligned} \tag{4.16}$$

Therefore, $\psi(s) \in L^2_{\text{las}}(R,l^2)$. This completes the proof. □

Lemma 4.10. *Every translation compact function $w(s)$ in $L^2_{\text{loc}}(R,l^2)$ is locally asymptotic smallness.*

Proof. Since $w(s)$ is tr.c. in $L^2_{\text{loc}}(R,l^2)$, we get that $\{w(s+t) \mid t \in R\}$ is precompact in $L^2_{\text{loc}}(R,l^2)$. By Proposition 4.3, we get that $\{w(s+t) \mid t \in R\}_{[0,1]}$ is precompact in $L([0,1];l^2)$. Thus, for any $\epsilon > 0$, there exists finite number $w_1(s), w_2(s), \ldots, w_K(s) \in L([0,1];l^2)$ such that for every $w \in \{w(s+t) \mid t \in \mathbb{R}\}_{[0,1]}$, there exist some $w_j(s)$, $1 \leq j \leq K$, such that

$$\int_0^1 \left\|w(s+t) - w_j(s)\right\|^2_{l^2} < \epsilon, \quad t \in \mathbb{R}. \tag{4.17}$$

For the ϵ given above, $w_j(s) \in L([0,1];l^2)$ implies that there exists positive integer N such that

$$\int_0^1 \sum_{|i|\geq N} \left|w_j(s)\right|^2 ds < \epsilon. \tag{4.18}$$

Therefore,

$$\begin{aligned}\int_0^1 \sum_{|i|\geq N} |w_i(s+t)|^2 ds &\leq 2\int_0^1 \sum_{|i|\geq N} \left|w_i(s+t) - w_{ji}(s)\right|^2 ds + 2\int_0^1 \sum_{|i|\geq N} \left|w_{ji}(s)\right|^2 ds \\ &\leq 4\epsilon,\end{aligned} \tag{4.19}$$

which implies $w(s)$ is locally asymptotic smallness. This completes the proof. □

We now establish the uniform estimates on the tails of solutions of (2.8)–(2.10) as $n \to \infty$.

Lemma 4.11. *Assume that* (H_1)–(H_3) *hold and* $(k_0, g_0) \in L^2_{\text{loc}}(R, l^2) \times L^2_{\text{loc}}(R, l^2)$ *is locally asymptotic smallness. Then for any* $\epsilon > 0$*, there exist positive integer* $N(\epsilon)$ *and* $T(\epsilon, R)$ *such that if* $\|(u(\tau), v(\tau))\|_{l^2\times l^2} \leq R$, $(u(t), v(t)) = U_{(k,g)}(t,\tau)(u(\tau), v(\tau))$, $(k, g) \in \mathscr{H}_w(k_0) \times \mathscr{H}_w(g_0)$ *satisfies*

$$\sum_{|i|\geq N} \left(|u_i(t)|^2 + |v_i(t)|^2\right) < \epsilon. \tag{4.20}$$

Proof. Choose a smooth function θ such that $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\begin{aligned}\theta(s) &= 0 \quad \text{for } 0 \leq s \leq 1, \\ \theta(s) &= 1 \quad \text{for } s \geq 2,\end{aligned} \tag{4.21}$$

and there exists a constant M_0 such that $|\theta'(s)| \leq M_0$ for $s \in \mathbb{R}^+$. Let N be a suitable large positive integer, $(\phi, \varphi) = (\theta(|i|/N)u_i, \theta(|i|/N)v_i)_{i\in\mathbb{Z}}$. Taking the inner product of (2.8) with $\beta\phi$ and (2.9) with $\alpha\varphi$ in l^2, we have

$$\begin{aligned}\langle \dot{u}, \beta\phi \rangle + \langle \nu A u, \beta\phi \rangle + \langle \lambda u, \beta\phi \rangle + \langle f(u, Bu), \beta\phi \rangle + \langle \alpha v, \beta\phi \rangle &= \langle k(t), \beta\phi \rangle, \\ \langle \dot{v}, \alpha\varphi \rangle + \langle \delta v, \alpha\varphi \rangle - \langle \beta u, \alpha\varphi \rangle &= \langle g(t), \alpha\varphi \rangle.\end{aligned} \tag{4.22}$$

From (H_1)–(H_3), we have

$$\langle \dot{u}, \beta\phi \rangle + \langle \dot{v}, \alpha\varphi \rangle \geq \frac{\eta_0}{2} \frac{d}{dt} \sum_{i\in\mathbb{Z}} \left(u_i^2 + v_i^2\right) \theta\left(\frac{|i|}{N}\right), \tag{4.23}$$

where η_0 is same as in Lemma 3.1

$$
\begin{aligned}
\langle \nu A u, \beta\phi\rangle &= \sum_{i\in\mathbb{Z}} \nu_i \beta_i (Bu)_i (B\phi)_i \\
&= \sum_{i\in\mathbb{Z}} \nu_i \beta_i (Bu)_i \left(\theta\left(\frac{|i+1|}{N}\right)u_{i+1} - \theta\left(\frac{|i|}{N}\right)u_i\right) \\
&\geq \sum_{i\in\mathbb{Z}} \nu_i \beta_i (Bu_i)^2 \theta\left(\frac{|i|}{N}\right) - \sum_{i\in\mathbb{Z}} \nu_i \beta_i \left|(Bu)_i \left(\theta\left(\frac{|i+1|}{N}\right) - \theta\left(\frac{|i|}{N}\right)\right)u_{i+1}\right| \\
&\geq \sum_{i\in\mathbb{Z}} \nu_i \beta_i (Bu_i)^2 \theta\left(\frac{|i|}{N}\right) - \frac{4\nu^0\beta^0 M_0 X^2}{N}, \quad \forall t \geq t_0,
\end{aligned}
\tag{4.24}
$$

t_0 as in (3.10)

$$
\begin{gathered}
\langle \lambda u, \beta\phi\rangle \geq \lambda_0\beta_0 \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)u_i^2, \\
\langle \alpha v, \beta\phi\rangle = \langle \beta u, \alpha\psi\rangle, \\
\langle \delta v, \alpha\psi\rangle \geq \delta_0\alpha_0 \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)v_i^2, \\
\langle k(t), \beta u\rangle \leq \beta^0 \sum_{i\in\mathbb{Z}} k_i(t)\theta\left(\frac{|i|}{N}\right)u_i \leq \frac{1}{2}\lambda_0\beta_0 \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)u_i^2 + \frac{1}{2}\frac{\beta^{0^2}}{\lambda_0\beta_0} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)k_i^2(t), \\
\langle g(t), \alpha\psi\rangle \leq \frac{1}{2}\delta_0\alpha_0 \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)v_i^2 + \frac{1}{2}\frac{\alpha^{0^2}}{\delta_0\alpha_0} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)g_i^2(t).
\end{gathered}
\tag{4.25}
$$

Summing up (4.22), from (4.23)–(4.25) we get

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}} \left(u_i^2 + v_i^2\right)\theta\left(\frac{|i|}{N}\right) + \frac{\gamma_0}{\eta_0}\sum_{i\in\mathbb{Z}} \left(u_i^2 + v_i^2\right)\theta\left(\frac{|i|}{N}\right) \\
&\quad \leq \frac{\beta^{0^2}}{\lambda_0\beta_0}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)k_i^2(t) + \frac{\alpha^{0^2}}{\delta_0\alpha_0}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)g_i^2(t) + \frac{4\nu^0\beta^0 M_0 X^2}{N}, \quad \forall t \geq t_0.
\end{aligned}
\tag{4.26}
$$

Thus,

$$
\begin{aligned}
\sum_{i\in\mathbb{Z}} \left(u_i^2 + v_i^2\right)\theta\left(\frac{|i|}{N}\right) &\leq \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)\left(u_i^2(\tau) + v_i^2(\tau)\right)e^{-(\gamma_0/\eta_0)(t-\tau)} \\
&\quad + \frac{\eta_0}{\gamma_0}\cdot\frac{4\nu^0\beta^0 M_0 X^2}{N} + \int_\tau^t \frac{\beta^{0^2}}{\lambda_0\beta_0} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)k_i^2(s)ds \\
&\quad + \int_\tau^t \frac{\alpha^{0^2}}{\delta_0\alpha_0} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right)g_i^2(s)ds.
\end{aligned}
\tag{4.27}
$$

We now estimate the integral term on the right-hand side of (4.27).

$$\begin{aligned}
\int_{\tau}^{t} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s)ds &\le \int_{t-1}^{t} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s)ds \\
&\quad + \int_{t-2}^{t-1} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s)ds + \cdots \\
&\le e^{-\gamma_0/\eta_0} \int_{0}^{1} e^{-(\gamma_0/\eta_0)s} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s+t-1)ds \\
&\quad + e^{-2\gamma_0/\eta_0} \int_{0}^{1} e^{-(\gamma_0/\eta_0)s} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s+t-2)ds + \cdots \\
&\le \left(1 + e^{-\gamma_0/\eta_0} + e^{-2\gamma_0/\eta_0} + \cdots\right) \sup_{t\in\mathbb{R}} \int_{0}^{1} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s+t)ds \\
&\le \frac{1}{1 - e^{(\gamma_0/\eta_0)}} \sup_{t\in\mathbb{R}} \int_{0}^{1} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_i^2(s+t)ds.
\end{aligned} \tag{4.28}$$

Similarly,

$$\int_{\tau}^{t} e^{-(\gamma_0/\eta_0)(t-s)} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) g_i^2(s)ds \le \frac{1}{1 - e^{\gamma_0/\eta_0}} \sup_{t\in\mathbb{R}} \int_{0}^{1} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) g_i^2(s+t)ds. \tag{4.29}$$

Since $(k_0(t), g_0(t))$ is locally asymptotic smallness, from (4.27)–(4.29) we get that for any $\epsilon > 0$, if $\|(u(\tau), v(\tau))\|_{l^2\times l^2}^2 \le R$, there exist $T = T(\epsilon, R) \ge \tau$ and sufficient large positive integer N such that

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} \left(u_i^2 + v_i^2\right) \theta\left(\frac{|i|}{N}\right) &\le 2\left(\frac{\eta_0}{\gamma_0} \cdot \frac{4\nu^0 \beta^0 M_0 X^2}{N} + \frac{\beta^{0^2}}{\lambda_0 \beta_0} \frac{1}{1 - e^{(\gamma_0/\eta_0)}} \sup_{t\in\mathbb{R}} \int_{0}^{1} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) k_{0i}^2(s+t)ds\right. \\
&\quad \left. + \frac{\alpha^{0^2}}{\delta_0 \alpha_0} \frac{1}{1 - e^{(\gamma_0/\eta_0)}} \sup_{t\in\mathbb{R}} \int_{0}^{1} \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{N}\right) g_{0i}^2(s+t)ds\right) < \epsilon, \quad \forall t \ge T.
\end{aligned} \tag{4.30}$$

The proof is completed. □

Lemma 4.12. *Assume that* (H_1)*–*(H_3) *hold, let* (u_{n0}, v_{n0}), $(u_0, v_0) \in l^2 \times l^2$. *If* $(u_{n0}, v_{n0}) \to (u_0, v_0)$ *in* $l^2 \times l^2$ *and* $(k_n, g_n) \rightharpoonup (k, g)$ *weakly in* $L^2_{\text{loc}}(R, l^2 \times l^2)$, *then for any* $t \ge \tau$, $\tau \in \mathbb{R}$,

$$U_{(k_n,g_n)}(u_{n0}, v_{n0}) \rightharpoonup U_{(k,g)}(u_0, v_0) \text{ weakly in } l^2 \times l^2, \quad n \longrightarrow \infty. \tag{4.31}$$

Proof. Let $(u_n, v_n)(t) = U_{(k_n,g_n)}(u_{n0}, v_{n0})$, $(u,v)(t) = U_{(k,g)}(u_0, v_0)$. Since $\{(u_{n0}, v_{n0})\}$ is bounded in $l^2 \times l^2$, by Lemma 3.2, we get that

$$\{(u_n, v_n)(t)\} \text{ is uniformly bounded in } l^2 \times l^2. \tag{4.32}$$

Therefore, for all $t \geq \tau$, $\tau \in \mathbb{R}$,

$$(u_n, v_n)(t) \rightharpoonup (u_w, v_w)(t) \text{ weakly in } l^2 \times l^2, \quad \text{as } n \longrightarrow \infty. \tag{4.33}$$

Note that $(u_n, v_n)(t)$ is the solution of (2.8) and (2.9) with time symbol $(k_n, g_n) \in L^{2,w}_{\text{loc}}(R, l^2 \times l^2)$, it follow from (4.32) that

$$(\dot{u}_n, \dot{v}_n)(t) \rightharpoonup (\dot{u}_w, \dot{v}_w)(t) \text{ weak starin } L^{\infty}\left(R, l^2 \times l^2\right), \quad \text{as } n \longrightarrow \infty. \tag{4.34}$$

In the following, we show that $(u_w, v_w)(t) = (u,v)(t)$. By the fact that $(u_n, v_n)(t)$ is the solution of (2.8) and (2.9), for any $\psi(t) \in C_0^{\infty}([\tau, t], l^2)$, we get that

$$\begin{gathered}
\int_{\tau}^{t} \dot{u}_{ni}\psi(t)dt + \int_{\tau}^{t} \nu_i (Au_n)_i \psi(t)dt + \int_{\tau}^{t} \lambda_i u_{ni}\psi(t)dt + \int_{\tau}^{t} f_i(u_{ni}, (Bu_n)_i)\psi(t)dt + \int_{\tau}^{t} \alpha_i v_{ni}\psi(t)dt \\
= \int_{\tau}^{t} k_{ni}(t)\psi(t)dt, \quad t \geq \tau, \\
\int_{\tau}^{t} \dot{v}_{ni}\psi(t)dt + \int_{\tau}^{t} \delta_i v_{ni}\psi(t)dt - \int_{\tau}^{t} \beta_i u_{ni}\psi(t)dt = \int_{\tau}^{t} g_{ni}(t)\psi(t)dt, \quad t \geq \tau.
\end{gathered} \tag{4.35}$$

Note that $(k_n, g_n) \rightharpoonup (k, g)$ weakly in $L^2_{\text{loc}}(R, l^2 \times l^2)$. Let $n \to \infty$ in (4.35), by (4.34) we get that $(u_w, v_w)(t)$ is the solution of (2.8) and (2.9) with the initial data (u_0, v_0). By the unique solvability of problem (2.8)–(2.10), we get that $(u_w, v_w)(t) = (u,v)(t)$. This completes the proof. □

Proof of Theorem A. From Lemmas 3.2, 4.11 and 4.12, and Theorem 4.2, we get the results. □

5. Upper Semicontinuity of Attractors

In this section, we present the approximation to the uniform attractor $\mathcal{A}_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)}$ obtained in Theory A by the uniform attractor of following finite-dimensional lattice systems in $R^{2n+1} \times R^{2n+1}$:

$$\begin{gathered}
\dot{u}_i + \nu_i (Au)_i + \lambda_i u_i + f_i(u_i, (Bu)_i) + \alpha_i v_i = k_i(t), \quad i \in Z_n, \ t > \tau, \\
\dot{v}_i + \delta_i v_i - \beta_i u_i = g_i(t), \quad i \in Z_n, \ t > \tau,
\end{gathered} \tag{5.1}$$

with the initial data

$$u(\tau) = (u_i(\tau))_{|i|\le n} = (u_{i,\tau})_{|i|\le n}, \quad v(\tau) = (v_i(\tau))_{|i|\le n} = (v_{i,\tau})_{|i|\le n}, \quad \tau \in \mathbb{R}, \tag{5.2}$$

and the periodic boundary conditions

$$(u_{n+1}, v_{n+1}) = (u_{-n}, v_{-n}), \qquad (u_{-n-1}, v_{n+1}) = (u_n, v_n). \tag{5.3}$$

Similar to systems (2.8)–(2.10), under the assumption (H_1)–(H_3), the approximation systems (5.1)–(5.2) with $k, g \in L_b^2(R, l^2)$ possess a unique solution $(u, v) = (u_i, v_i)_{|i|\le n} \in C([\tau, +\infty), R^{2n+1} \times R^{2n+1})$, which continuously depends on initial data. Therefore, we can associate a family of processes $\{U^n_{(k,g)}(t, \tau)\}_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)}$ which satisfy similar properties (3.8)–(3.9). Similar to Lemma 3.2, we have the following result.

Lemma 5.1. *Assume that $k_0, g_0 \in L_b^2(R, l^2)$, and (H_1)–(H_3) hold. Let $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$. Then, there exists a bounded uniform absorbing set B_1 for the family of processes $\{U^n_{(k,g)}(t, \tau)\}_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)}$, that is, for any bounded set $B_n \subset R^{2n+1} \times R^{2n+1}$, there exists $t_0 = t_0(\tau, B_n) \ge \tau$, for $t \ge t_0$,*

$$\bigcup_{(k,g)\in\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)} U^n_{(k,g)}(t, \tau) B_n \subset B_1. \tag{5.4}$$

In particular, B_1 is independent of (k, g) and n.

Since (5.1) is finite-dimensional systems, it is easy to know that under the assumption of Lemma 5.1, the family of processes $\{U^n_{(k,g)}(t, \tau)\}$, $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$ is uniformly (w.r.t. $\mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$) ω-limit compact. Similar to Lemma 4.12, if $(u^n_{m0}, v^n_{m0}) \to (u^n_0, v^n_0)$ in $l^2 \times l^2$, $(k_m, g_m) \rightharpoonup (k, g)$ weakly in $L^2_{\text{loc}}(R, R^{2n+1} \times R^{2n+1})$, then for any $t \ge \tau$, $\tau \in \mathbb{R}$,

$$U^n_{(k_m,g_m)}(u^n_{m0}, v^n_{m0}) \rightharpoonup U^n_{(k,g)}(u^n_0, v^n_0) \text{ weakly in } l^2 \times l^2, \quad m \longrightarrow \infty. \tag{5.5}$$

Lemma 5.2. *Assume that $(k_0(s), g_0(s)) \in L_b^2(R, l^2) \times L_b^2(R, l^2)$ and (H_1)–(H_3) hold. Then the process $\{U^n_{(k_0,g_0)}\}$ corresponding to problems (5.1)-(5.2) with external term $(k_0(s), g_0(s))$ possesses compact uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathcal{A}^n_0$ in $l^2 \times l^2$ which coincides with uniform (w.r.t. $(k(s), g(s)) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$) attractor $\mathcal{A}^n_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)}$ for the family of processes $\{U^n_{(k,g)}(t, \tau)\}$, $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$, that is,*

$$\mathcal{A}^n_0 = \mathcal{A}^n_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)} = \omega_{0,\mathcal{A}^n_{\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)}}(B_1) = \bigcup_{(k,g)\in\mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)} \mathcal{K}_{n,(k,g)}(0), \tag{5.6}$$

where B_1 is the uniform (w.r.t. $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$) absorbing set in $R^{2n+1} \times R^{2n+1}$, and $\mathcal{K}_{(k,g)}$ is kernel of the process $\{U_{(k,g)}(t, \tau)\}$. The uniform attractor uniformly (w.r.t. $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$) attracts the bounded set in $R^{2n+1} \times R^{2n+1}$.

Proof of Theorem B. If $(u_0^n, v_0^n) \in \mathcal{A}_0^n$, it follows from Lemma 5.2 that there exist $(k^n, g^n) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$ and a bounded complete solution $(u^n(\cdot), v^n(\cdot)) \in C(R, R^{2n+1} \times R^{2n+1})$ such that

$$\begin{gathered} (u^n(t), v^n(t)) = U^n_{(k^n,g^n)}(t,0)(u_0^n, v_0^n), \qquad (u_0^n, v_0^n) = (u^n(0), v^n(0)), \\ (u^n(t), v^n(t)) \in \mathcal{A}_0^n, \quad \forall t \in \mathbb{R},\ n = 1, 2, \ldots. \end{gathered} \tag{5.7}$$

Since $(k^n, g^n) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$, there exist $(k, g) \in \mathcal{H}_w(k_0) \times \mathcal{H}_w(g_0)$ and a subsequence of $\{(k^n, g^n)\}_{n=1}^{\infty}$, which is still denote by $\{(k^n, g^n)\}_{n=1}^{\infty}$, such that

$$(k^n, g^n) \rightharpoonup (k, g) \text{ weakly in } L^2_{\text{loc}}\left(R, l^2 \times l^2\right), \quad \text{as } n \longrightarrow \infty. \tag{5.8}$$

From Lemma 5.1, we get that

$$\|(u^n, v^n)(t)\|_{l^2 \times l^2} \le C_1 \quad \forall t \in \mathbb{R},\ n = 1, 2, \ldots, \tag{5.9}$$

which imply that

$$\|\dot{u}^n(t)\| \le C_2, \qquad \|\dot{v}^n(t)\| \le C_2. \tag{5.10}$$

Thus,

$$\begin{gathered} \|u^n(t) - u^n(s)\| \le \|\dot{u}^n\||t - s| \le C_2|t - s|, \\ \|v^n(t) - v^n(s)\| \le \|\dot{v}^n\||t - s| \le C_2|t - s|. \end{gathered} \tag{5.11}$$

Let $I_j (j = 1, 2, \ldots)$ be a sequence of compact intervals of R such that $I_j \subset I_{j+1}$ and $\bigcup_j I_j = R$. From (5.9) and (5.11), using Ascoli's theorem, we get that for each $t \in I_j$, there exists a subsequence of $\{(u^n, v^n)(t)\}$ (still denoted by $\{(u^n, v^n)(t)\}$) and $(u_t, v_t) \in l^2 \times l^2$ such that

$$(u^n, v^n)(t) \rightharpoonup (u_t, v_t)(t) \text{ weakly in } l^2 \times l^2, \quad \text{as } n \longrightarrow \infty. \tag{5.12}$$

Proceeding as in the proof of Lemma 4.11, we get that the weak convergence is actually strong convergence, and therefore $\{(u^n, v^n)(t)\}$ is precompact in $C(I_j, l^2 \times l^2)$ for each $j = 1, 2, \ldots$. Then we infer that there exists a subsequence $\{(u^{n_1}, v^{n_1})(t)\}$ of $\{(u^n, v^n)(t)\}$ and $(u_1, v_1)(t)$ such that $\{(u^{n_1}, v^{n_1})(t)\}$ converges to $(u_1, v_1)(t) \in C(I_1, l^2 \times l^2)$. Using Ascoli's theorem again, we get, by induction, that there is a subsequence $\{(u^{n_{j+1}}, v^{n_{j+1}})(t)\}$ of $\{(u^{n_j}, v^{n_j})(t)\}$ such that $\{(u^{n_{j+1}}, v^{n_{j+1}})(t)\}$ converges to $(u_{j+1}, v_{j+1})(t)$ in $C(I_{j+1}, l^2 \times l^2)$, where $(u_{j+1}, v_{j+1})(t)$ is an extension of $(u_j, v_j)(t)$ to I_{j+1}. Finally, taking a diagonal subsequence in the usual way, we find that there exist a subsequence $\{(u^{n_n}, v^{n_n})(t)\}$ of $\{(u^n, v^n)(t)\}$ and $(u, v)(t) \in C(R, l^2 \times l^2)$ such that for any compact interval $I \subset \mathbb{R}$

$$(u^{n_n}, v^{n_n})(t) \longrightarrow (u, v)(t) \in C\left(I, l^2 \times l^2\right), \quad \text{as } n \longrightarrow \infty. \tag{5.13}$$

From (5.9) we get that

$$\|(u,v)(t)\|_{l^2\times l^2} \le C_1 \quad \forall t \in \mathbb{R}. \tag{5.14}$$

Next, we show that $(u(t),v(t))$ is the solution of (2.8)–(2.10). It follows from (5.10) that

$$(\dot{u}^n(t),\dot{v}^n(t)) \rightharpoonup (\dot{u}(t),\dot{v}(t)) \text{ weak star in } L^\infty\left(R,l^2\times l^2\right), \quad \text{as } n \longrightarrow \infty. \tag{5.15}$$

For fixed $i \in \mathbb{Z}$, let $n > |i|$. Since $(u^n(\cdot),v^n(\cdot))$ is the solution of (5.1)–(5.2) with $(k^n,g^n) \in \mathcal{H}_w(k_0)\times\mathcal{H}_w(g_0)$, we have

$$\begin{gathered}\dot{u}_i^n(t) = -\nu_i(Au^n)_i - \lambda_i u_i^n - f_i\left(u_i^n,(Bu^n)_i\right) - \alpha_i v_i^n + k_i^n(t), \quad t\in\mathbb{R},\\ \dot{v}_i^n(t) = -\delta_i v_i^n + \beta_i u_i^n + g_i(t), \quad t\in\mathbb{R}.\end{gathered} \tag{5.16}$$

Thus, for each $\psi(t) \in C_0^\infty(I,l^2)$, we have

$$\begin{aligned}\int_I \dot{u}_i^n(t)\psi(t)dt &= -\int_I \nu_i(Au^n)_i\psi(t)dt - \int_I \lambda_i u_i^n\psi(t)dt - \int_I f_i\left(u_i^n,(Bu^n)_i\right)\psi(t)dt\\ &\quad - \int_I \alpha_i v_i^n\psi(t)dt + \int_I k_i^n(t)\psi(t)dt, \quad t\in\mathbb{R},\\ \int_I \dot{v}_i^n(t)\psi(t)dt &= -\int_I \delta_i v_i^n\psi(t)dt + \int_I \beta_i u_i^n\psi(t)dt + \int_I g_i^n(t)\psi(t)dt, \quad t\in\mathbb{R}.\end{aligned} \tag{5.17}$$

Letting $n \to \infty$, by (5.8), (5.13), (5.15) and (5.17) we find that (u,v) satisfies

$$\begin{gathered}\dot{u}_i(t) = -\nu_i(Au)_i - \lambda_i u_i - f_i\left(u_i^n,(Bu)_i\right) - \alpha_i v_i + k_i(t), \quad \forall t\in I,\ i\in\mathbb{Z},\\ \dot{v}_i(t) = -\delta_i v_i + \beta_i u_i + g_i(t), \quad \forall t\in I,\ i\in\mathbb{Z}.\end{gathered} \tag{5.18}$$

Since I is arbitrary, we note that (5.18) are valid for all $t \in \mathbb{R}$. From (5.14) we find that (u,v) is a bounded complete solution of (2.8)–(2.10). Therefore, $(u(0),v(0)) \in \mathcal{A}_0$. By (5.13) we get that

$$(u^{n_n}(0),v^{n_n}(0)) \longrightarrow (u(0),v(0)) \in \mathcal{A}_0. \tag{5.19}$$

The proof is complete. □

Remark 5.3. All the result of this paper is valid for the systems in [20, 21].

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