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Research Article

On the Existence of Locally Attractive Solutions of a Nonlinear Quadratic Volterra Integral Equation of Fractional Order

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The authors employs a hybrid fixed point theorem involving the multiplication of two operators for proving an existence result of locally attractive solutions of a nonlinear quadratic Volterra integral equation of fractional (arbitrary) order. Investigations will be carried out in the Banach space of real functions which are defined, continuous, and bounded on the real half axis $\mathbb{R}_+$.

1. Introduction

The theory of differential and integral equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to differential and integral equations of fractional order (cf., e.g., [1–6]). These papers contain various types of existence results for equations of fractional order.

In this paper, we study the existence of locally attractive solutions of the following nonlinear quadratic Volterra integral equation of fractional order:

$$x(t) = \left[f(t, x(t))\right]\left(q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(t, s, x(s))}{(t-s)^{1-\alpha}}ds\right), \tag{1.1}$$

for all $t \in \mathbb{R}_+$ and $\alpha \in (0, 1)$, in the space of real functions defined, continuous, and bounded on an unbounded interval.

It is worthwhile mentioning that up to now integral equations of fractional order have only been studied in the space of real functions defined on a bounded interval. The result obtained in this paper generalizes several ones obtained earlier by many authors.

In fact, our result in this paper is motivated by the extension of the work of Hu and Yan [7]. Also, We proceed and generalize the results obtained in the papers [8, 9].

2. Notations, Definitions, and Auxiliary Facts

Denote by $L^1(a,b)$ the space of Lebesgue integrable functions on the interval (a,b), which is equipped with the standard norm. Let $x \in L^1(a,b)$ and let $\alpha > 0$ be a fixed number. The Riemann-Liouville fractional integral of order α of the function $x(t)$ is defined by the formula:

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{x(s)}{(t-s)^{1-\alpha}}ds, \quad t \in (a,b), \tag{2.1}$$

where $\Gamma(\alpha)$ denotes the gamma function.

It may be shown that the fractional integral operator, I^{α} transforms the space $L^1(a,b)$ into itself and has some other properties (see [10–12]).

Let $X = \mathrm{BC}(\mathbb{R}_+)$ be the space of continuous and bounded real-valued functions on $\mathbb{R}_+$ and let Ω be a subset of X. Let $P : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (Px)(t), \tag{2.2}$$

for all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator equation (2.2) on $\mathbb{R}_+$. We need the following definitions in the sequel.

Definition 2.1. We say that solutions of (2.2) are locally attractive if there exists an $x_0 \in \mathrm{BC}(\mathbb{R}_+)$ and an $r > 0$ such that for all solutions $x = x(t)$ and $y = y(t)$ of (2.2) belonging to $B_r(x_0) \cap \Omega$ we have that:

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0. \tag{2.3}$$

Definition 2.2. An operator $P : X \to X$ is called Lipschitz if there exists a constant k such that $\|Px - Py\| \le k\|x - y\|$ for all $x, y \in X$. The constant k is called the Lipschitz constant of P on X.

Definition 2.3 (Dugundji and Granas [13]). An operator P on a Banach space X into itself is called compact if for any bounded subset S of X, $P(S)$ is a relatively compact subset of X. If P is continuous and compact, then it is called completely continuous on X.

We seek the solutions of (1.1) in the space $\mathrm{BC}(\mathbb{R}_+)$ of continuous and bounded real-valued functions defined on $\mathbb{R}_+$. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $\mathrm{BC}(\mathbb{R}_+)$ by

$$\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}, \quad \big(xy\big)(t) = x(t)y(t), \quad t \in \mathbb{R}_+. \tag{2.4}$$

Clearly, $BC(\mathbb{R}_+)$ becomes a Banach space with respect to the above norm and the multiplication in it. By $L^1(\mathbb{R}_+)$ we denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|dt. \tag{2.5}$$

We employ a hybrid fixed point theorem of Dhage [14] for proving the existence result.

Theorem 2.4 (Dhage [14]). *Let S be a closed-convex and bounded subset of the Banach space X and let $F, G : S \to S$ be two operators satisfying:*

(a) *F is Lipschitz with the Lipschitz constant k,*

(b) *G is completely continuous,*

(c) *$FxGx \in S$ for all $x \in S$, and*

(d) *$Mk < 1$ where $M = \|G(S)\| = \sup\{\|Gx\| : x \in S\}$.*

Then the operator equation

$$FxGx = x \tag{2.6}$$

has a solution and the set of all solutions is compact in S.

3. Existence Result

We consider the following set of hypotheses in the sequel.

$(H1)$ The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a bounded function $l : \mathbb{R}_+ \to \mathbb{R}_+$ with bound L satisfying

$$|f(t,x) - f(t,y)| \le l(t)|x - y| \tag{3.1}$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

$(H2)$ The function $f_1 : \mathbb{R}_+ \to \mathbb{R}$ defined by $f_1 = |f(t,0)|$ is bounded with

$$f_0 = \sup\{f_1(t) : t \in \mathbb{R}_+\}. \tag{3.2}$$

$(H3)$ The function $q : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\lim_{t\to\infty} q(t) = 0$.

$(H4)$ The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exist a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ with $h(0) = 0$ and such that

$$|g(t,s,x) - g(t,s,y)| \le m(t)h(|x - y|) \tag{3.3}$$

for all $t, s \in \mathbb{R}_+$ such that $s \le t$ and for all $x, y \in \mathbb{R}$.

For further purposes let us define the function $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ by putting

$$g_1(t) = \max\{|g(t,s,0)| : 0 \le s \le t\}. \tag{3.4}$$

Obviously the function g_1 is continuous on $\mathbb{R}_+$.

In what follows we will assume additionally that the following conditions are satisfied.

$(H5)$ The functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formulas

$$a(t) = m(t)t^{\alpha}, \qquad b(t) = g_1(t)t^{\alpha}, \tag{3.5}$$

are bounded on $\mathbb{R}_+$ and vanish at infinity, that is, $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} b(t) = 0$.

Remark 3.1. Note that if the hypotheses $(H3)$ and $(H5)$ hold, then there exist constants $K_1 > 0$ and $K_2 > 0$ such that:

$$K_1 = \sup\{q(t) : t \in \mathbb{R}_+\}, \qquad K_2 = \sup\left\{\frac{a(t)h(r) + b(t)}{\Gamma(\alpha+1)} : t, r \in \mathbb{R}_+\right\}. \tag{3.6}$$

Theorem 3.2. *Assume that the hypotheses $(H1)$–$(H5)$ hold. Furthermore, if $L(K_1 + K_2) < 1$, where K_1 and K_2 are defined in Remark 3.1, then* (1.1) *has at least one solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of* (1.1) *are locally attractive on $\mathbb{R}_+$.*

Proof. Set $X = \mathrm{BC}(\mathbb{R}_+, \mathbb{R})$. Consider the closed ball $B_r(0)$ in X centered at origin 0 and of radius r, where $r = f_0(K_1 + K_2)/(1 - L(K_1 + K_2)) > 0$.

Let us define two operators F and G on $B_r(0)$ by

$$\begin{aligned} Fx(t) &= f(t, x(t)), \\ Gx(t) &= q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(t,s,x(s))}{(t-s)^{1-\alpha}} ds, \end{aligned} \tag{3.7}$$

for all $t \in \mathbb{R}_+$.

According to the hypothesis $(H1)$, the operator F is well defined and the function Fx is continuous and bounded on $\mathbb{R}_+$. Also, since the function q is continuous on $\mathbb{R}_+$, the function Gx is continuous and bounded in view of hypothesis $(H4)$. Therefore F and G define the operators $F, G : B_r(0) \to X$. We will show that F and G satisfy the requirements of Theorem 2.4 on $B_r(0)$.

The operator F is a Lipschitz operator on $B_r(0)$. In fact, let $x, y \in B_r(0)$ be arbitrary. Then by hypothesis $(H1)$, we get

$$|Fx(t) - Fy(t)| = |f(t,x(t)) - f(t,y(t))| \le l(t)|x(t) - y(t)| \le L\|x - y\|, \tag{3.8}$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t,

$$\|Fx - Fy\| \le L\|x - y\|, \tag{3.9}$$

for all $x, y \in B_r(0)$. This shows that F is a Lipschitz on $B_r(0)$ with the Lipschitz constant L.

Next, we show that G is a continuous and compact operator on $B_r(0)$. First we show that G is continuous on $B_r(0)$. To do this, let us fix arbitrary $\epsilon > 0$ and take $x, y \in B_r(0)$ such that $\|x - y\| \le \epsilon$. Then we get

$$\begin{aligned}
\left|(Gx)(t) - (Gy)(t)\right| &\le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left|g(t,s,x(s)) - g(t,s,y(s))\right|}{(t-s)^{1-\alpha}} ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{m(t)h(\left|x(s) - y(s)\right|)}{(t-s)^{1-\alpha}} ds \\
&\le \frac{m(t)t^\alpha}{\Gamma(\alpha+1)} h(r) \\
&\le \frac{a(t)}{\Gamma(\alpha+1)} h(r).
\end{aligned} \tag{3.10}$$

Since $h(r)$ is continuous on $\mathbb{R}_+$, then it is bounded on $\mathbb{R}_+$, and there exists a nonnegative constant, say h^*, such that $h^* = \sup\{h(r) : r > 0\}$. Hence, in view of hypothesis $(H5)$, we infer that there exists $T > 0$ such that $a(t) \le \Gamma(\alpha+1)\epsilon/h^*$ for $t > T$. Thus, for $t > T$ we derive that

$$\left|(Gx)(t) - (Gy)(t)\right| \le \epsilon. \tag{3.11}$$

Furthermore, let us assume that $t \in [0, T]$. Then, evaluating similarly to the above we obtain the following estimate:

$$\left|(Gx)(t) - (Gy)(t)\right| \le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left|g(t,s,x(s)) - g(t,s,y(s))\right|}{(t-s)^{1-\alpha}} ds \le \frac{T^\alpha}{\Gamma(\alpha+1)} \omega_r^T(g,\epsilon), \tag{3.12}$$

where $\omega_r^T(g,\epsilon) = \sup\{|g(t,s,x) - g(t,s,y)| : t, s \in [0,T],\ x, y \in [-r, r],\ |x - y| \le \epsilon\}$.

Therefore, from the uniform continuity of the function $g(t,s,x)$ on the set $[0,T] \times [0,T] \times [-r, r]$ we derive that $\omega_r^T(g,\epsilon) \to 0$ as $\epsilon \to 0$. Hence, from the above-established facts we conclude that the operator G maps the ball $B_r(0)$ continuously into itself.

Now, we show that G is compact on $B_r(0)$. It is enough to show that every sequence $\{Gx_n\}$ in $G(B_r(0))$ has a Cauchy subsequence. In view of hypotheses $(H3)$ and $(H4)$, we infer that:

$$\begin{aligned}
|Gx_n(t)| &\le |q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,x_n(s))|}{(t-s)^{1-\alpha}}ds \\
&\le |q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,x_n(s)) - g(t,s,0)|}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,0)|}{(t-s)^{1-\alpha}}ds \\
&\le |q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{m(t)h(|x_n(s)|)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g_1(t)}{(t-s)^{1-\alpha}}ds \\
&\le |q(t)| + \frac{m(t)t^\alpha}{\Gamma(\alpha+1)}h(r) + \frac{g_1(t)t^\alpha}{\Gamma(\alpha+1)} \\
&\le |q(t)| + \frac{a(t)h(r)+b(t)}{\Gamma(\alpha+1)} \\
&\le K_1 + K_2,
\end{aligned} \tag{3.13}$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|Gx_n\| \le K_1 + K_2$ for all $n \in \mathbb{N}$. This shows that $\{Gx_n\}$ is a uniformly bounded sequence in $G(B_r(0))$. We show that it is also equicontinuous. Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} q(t) = 0$, there is constant $T > 0$ such that $|q(t)| < \epsilon/2$ for all $t \ge T$.

Let $t_1, t_2 \in \mathbb{R}_+$ be arbitrary. If $t_1, t_2 \in [0,T]$, then we have

$$\begin{aligned}
&|Gx_n(t_2) - Gx_n(t_1)| \\
&\quad\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}}ds - \int_0^{t_1} \frac{g(t_1,s,x_n(s))}{(t_1-s)^{1-\alpha}}ds\right| \\
&\quad\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}}ds + \int_{t_1}^{t_2} \frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}}ds + \int_0^{t_1} \frac{g(t_1,s,x_n(s))}{(t_1-s)^{1-\alpha}}ds\right| \\
&\quad\le |q(t_2) - q(t_1)| \\
&\qquad+ \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}\left|\frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}} - \frac{g(t_1,s,x_n(s))}{(t_2-s)^{1-\alpha}}\right|ds\right. \\
&\qquad\qquad \left.+ \int_0^{t_1}\left|\frac{g(t_1,s,x_n(s))}{(t_2-s)^{1-\alpha}} - \frac{g(t_1,s,x_n(s))}{(t_1-s)^{1-\alpha}}\right|ds + \int_{t_1}^{t_2}\frac{|g(t_2,s,x_n(s))|}{(t_2-s)^{1-\alpha}}ds\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|q(t_2)-q(t_1)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg(\int_0^{t_1}\frac{\left|g(t_2,s,x_n(s))-g(t_1,s,x_n(s))\right|}{(t_2-s)^{1-\alpha}}ds \\
&\qquad + \int_0^{t_1}\left|g(t_1,s,x_n(s))\right|\left[\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right]ds+\int_{t_1}^{t_2}\frac{\left|g(t_2,s,x_n(s))\right|}{(t_2-s)^{1-\alpha}}ds\Bigg) \\
&\leq \left|q(t_2)-q(t_1)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg(\int_0^{t_1}\left[\left|g(t_2,s,x_n(s))-g(t_1,s,x_n(s))\right|\right]\frac{1}{(t_2-s)^{1-\alpha}}ds \\
&\qquad + \int_0^{t_1}\left(\left|g(t_1,s,x_n(s))-g(t_1,s,0)\right|+\left|g(t_1,s,0)\right|\right)\left[\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right]ds \\
&\qquad + \int_{t_1}^{t_2}\frac{\left|g(t_2,s,x_n(s))-g(t_2,s,0)\right|+\left|g(t_2,s,0)\right|}{(t_2-s)^{1-\alpha}}ds\Bigg) \\
&\leq \left|q(t_2)-q(t_1)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Bigg(\int_0^{t_1}\left[\left|g(t_2,s,x_n(s))-g(t_1,s,x_n(s))\right|\right]\frac{1}{(t_2-s)^{1-\alpha}}ds \\
&\qquad + \int_0^{t_1}\left[m(t_1)h(|x_n(s)|)+g_1(t_1)\right]\left[\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right]ds \\
&\qquad + \int_{t_1}^{t_2}\frac{m(t_2)h(|x_n(s)|)+g_1(t_2)}{(t_2-s)^{1-\alpha}}ds\Bigg) \\
&\leq \left|q(t_2)-q(t_1)\right|+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[\left|g(t_2,s,x_n(s))-g(t_1,s,x_n(s))\right|\right]\frac{1}{(t_2-s)^{1-\alpha}}ds \\
&\quad + \frac{m(t_1)h(r)+g_1(t_1)}{\Gamma(\alpha+1)}\left[t_1^{\alpha}-t_2^{\alpha}+(t_2-t_1)^{\alpha}\right]+\frac{m(t_2)h(r)+g_1(t_2)}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.
\end{aligned}
\tag{3.14}
$$

From the uniform continuity of the function $q(t)$ on $[0,T]$ and the function g in $[0,T]\times[0,T]\times[-r,r]$, we get $|Gx_n(t_2)-Gx_n(t_1)| \to 0$ as $t_1 \to t_2$.

If $t_1, t_2 \geq T$, then we have

$$
\begin{aligned}
|Gx_n(t_2)-Gx_n(t_1)| &\leq \left|q(t_2)-q(t_1)\right|+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}}ds-\int_0^{t_1}\frac{g(t_1,s,x_n(s))}{(t_1-s)^{1-\alpha}}ds\right| \\
&\leq \left|q(t_1)\right|+\left|q(t_2)\right|+\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{g(t_2,s,x_n(s))}{(t_2-s)^{1-\alpha}}ds-\int_0^{t_1}\frac{g(t_1,s,x_n(s))}{(t_1-s)^{1-\alpha}}ds\right| \\
&< \epsilon,
\end{aligned}
\tag{3.15}
$$

as $t_1 \to t_2$.

Similarly, if $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < T < t_2$, then we have

$$|Gx_n(t_2) - Gx_n(t_1)| \leq |Gx_n(t_2) - Gx_n(T)| + |Gx_n(T) - Gx_n(t_1)|. \tag{3.16}$$

Note that if $t_1 \to t_2$, then $T \to t_2$ and $t_1 \to T$. Therefore from the above obtained estimates, it follows that:

$$|Gx_n(t_2) - Gx_n(T)| \longrightarrow 0, \quad |Gx_n(T) - Gx_n(t_1)| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \tag{3.17}$$

As a result, $|Gx_n(t_2) - Gx_n(T)| \to 0$ as $t_1 \to t_2$. Hence $\{Gx_n\}$ is an equicontinuous sequence of functions in X. Now an application of the Arzelá-Ascoli theorem yields that $\{Gx_n\}$ has a uniformly convergent subsequence on the compact subset $[0,T]$ of $\mathbb{R}$. Without loss of generality, call the subsequence of the sequence itself.

We show that $\{Gx_n\}$ is Cauchy sequence in X. Now $|Gx_n(t) - Gx(t)| \to 0$ as $n \to \infty$ for all $t \in [0,T]$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for $m, n \geq n_0$, then we have

$$\begin{aligned}
|Gx_m(t) - Gx_n(t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t \frac{g(t,s,x_m(s)) - g(t,s,x_n(t))}{(t-s)^{1-\alpha}}ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,x_m(s)) - g(t,s,x_n(t))|}{(t-s)^{1-\alpha}}ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{m(t)h(|x_m(s) - x_n(s)|)}{(t-s)^{1-\alpha}}ds \\
&\leq \frac{m(t)t^\alpha h(r)}{\Gamma(\alpha+1)} \\
&\leq \frac{a(t)h^*}{\Gamma(\alpha+1)} \\
&< \epsilon.
\end{aligned} \tag{3.18}$$

This shows that $\{Gx_n\} \subset G(B_r(0)) \subset X$ is Cauchy. Since X is complete, then $\{Gx_n\}$ converges to a point in X. As $G(B_r(0))$ is closed, $\{Gx_n\}$ converges to a point in $G(B_r(0))$. Hence, $G(B_r(0))$ is relatively compact and consequently G is a continuous and compact operator on $B_r(0)$.

Next, we show that $FxGx \in B_r(0)$ for all $x \in B_r(0)$. Let $x \in B_r(0)$ be arbitrary, then

$$
\begin{aligned}
|Fx(t)Gx(t)| &\le |Fx(t)||Gx(t)| \\
&\le \left|f(t,x(t))\right|\left(\left|q(t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|g(t,s,x(s))\right|}{(t-s)^{1-\alpha}}ds\right) \\
&\le \left[\left|f(t,x(t)) - f(t,0)\right| + \left|f(t,0)\right|\right] \\
&\quad + \left(\left|q(t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|g(t,s,x(s)) - g(t,s,0)\right| + \left|g(t,s,0)\right|}{(t-s)^{1-\alpha}}ds\right) \\
&\le \left[l(t)|x(t)| + f_1(t)\right] + \left(\left|q(t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{m(t)h(|x(t)|) + g_1(t)}{(t-s)^{1-\alpha}}ds\right) \\
&\le \left[L\|x\| + f_0\right] + \left(\left|q(t)\right| + \frac{m(t)t^{\alpha}h(r) + g_1(t)t^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&\le \left[L\|x\| + f_0\right] + \left(\left|q(t)\right| + \frac{a(t)h(r) + b(t)}{\Gamma(\alpha+1)}\right) \\
&\le \left[L\|x\| + f_0\right] + (K_1 + K_2) \\
&\le L(K_1 + K_2)\|x\| + f_0(K_1 + K_2) \\
&= \frac{f_0(K_1 + K_2)}{1 - L(K_1 + K_2)} \\
&= r,
\end{aligned}
\tag{3.19}
$$

for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|FxGx\| \le r$ for all $x \in B_r(0)$. Hence hypothesis (c) of Theorem 2.4 holds.

Also we have

$$
\begin{aligned}
M &= \|G(B_r(0))\| \\
&= \sup\{\|Gx\| : x \in B_r(0)\} \\
&= \sup\left\{\sup_{t\ge 0}\left\{\left|q(t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|g(t,s,x(s))\right|}{(t-s)^{1-\alpha}}ds\right\} : x \in B_r(0)\right\} \\
&\le \sup_{t\ge 0}\left|q(t)\right| + \sup_{t\ge 0}\left[\frac{a(t)h(r) + b(t)}{\Gamma(\alpha+1)}\right] \\
&\le K_1 + K_2,
\end{aligned}
\tag{3.20}
$$

and therefore $Mk = L(K_1 + K_2) < 1$. Now we apply Theorem 2.4 to conclude that (1.1) has a solution on $\mathbb{R}_+$

Finally, we show the local attractivity of the solutions for (1.1). Let x and y be any two solutions of (1.1) in $B_r(0)$ defined on $\mathbb{R}_+$, then we get

$$\begin{aligned}|x(t)-y(t)| &\leq \left|f(t,x(t))\left(q(t)+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(t,s,x(s))}{(t-s)^{1-\alpha}}ds\right)\right| \\ &\quad+\left|f(t,y(t))\left(q(t)+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(t,s,y(s))}{(t-s)^{1-\alpha}}ds\right)\right| \\ &\leq |f(t,x(t))|\left(|q(t)|+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,x(s))|}{(t-s)^{1-\alpha}}ds\right) \\ &\quad+|f(t,y(t))|\left(|q(t)|+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{|g(t,s,y(s))|}{(t-s)^{1-\alpha}}ds\right) \\ &\leq 2(Lr+f_0)\left(|q(t)|+\frac{a(t)h(r)+b(t)}{\Gamma(\alpha+1)}\right),\end{aligned} \tag{3.21}$$

for all $t \in \mathbb{R}_+$. Since $\lim_{t\to\infty} q(t) = 0$, $\lim_{t\to\infty} a(t) = 0$ and $\lim_{t\to\infty} b(t) = 0$, for $\epsilon > 0$, there are real numbers $T' > 0$, $T'' > 0$ and $T''' > 0$ such that $|q(t)| < \epsilon$ for $t \geq T'$, $a(t) < h^*\epsilon/\Gamma(\alpha+1)$ for all $t \geq T''$ and $b(t) < \epsilon/\Gamma(\alpha+1)$ for all $t \geq T'''$. If we choose $T^* = \max\{T', T'', T'''\}$, then from the above inequality it follows that $|x(t)-y(t)| \leq \epsilon^*$ for $t \geq T^*$, where $\epsilon^* = 6(Lr+f_0)\epsilon > 0$. This completes the proof. □

4. An Example

In this section we provide an example illustrating the main existence result contained in Theorem 3.2.

Example 4.1. Consider the following quadratic Volterra integral equation of fractional order:

$$x(t) = \left[t+t^2x(t)\right]\left(te^{-t^2/2}+\frac{1}{\Gamma(2/3)}\int_0^t \frac{x^{2/3}(s)e^{-(3t+s)}+1/\left(10t^{8/3}+1\right)}{(t-s)^{1/3}}ds\right), \tag{4.1}$$

where $t \in \mathbb{R}_+$.

Observe that the above equation is a special case of (1.1). Indeed, if we put $\alpha = 2/3$ and

$$\begin{aligned}f(t,x) &= t+t^2x, \\ q(t) &= te^{-t^2/2}, \\ g(t,s,x) &= x^{2/3}(s)e^{-(3t+s)}+\frac{1}{10t^{8/3}+1}.\end{aligned} \tag{4.2}$$

Then we can easily check that the assumptions of Theorem 3.2 are satisfied. In fact, we have that the function $f(t,x)$ is continuous and satisfies assumption $(H1)$, where $l(t) = t^2$

and $\|f(t,0)\| = f(t,0) = t = f_1$ as in assumption $(H2)$. We have that the function $q(t)$ is continuous and it is easily seen that $q(t) \to 0$ as $t \to \infty$, thus assumption $(H3)$ is satisfied. Next, let us notice that the function $g(t,s,x)$ satisfies assumption $(H4)$, where $m(t) = e^{-3t}$, $h(r) = r^{2/3}$ and $g(t,s,0) = 1/(10t^{8/3}+1)$. Thus $g_1 = g(t,s,0)$. To check that assumption $(H5)$ is satisfied let us observe that the functions a, b appearing in that assumption take the form:

$$a(t) = t^{2/3}e^{-3t}, \qquad b(t) = \frac{t^{2/3}}{10t^{8/3}+1}. \tag{4.3}$$

Thus it is easily seen that $a(t), b(t) \to 0$ as $t \to \infty$. Finally, let us note that in Remark 3.1 there are two constants $K_1, K_2 > 0$ such that $L(K_1 + K_2) < 1$. It is also easy to check that $K_1 = q(1) = e^{-1/2} = 0.60653\ldots$, $K_2 = (e^{-3} + 0.1)/0.8856 = 0.16913\ldots$ and $L = 1$. Then $L(K_1 + K_2) = 0.77566\ldots < 1$. Hence, taking into account that $\Gamma(5/3) > 0.8856$ (cf. [4]), all the assumptions of Theorem 3.2 are satisfied and (4.1) has a solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of (4.1) are uniformly locally attractive in the sense of Definition 2.1.

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