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Research Article

A Study on the p-Adic Integral Representation on $\mathbb{Z}_p$ Associated with Bernstein and Bernoulli Polynomials

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We consider the Bernstein polynomials on $\mathbb{Z}_p$ and investigate some interesting properties of Bernstein polynomials related to Stirling numbers and Bernoulli numbers.

1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. Then, Bernstein operator for $f \in C[0,1]$ is defined as

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x), \tag{1.1}$$

for $k, n \in \mathbb{Z}$, where $B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is called Bernstein polynomial of degree n. Some researchers have studied the Bernstein polynomials in the area of approximation theory (see [1–6]).

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{1.2}$$

(see [4, 7–15]).

In the special case, if we set $f(x) = x^n$ in (1.2), we have

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu(x). \tag{1.3}$$

In this paper, we consider Bernstein polynomials on $\mathbb{Z}_p$ and we investigate some interesting properties of Bernstein polynomials related to Stirling numbers and Bernoulli numbers.

2. Bernstein Polynomials Related to Stirling Numbers and Bernoulli Numbers

In this section, for $f \in UD(\mathbb{Z}_p)$, we consider Bernstein type operator on $\mathbb{Z}_p$ as follows:

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_k(x), \tag{2.1}$$

for $n \in \mathbb{Z}_+$, where $B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is called Bernstein polynomial of degree n. We consider Newton's forward difference operator as follows:

$$\begin{aligned} &\Delta f(x) = f(x+1) - f(x), \\ &\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^n - k f(x+k) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+n-k). \end{aligned} \tag{2.2}$$

For $x = 0$,

$$\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(n-k) = \sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n-k} f(k). \tag{2.3}$$

Then, we have

$$f(n) = (1+\Delta)^n f(0) = \sum_{l=0}^{n} \binom{n}{l} \Delta^l f(0). \tag{2.4}$$

From (2.4), we note that

$$f(x)=\sum_{n=0}^{\infty}\binom{x}{n}\Delta^n f(0), \tag{2.5}$$

where

$$\Delta^n f(0)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(n-k). \tag{2.6}$$

The Stirling number of the first kind is defined by

$$\prod_{k=1}^{n}(1+kz)=\sum_{k=0}^{n}S_1(n,k)z^k, \tag{2.7}$$

and the Stirling number of the second kind is also defined by

$$\prod_{k=1}^{n}\left(\frac{1}{1+kz}\right)=\sum_{k=0}^{n}S_2(n,k)z^k. \tag{2.8}$$

By (2.5), (2.6), (2.7), and (2.8), we see that

$$S_2(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n, \tag{2.9}$$

where $\Delta^n 0^m=\sum_{k=0}^{n}\binom{n}{k}(-1)^k(n-k)^m$. Note that, for $k\in\mathbb{Z}_+$ and $x\in[0,1]$,

$$\begin{aligned}F^{(k)}(t,x)=\frac{t^k e^{(1-x)t}x^k}{k!}&=x^k\sum_{n=0}^{\infty}\binom{n+k}{k}(1-x)^n\frac{t^{n+k}}{(n+k)!}\\&=\sum_{n=k}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\frac{t^n}{(n)!}=\sum_{n=0}^{\infty}B_{k,n}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Thus, we note that $t^k e^{(1-x)t}x^k/k!$ is the generating function of Bernstein polynomial. It is easy to show that

$$\frac{1}{\binom{n}{k}}\int_{\mathbb{Z}_p}B_{k,n}(x)d\mu(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\int_{\mathbb{Z}_p}x^{l+k}d\mu(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l B_{n+k}. \tag{2.11}$$

By (2.11), we obtain the following theorem.

Theorem 2.1. *For $n, k \in \mathbb{Z}_+$ with $n \geq k$, one has*

$$\frac{1}{\binom{n}{k}} \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu(x) = \sum_{m=0}^{\infty} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l B_{n+k}, \tag{2.12}$$

where B_n are the nth Bernoulli numbers.

In [12], it is known that

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S_2(n,k), \tag{2.13}$$

$$\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x) = x^i, \tag{2.14}$$

for $i \in \mathbb{N}$. By (1.1) and (2.14), we see that

$$\begin{aligned} x^i &= \sum_{m=0}^{\infty} \binom{n-i+m-1}{m} (-1)^m x^{n-i-m} (1-x)^m \sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x) \\ &= \sum_{m=0}^{\infty} \sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} \binom{n-i+m-1}{m} \binom{n}{k} (-1)^m x^{n-i-m+k} (1-x)^{n+m-k} \\ &= \sum_{m=0}^{\infty} \sum_{k=i}^{n} \sum_{l=0}^{n+m-k} \binom{n-i+m-1}{m} \binom{n+m-k}{l} \binom{n}{k} \\ &\quad \times (-1)^{l+m} x^{l+n-i-m+k}, \end{aligned} \tag{2.15}$$

for $i \in \mathbb{N}$. By (2.15), we obtain the following theorem.

Theorem 2.2. *For $n, k \in \mathbb{Z}_+$, and $i \in \mathbb{N}$, one has*

$$B_i = \sum_{m=0}^{\infty} \sum_{k=i}^{n} \sum_{l=0}^{m+n-k} \binom{n-i+m-1}{m} \binom{m+n-k}{l} \binom{n}{k} (-1)^{l+m} B_{l+n-i-m+k}. \tag{2.16}$$

From (2.13) and (2.14), we note that

$$\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x) = \sum_{k=0}^{i} \binom{x}{k} k! S_2(i,k). \tag{2.17}$$

In [16], it is known that

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) = \frac{1}{n+1}. \tag{2.18}$$

By (2.17), (2.18), and Theorem 2.2, we have

$$B_n = \sum_{k=0}^{m} \frac{k!}{k+1}(-1)^k S_2(k, n-k). \tag{2.19}$$

From the definition of the Stirling numbers of the first kind, we drive that

$$\binom{x}{n} n! = (x)_n = \sum_{k=0}^{n} S_1(n,k) x^k. \tag{2.20}$$

By (2.17), (2.19), and (2.20), we obtain the following theorem.

Theorem 2.3. *For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has*

$$\sum_{k=i-1}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x) = \sum_{k=0}^{i} \sum_{l=0}^{k} S_1(n,l) S_2(i,k) x^l. \tag{2.21}$$

By Theorems 2.2 and 2.3, we obtain the following corollary.

Corollary 2.4. *For $k \in \mathbb{N}$, one has*

$$B_i(x) = \sum_{k=0}^{i} \sum_{l=0}^{k} S_1(n,l) S_2(i,k) B_l, \tag{2.22}$$

where B_i are the ith Bernoulli numbers.

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