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Research Article

Asymptotic Bounds for Linear Difference Systems

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This paper describes asymptotic properties of solutions of some linear difference systems. First we consider system of a general form and estimate its solutions by use of a solution of an auxiliary scalar difference inequality assuming that this solution admits certain properties. Then applying this result to linear difference systems of a variable order with constant (or bounded) coefficients we derive effective asymptotic criteria for such systems. Beside it, we give applications of these results to numerical analysis of vector differential equations with infinite lags.

1. Introduction

Stability and asymptotic investigation of linear difference equations is a conventional topic which covers the study of various types of these equations. In this paper, we consider the linear difference system:

$$\mathbf{y}(n+1) = A(n)\mathbf{y}(n) + \sum_{k=0}^{p} B_k(n)\mathbf{y}(\alpha(n)+k), \quad n \in \mathbb{N}(n_0), \tag{1.1}$$

where $\mathbb{N}(n_0) = \{n_0, n_0+1, \ldots\}$, $n_0 \in \mathbb{N}$, $p \in \mathbb{N}$, $A(n)$ and $B_k(n)$ are given $m \times m$ real matrices, and $\alpha : \mathbb{N}(n_0) \to \mathbb{Z}$ is nondecreasing, unbounded as $n \to \infty$, and satisfying

$$\alpha(n) + p < n \quad \forall n \in \mathbb{N}(n_0). \tag{1.2}$$

Equation (1.1) represents a general pattern, where some other specifications of entry parameters and their properties are necessary. Before doing this we note that many papers on asymptotics of difference equations involve difference systems fitting this pattern under special choices of parameters. For example, if $A(n)$ is an asymptotically constant matrix and $B_k(n)$, $k = 0, 1, \ldots, p$ are identically zero matrices, then (1.1) is of Poincaré type (studied, e.g., in [1–3] with respect to its asymptotic properties). Similarly, asymptotic stability of the system (1.1) is another frequent topic discussed in the framework of asymptotic analysis of difference systems. We can mention, for example, the papers [4, 5] or [6], where various particular cases of (1.1) have been investigated.

The primary goal of this paper does not consist in the generalization of these results. Although we are going to formulate the asymptotic bound of solutions for a system of the general form (1.1), we show that conditions of this criterion are satisfied especially when (1.1) is of a variable order which becomes infinite as $n \to \infty$ (i.e., when $n - \alpha(n) \to \infty$ as $n \to \infty$). Such difference equations are studied only rarely (especially in the scalar case—see, e.g., [7]). Our wish is to describe some of their asymptotic properties.

The idea of our approach consists in a modification of suitable techniques developed for asymptotic analysis of differential equations with infinite lag (which are essentially continuous counterparts of difference equations of infinite order). Considering the scalar case, some general principles of this approach were initiated in [8] and applied in [9].

This paper is structured as follows. In Section 2, we mention some related mathematical tools which are necessary in our asymptotic investigation of (1.1). Section 3 presents an asymptotic bound of all solutions of (1.1). This bound is expressed via a solution of some auxiliary scalar difference inequality introduced in Section 2 and its validity requires some properties of this solution. In Section 4, we consider some particular cases of (1.1) and derive effective asymptotic formulae for their solutions following from the asymptotic result formulated in Section 3. Furthermore, we apply some of our results to the simplest (Euler) discretization of the vector pantograph equation and compare obtained result with corresponding asymptotic property of the exact pantograph equation.

2. Preliminaries

Let $|\cdot|$ denote a vector norm on $\mathbb{R}^m$ and let $\|\cdot\|$ be an induced matrix norm. Throughout this paper we assume that the auxiliary linear system

$$\mathbf{y}(n+1) = A(n)\mathbf{y}(n) \tag{2.1}$$

is uniformly asymptotically stable (for a precise definition and related properties we refer to [10]). We recall here that the system (2.1) is uniformly asymptotically stable if and only if there exist real scalars $M > 0$ and $0 < \mu < 1$ such that

$$\left\| Y(n)Y^{-1}(s) \right\| \leq M\mu^{n-s} \quad \forall n \geq s \in \mathbb{N}(n_0), \tag{2.2}$$

where $Y(n)$ is the fundamental matrix of (2.1). Then along with the investigated difference system (1.1) we consider the scalar difference inequality:

$$\sum_{k=0}^{p} \|B_k(n)\| \omega(\alpha(n) + k) \leq (1 - \mu)\omega(n). \tag{2.3}$$

This inequality formally arises from (1.1) and later we show that it can provide an upper bound sequence for any solution $\mathbf{y}(n)$ of (1.1). Before doing this we have to specify some properties of $\omega(n)$ which turn out to be necessary for the validity of such estimates. We define the function $\beta : \mathbb{N}(n_0) \to \mathbb{N}(n_0)$ by the relation

$$\beta(n) = \sup\{w \in \mathbb{N}(n_0),\ \alpha(w) + p \leq n\} \tag{2.4}$$

and the orbit $\Omega(n_0) = \{\beta^j(n_0),\quad j \in \mathbb{N}\}$, where β^j means the jth iterate of β. Since the property (1.2) implies that $\beta(n) > n$ for all $n \in \mathbb{N}(n_0)$, this orbit is unbounded (as $j \to \infty$). Further, we introduce the operator

$$\Delta^{-}\omega(n) = \frac{(\Delta\omega(n) - |\Delta\omega(n)|)}{2} \tag{2.5}$$

and assume that there exists a positive solution $\omega(n)$ of (2.3) such that

$$\Delta^{-}\omega(n) \leq \Delta^{-}\omega(n+1), \quad \forall n \in \mathbb{N}(n_0), \tag{2.6}$$

$$\sum_{n \in \Omega(n_0)} \frac{\Delta^{-}\omega(n)}{\omega(\beta(n))} > -\infty. \tag{2.7}$$

The discussion on the existence of $\omega(n)$ having such properties is performed in Section 4. Now we mention only an obvious fact, namely, that if $\omega(n)$ is nondecreasing, then the assumptions (2.6) and (2.7) become trivial.

Finally, we consider the functional relation

$$\varphi(\beta(n)) = \varphi(n) + 1, \quad n \in \Omega(n_0) \tag{2.8}$$

known as the Abel equation (see, e.g., [11]). This relation enables us to set up an increasing sequence of the values $\varphi(\beta^j(n_0))$ starting from the value $\varphi(n_0)$. In addition, we consider an arbitrary extension of this solution $\varphi(n)$ of (2.8) from $\Omega(n_0)$ to $\mathbb{N}(n_0)$ requiring

$$\varphi(n) \leq \varphi(n+1), \quad n \in \mathbb{N}(n_0). \tag{2.9}$$

As we can see later, this sequence may also be involved in an upper bound term for solutions $\mathbf{y}(n)$ of (1.1).

3. Asymptotic Estimate for the Solutions of (1.1)

The goal of this section is to derive an asymptotic bound of solutions of the system (1.1). The following holds.

Theorem 3.1. *Consider the difference equation* (1.1)*, where $A(n)$ is nonsingular for all $n \in \mathbb{N}(n_0)$ and assume that the auxiliary difference system* (2.1) *is uniformly asymptotically stable. Further, consider the inequality* (2.3) *with the real scalar $0 < \mu < 1$ given by* (2.2) *and let $\omega(n)$ be a positive* *monotonous solution of* (2.3) *having the properties* (2.6) *and* (2.7)*. Finally, let $\varphi(n)$ be a sequence satisfying* (2.8) *and* (2.9)*. If $\mathbf{y}(n)$ is a solution of* (1.1)*, then*

$$\mathbf{y}(n) = O\Big(\omega(n) M^{\varphi(n)}\Big) \quad \text{as } n \longrightarrow \infty, \tag{3.1}$$

where the positive real constant M is given by (2.2)*.*

Proof. The vector $\mathbf{z}(n) = \mathbf{y}(n)/\omega(n)$ provides a solution of the linear difference system:

$$\omega(n+1)\mathbf{z}(n+1) = A(n)\omega(n)\mathbf{z}(n) + \sum_{k=0}^{p} B_k(n)\omega(\alpha(n)+k)\mathbf{z}(\alpha(n)+k), \quad n \in \mathbb{N}(n_0). \tag{3.2}$$

If we multiply (3.2) by $Y^{-1}(n+1)$ ($Y^{-1}(n)$ is the inverse matrix to the fundamental matrix $Y(n)$ of (2.1)) from the left and rewrite it with n replaced by s, we obtain

$$\Delta\Big(Y^{-1}(s)\omega(s)\mathbf{z}(s)\Big) = Y^{-1}(s+1)\sum_{k=0}^{p} B_k(s)\omega(\alpha(s)+k)\mathbf{z}(\alpha(s)+k). \tag{3.3}$$

Indeed, applying the difference operator on the left-hand side of (3.3) and using $Y(s+1) = A(s)Y(s)$ we arrive at (3.2).

We define recursively $n_{\ell+1} := \beta(n_\ell)$ and put $I_0 := \mathbb{N}(\alpha(n_0), n_0)$, $I_{\ell+1} := \mathbb{N}(n_\ell, n_{\ell+1})$, $\ell = 0, 1, 2, \ldots$ (here we use the notation $\mathbb{N}(a,b) = \{a, a+1, \ldots, b\}$, where $a, b \in \mathbb{N}$, $a < b$). Thus $\mathbb{N}(n_0) = \bigcup_{\ell=1}^{\infty} I_\ell$.

Let $n \in I_{\ell+1}$, $n > n_\ell$. Summing the relation (3.3) from n_ℓ to $n-1$ we get

$$\omega(n)\mathbf{z}(n) = Y(n)Y^{-1}(n_\ell)\omega(n_\ell)\mathbf{z}(n_\ell) + \sum_{s=n_\ell}^{n-1} Y(n)Y^{-1}(s+1)\sum_{k=0}^{p} B_k(s)\omega(\alpha(s)+k)\mathbf{z}(\alpha(s)+k). \tag{3.4}$$

Notice that $\alpha(s) + p \leq n_\ell$ for any $s \in I_{\ell+1}$. Then considering appropriate norms and employing inequalities (2.2)-(2.3) we arrive at the estimate

$$\begin{aligned} |\mathbf{z}(n)| &\leq MS_\ell\left(\mu^{n-n_\ell}\frac{\omega(n_\ell)}{\omega(n)} + \frac{1}{\omega(n)}\sum_{s=n_\ell}^{n-1}\mu^{n-s-1}(1-\mu)\omega(s)\right) \\ &= MS_\ell\left(\mu^{n-n_\ell}\frac{\omega(n_\ell)}{\omega(n)} + \frac{1}{\omega(n)}\sum_{s=n_\ell}^{n-1}\omega(s)\Delta\mu^{n-s}\right), \end{aligned} \tag{3.5}$$

where $S_\ell := \sup\{|\mathbf{z}(s)|,\ s \in \bigcup_{r=0}^{\ell} I_r\}$. Then summation by parts formula yields

$$\sum_{s=n_\ell}^{n-1}\omega(s)\Delta\mu^{n-s} = \omega(n) - \omega(n_\ell)\mu^{n-n_\ell} - \sum_{s=n_\ell}^{n-1}\mu^{n-s-1}\Delta\omega(s). \tag{3.6}$$

Substituting this back into (3.5) we have

$$\begin{aligned}
|\mathbf{z}(n)| &\leq MS_\ell\left(1-\frac{1}{\omega(n)}\sum_{s=n_\ell}^{n-1}\mu^{n-s-1}\Delta\omega(s)\right)\\
&\leq MS_\ell\left(1-\frac{1}{\omega(n)}\sum_{s=n_\ell}^{n-1}\mu^{n-s-1}\Delta^-\omega(s)\right)\\
&= MS_\ell\left(1-\frac{1}{(1-\mu)\omega(n)}\sum_{s=n_\ell}^{n-1}\Delta\mu^{n-s}\Delta^-\omega(s)\right)\\
&\leq MS_\ell\left(1-\frac{\Delta^-\omega(n_\ell)}{(1-\mu)\omega(n)}\sum_{s=n_\ell}^{n-1}\Delta\mu^{n-s}\right)\\
&\leq MS_\ell\left(1-\frac{\Delta^-\omega(n_\ell)}{(1-\mu)\omega(\beta(n_\ell))}\right),
\end{aligned}\tag{3.7}$$

where we have used the property (2.6) and relations

$$\sum_{s=n_\ell}^{n-1}\Delta\mu^{n-s}=1-\mu^{n-n_l}<1,\qquad \frac{-\Delta^-\omega(n_\ell)}{\omega(n)}\leq\frac{-\Delta^-\omega(n_\ell)}{\omega(\beta(n_\ell))}.\tag{3.8}$$

Since this estimate holds for any $n\in I_{\ell+1}$, we can write

$$S_{\ell+1}\leq MS_\ell\left(1-\frac{\Delta^-\omega(n_\ell)}{(1-\mu)\omega(\beta(n_\ell))}\right).\tag{3.9}$$

Then (2.7) implies that $S_{\ell+1}\leq KM^\ell$; hence

$$|\mathbf{z}(n)|\leq KM^\ell\tag{3.10}$$

for a suitable real $K>0$ and any $n\in I_{\ell+1}$, that is, any n satisfying $\beta^\ell(n_0)\leq n\leq\beta^{\ell+1}(n_0)$. Applying the function $\varphi(n)$ satisfying (2.8) and (2.9) to these inequalities we get

$$\varphi\Big(\beta^\ell(n_0)\Big)\leq\varphi(n)\leq\varphi\Big(\beta^{\ell+1}(n_0)\Big),\tag{3.11}$$

that is,

$$\ell\leq\varphi(n)-\varphi(n_0)\leq\ell+1\tag{3.12}$$

by the use of $\varphi(\beta^{\ell}(n_0)) = \varphi(n_0) + \ell$. To summarize this,

$$|\mathbf{z}(n)| \le LM^{\varphi(n)} \tag{3.13}$$

for a real constant $L > 0$ and all $n \in \mathbb{N}(n_0)$. The property (3.1) is proved. □

As we have remarked earlier, a discussion of conditions of Theorem 3.1 will be performed in the following section. We mention here only an immediate consequence of this assertion. If we choose a constant sequence $\omega(n)$ (which obviously satisfies both assumptions (2.6) and (2.7)) and substitute it into (2.3), we get the following.

Corollary 3.2. *Consider the difference equation* (1.1), *where* $A(n)$ *is nonsingular for all* $n \in \mathbb{N}(n_0)$ *and assume that the auxiliary difference system* (2.1) *is uniformly asymptotically stable; that is,* (2.2) *holds with* $0 < \mu < 1$ *and* $M > 0$. *If* $M \le 1$ *and*

$$\sum_{k=0}^{p} \|B_k(n)\| \le 1 - \mu, \tag{3.14}$$

then any solution $\mathbf{y}(n)$ *of* (1.1) *is bounded.*

4. Applications

The difference equations of the type (1.1) play a significant role in numerical analysis of linear differential equations with a delayed argument. Consider the equation

$$\mathbf{x}'(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{x}(\tau(t)), \quad t \in I = [t_0, \infty) \tag{4.1}$$

with continuous matrix functions C, D and continuous delay argument τ satisfying $\tau(t) < t$ for all $t > t_0$. Important numerical discretizations of (4.1) originate from the class of Θ-methods. If we consider the mesh points $t_n = t_0 + nh$ ($n \in \mathbb{N}$, $h > 0$ is the stepsize) and $y(n)$ means the approximation of the exact value $x(t_n)$, then the Θ-method applied to (4.1) leads just to the difference equation (1.1), where $A(n)$, $B_k(n)$, and $\alpha(n)$ depend on $C(t)$, $D(t)$ and $\tau(t)$. Since $\tau(t_n)$ generally does not belong to the given mesh, we have to employ an interpolation (standard procedures are piecewise constant and piecewise linear interpolation). The choice of a suitable interpolation influences the value of parameter p in (1.1); usual values are $p = 0$, $p = 1$, and $p = 2$. General references concerning the Θ-methods and other discretizations of delay differential equations are provided by books [12, 13], where relevant formulae and some of their properties can be found.

We illustrate the previous considerations by the one-leg Θ-method ($0 \le \Theta \le 1$) with a piecewise linear interpolation applied to the pantograph equation:

$$\mathbf{x}'(t) = C\mathbf{x}(t) + D\mathbf{x}(\lambda t), \quad 0 < \lambda < 1, \ t \ge 0, \tag{4.2}$$

where C and D are constant real matrices. The corresponding discretization becomes

$$\begin{aligned}\mathbf{y}(n+1) = \mathbf{y}(n) + hC(\Theta\mathbf{y}(n+1) + (1-\Theta)\mathbf{y}(n)) + hD\big(\gamma_1(n)\mathbf{y}(\lfloor\lambda n\rfloor) + \gamma_2(n)\mathbf{y}(\lfloor\lambda n\rfloor + 1)\big) \\ + hD\big(\gamma_3(n)\mathbf{y}(\lfloor\lambda(n+1)\rfloor) + \gamma_4(n)\mathbf{y}(\lfloor\lambda(n+1)\rfloor + 1)\big),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\gamma_1(n) = (1-\Theta)(1 + \lfloor\lambda n\rfloor - \lambda n), \qquad \gamma_2(n) = 1 - \Theta - \gamma_1(n), \\ \gamma_3(n) = \Theta(1 + \lfloor\lambda(n+1)\rfloor - \lambda(n+1)), \qquad \gamma_4(n) = \Theta - \gamma_3(n),\end{aligned} \tag{4.4}$$

$n = 0, 1, \ldots$, and for the well-posedness of the method we assume that $I - h\Theta C$ is nonsingular. Then (4.3) is obviously the difference system of the type (1.1), where $A(n) = A$ is constant matrix, $B_k(n)$ are variable, but bounded matrix functions, $\alpha(n) = \lfloor\lambda n\rfloor$ ($\lfloor\ \rfloor$ means an integer part), and $p = 2$.

On this account, we are going to discuss the question of a possible usefulness of Theorem 3.1 for asymptotic description of such a system. First we state the following.

Lemma 4.1. *Consider the inequality* (2.3)*, where* $0 < \mu < 1$*,* $\|B_k(n)\| \le b_k$*, and* $\alpha(n) = \lfloor\lambda n\rfloor$*,* $0 < \lambda < 1$*,* $n \in \mathbb{N}(n_0)$*,* $k = 0, 1, \ldots, p$*. Let*

$$b = \sum_{k=0}^{p} b_k, \qquad r = \log_\lambda \frac{1-\mu}{b}. \tag{4.5}$$

Then

$$\omega(n) = \begin{cases} \left(n - \dfrac{p}{1-\lambda}\right)^r & \text{if } b \ge 1 - \mu, \\ \left(n + \dfrac{p}{1-\lambda}\right)^r & \text{if } b < 1 - \mu \end{cases} \tag{4.6}$$

is the monotonous solution of (2.3) *satisfying the properties* (2.6) *and* (2.7)*.*

Proof. First we prove that $\omega(n)$ is the solution of (2.3). Let $b \ge 1-\mu$. Then $r \ge 0$ and substituting into (2.3) we have

$$\sum_{k=0}^{p} \|B_k(n)\|\omega(\lfloor\lambda n\rfloor + k) \le b\left(\lfloor\lambda n\rfloor + p - \frac{p}{1-\lambda}\right)^r \le b\left(\lambda n - \lambda\frac{p}{1-\lambda}\right)^r = (1-\mu)\omega(n). \tag{4.7}$$

The case $b < 1 - \mu$ can be verified analogously. Further, we show that $\omega(n)$ satisfies (2.6) and (2.7). It is enough to consider the case $b < 1 - \mu$ only, because in the opposite case $\Delta^{-}\omega(n)$ is identically zero. If $b < 1 - \mu$, then

$$\Delta^{-}\omega(n) = \Delta\omega(n) = \left(n + 1 + \frac{1}{1-\lambda}\right)^r - \left(n + \frac{1}{1-\lambda}\right)^r \tag{4.8}$$

and (2.6) obviously holds with respect to $r < 0$. To verify (2.7) we utilize the relation

$$\lambda^{-1}(n-p) - 1 < \beta(n) < \lambda^{-1}(n+1) \tag{4.9}$$

following from the definition of $\beta(n)$. Then using the mean value theorem we get

$$\begin{aligned} -\Delta^{-}\omega(n) &= \left(n + \frac{1}{1-\lambda}\right)^{r} - \left(n+1+\frac{1}{1-\lambda}\right)^{r} \le -r\left(n+\frac{1}{1-\lambda}\right)^{r-1}, \\ \omega(\beta(n)) &> \omega\left(\lambda^{-1}(n+1)\right) = \left(\lambda^{-1}n + \frac{1}{\lambda(1-\lambda)}\right)^{r}. \end{aligned} \tag{4.10}$$

From here we can deduce that

$$\frac{-\Delta^{-}\omega(n)}{\omega(\beta(n))} \le \frac{-r(n+1/(1-\lambda))^{r-1}}{\lambda^{-r}(n+1/(1-\lambda))^{r}} = O\left(\frac{1}{n}\right) \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

Further, it follows from (4.9) that $\sum_{j=1}^{\infty} 1/\beta^{j}(n_0)$ converges, and hence (2.7) holds. □

We show that Lemma 4.1 enables us to formulate an effective asymptotic result for solutions of the discretization (4.3). As remarked earlier, this recurrence relation can be rewritten as the difference system (1.1) in a special form

$$\mathbf{y}(n+1) = A\mathbf{y}(n) + \sum_{k=0}^{p} B_k(n)\mathbf{y}(\lfloor \lambda n \rfloor + k), \quad 0 < \lambda < 1,\ n \in \mathbb{N}(n_0) \tag{4.12}$$

with a constant matrix A and bounded matrices $B_k(n)$. Let $\rho(A)$ be the spectral radius of the matrix A and assume that A has a complete set of eigenvectors. Then the system (2.1) is uniformly asymptotically stable if and only if $\rho(A) < 1$ and the inequality (2.2) becomes

$$\left\| Y(n)Y^{-1}(s) \right\| \le M(\rho(A))^{n-s} \quad \forall n \ge s \in \mathbb{N}(n_0) \tag{4.13}$$

(see [10]). Similarly we can modify the relation (4.5) with μ being replaced by $\rho(A)$. It remains to dispose with the form of $\varphi(n)$ satisfying (2.8) and (2.9). Because of the inequality (4.9) we consider two auxiliary Abel equations with a continuous argument, namely,

$$\begin{aligned} \varphi_{*}\left(\lambda^{-1}(t+1)\right) &= \varphi_{*}(t) + 1, \\ \varphi^{*}\left(\lambda^{-1}(t-p) - 1\right) &= \varphi^{*}(t) + 1. \end{aligned} \tag{4.14}$$

We can easily check that the functions

$$\varphi_{*}(t) = -\log_{\lambda}\left(t + \frac{1}{1-\lambda}\right) + C_1, \qquad \varphi^{*}(t) = -\log_{\lambda}\left(t - \frac{p+\lambda}{1-\lambda}\right) + C_2, \tag{4.15}$$

where C_1, C_2 are arbitrary real constants, satisfy these equations on corresponding domains. Taking into account the property (2.9) we can see that asymptotics of $\varphi(n)$ satisfying (2.8) and (2.9) are described via $-\log_\lambda n$. This observation is sufficient for our purpose.

Summarizing previous considerations, we have derived the following.

Corollary 4.2. *Consider the difference equation* (4.12)*, where A is a nonsingular constant matrix with a complete set of eigenvectors such that $\rho(A) < 1$ and let $\|B_k(n)\| \le b_k$, $n \in \mathbb{N}(n_0)$, $k = 0, 1, \ldots, p$. If $\mathbf{y}(n)$ is a solution of* (4.12)*, then*

$$\mathbf{y}(n) = O\left(n^r M^{-\log_\lambda n}\right) \quad \text{as } n \longrightarrow \infty, \tag{4.16}$$

where $r = \log_\lambda((1 - \rho(A))/b)$, $b = \sum_{k=0}^{p} b_k$ and M is given by (4.13)*.*

Remark 4.3. The previous ideas and procedures can be extended into a more general case. Consider the differential equation (4.2) with the delayed argument $\tau(t) = \lambda t$ replaced by a general $\tau(t)$ such that $\tau'(t)$ is continuous and nonincreasing on I and $0 < \tau'(t) \le \lambda < 1$ for all $t \in I$. Then using the above mentioned discretization we arrive at (4.12) with $\lfloor \lambda n \rfloor$ being replaced by $\alpha(n)$ having some specific properties (in particular, $n - \alpha(n) \to \infty$ as $n \to \infty$). Employing the Schröder transformation $u = \eta(t)$, where $\eta(t)$ is a differentiable solution of the relation $\eta(\tau(t)) = \lambda\eta(t)$, we can similarly as in the case $\alpha(n) = \lfloor \lambda n \rfloor$ verify that the property (4.16) remains preserved with n being replaced by $\eta(n)$ on its right-hand side. A detailed modification of the above utilized technique to this general case is only a technical matter and we omit it.

We recall that the asymptotic estimate (4.16) follows from the general Theorem 3.1. Our next intention is to show that considering a simplified form of (4.12) we can modify the proof of Theorem 3.1 to obtain a stronger estimate of solutions.

Let $p = 0$ and $B_0(n) = B$ in (4.12) (a numerical interpretation of such an equation is discussed later). If the matrix A has a complete set of eigenvectors, then it is diagonalizable. Therefore we can choose the basis such that A is diagonal and consider the difference system

$$\mathbf{y}(n+1) = A\mathbf{y}(n) + B\mathbf{y}(\lfloor \lambda n \rfloor), \quad 0 < \lambda < 1, \ n \in \mathbb{N}(n_0), \tag{4.17}$$

where $\mathbf{y} = (y_i)$, $A = \operatorname{diag}(a_i)$, and $B = (b_{ij})$ are constant (possibly complex) matrices. To formulate a stronger estimate of its solutions, instead of the scalar inequality (2.3) we employ the vector inequality

$$|B|\boldsymbol{\varphi}(\lfloor \lambda n \rfloor) \le (I - |A|)\boldsymbol{\varphi}(n) \tag{4.18}$$

and discuss the existence of a positive solution $\boldsymbol{\varphi} = (\varphi_i)$ of (4.18). We note that the symbols $|A|$, $|B|$ mean matrices $|A| = \operatorname{diag}(|a_i|)$, $|B| = (|b_{ij}|)$ and the inequality (4.18) between two vectors represents m inequalities between their corresponding coordinates. Similarly, by a positive (nonnegative) vector or matrix we understand such a vector or matrix whose all coordinates or elements are positive (nonnegative).

First we recall the following notion.

Definition 4.4. We say that $m \times m$ matrix $Q = (q_{ij})$ is reducible if there exists a partition of the set $\{1, 2, \ldots, m\}$ into two complementary systems $\{i_1, \ldots, i_u\}$ and $\{j_1, \ldots, j_v\}$ $(u + v = m)$ such that $q_{ij} = 0$ whenever $i \in \{i_1, \ldots, i_u\}$ and $j \in \{j_1, \ldots, j_v\}$. Otherwise Q is called irreducible.

Irreducibility plays an important role in the spectral theory of nonnegative matrices. We utilize here the following assertion which is often referred to as the Frobenius theorem (see, e.g., [14]).

Theorem 4.5. *Every irreducible nonnegative matrix Q has a positive eigenvalue such that the corresponding eigenvector is positive. This eigenvalue is the spectral radius of Q and it is the simple root of the characteristic equation.*

If $\rho(A) < 1$, we rewrite the inequality (4.18) as

$$Q\boldsymbol{\psi}(\lfloor \lambda n \rfloor) \leq \boldsymbol{\psi}(n), \tag{4.19}$$

where $Q = (I - |A|)^{-1}|B|$. In addition, if B is irreducible, then Q is irreducible and nonnegative. Hence, by the Frobenius theorem, Q has the eigenvalue $\rho(Q)$ with the corresponding positive eigenvector $\boldsymbol{\xi}$. Using this notation we have the following.

Proposition 4.6. *Let $A = \operatorname{diag}(a_i)$ be a diagonal matrix with $|a_i| < 1$, $i = 1, \ldots, m$ and let $B = (b_{ij})$ be irreducible. Then the inequality* (4.18) *admits the positive solution:*

$$\boldsymbol{\psi}(n) = \begin{cases} \boldsymbol{\xi}(\rho(Q))^{-\log_\lambda n} & \text{if } \rho(Q) \geq 1, \\ \boldsymbol{\xi}(\rho(Q))^{-\log_\lambda (n + 1/(1-\lambda))} & \text{if } \rho(Q) < 1. \end{cases} \tag{4.20}$$

Proof. Let $\rho(Q) \geq 1$. Then all coordinates of $\boldsymbol{\psi}(n)$ are nondecreasing and we can write

$$|B|\boldsymbol{\psi}(\lfloor \lambda n \rfloor) = |B|\boldsymbol{\xi}(\rho(Q))^{-\log_\lambda(\lfloor \lambda n \rfloor)} \leq |B|\boldsymbol{\xi}(\rho(Q))^{-\log_\lambda(\lambda n)}. \tag{4.21}$$

It suffices to show that

$$|B|\boldsymbol{\xi}(\rho(Q))^{-\log_\lambda(\lambda n)} = (I - |A|)\boldsymbol{\xi}(\rho(Q))^{-\log_\lambda n} \tag{4.22}$$

which is equivalent to

$$Q\boldsymbol{\xi} = (\rho(Q))\boldsymbol{\xi}. \tag{4.23}$$

The required spectral properties of Q now follow from the Frobenius theorem.

The case $\rho(Q) < 1$ can be proved quite analogously. $\square$

Utilizing Proposition 4.6 we can prove the following.

Theorem 4.7. *Consider the difference system* (4.17)*, where* $A = \operatorname{diag}(a_i)$ *and* $B = (b_{ij})$ *are constant matrices such that* $|a_i| < 1$, $i = 1, \dots, m$ *and* B *is irreducible. Then*

$$\mathbf{y}(n) = O\left(n^{-\log_\lambda \rho(Q)}\right), \quad Q = (I - |A|)^{-1}|B| \quad \text{as } n \longrightarrow \infty \tag{4.24}$$

holds for any solution $\mathbf{y}(n)$ *of* (4.17).

Proof. The proof technique is a modification of the procedures employed in the proof of Theorem 3.1. Therefore we outline a sketch of it. The system (4.17) can be rewritten as

$$y_i(n+1) = a_i y_i(n) + \sum_{j=1}^{m} b_{ij} y_j(\lfloor \lambda n \rfloor), \quad i = 1, \dots, m. \tag{4.25}$$

If $a_i \neq 0$, we can divide the ith equation from (4.25) by a_i^{n+1} to obtain

$$a_i^{-n-1} y_i(n+1) = a_i^{-n} y_i(n) + a_i^{-n-1} \sum_{j=1}^{m} b_{ij} y_j(\lfloor \lambda n \rfloor), \tag{4.26}$$

that is,

$$\Delta\left(a_i^{-s} y_i(s)\right) = a_i^{-s-1} \sum_{j=1}^{m} b_{ij} y_j(\lfloor \lambda s \rfloor). \tag{4.27}$$

Let the symbols n_ℓ, I_ℓ, and S_ℓ have the same meaning as in the proof of Theorem 3.1 (with $\beta(n)$ given by (2.4), where $\alpha(n) = \lfloor \lambda n \rfloor$ and $p = 0$). If we choose $n \in I_{\ell+1}$, $n > n_\ell$, and sum (4.27) from n_ℓ to $n-1$, we arrive at

$$y_i(n) = a_i^{n-n_\ell} y_i(n_\ell) + \sum_{s=n_\ell}^{n-1} a_i^{n-s-1} \sum_{j=1}^{m} b_{ij} y_j(\lfloor \lambda s \rfloor). \tag{4.28}$$

Put $z_i(n) = y_i(n)/\psi_i(n)$, where $\boldsymbol{\psi}(n) = (\psi_i(n))$ is given by (4.20). Then

$$z_i(n) = \frac{a_i^{n-n_\ell} \psi_i(n_\ell)}{\psi_i(n)} z_i(n_\ell) + \sum_{s=n_\ell}^{n-1} \frac{a_i^{n-s-1}}{\psi_i(n)} \sum_{j=1}^{m} b_{ij} \psi_j(\lfloor \lambda s \rfloor) z_j(\lfloor \lambda s \rfloor), \tag{4.29}$$

hence

$$\begin{aligned} |z_i(n)| &\leq S_\ell \frac{|a_i|^{n-n_\ell} \psi_i(n_\ell)}{\psi_i(n)} + S_\ell \sum_{s=n_\ell}^{n-1} \frac{|a_i|^{n-s-1}}{\psi_i(n)} \sum_{j=1}^{m} |b_{ij}| \psi_j(\lfloor \lambda s \rfloor) \\ &\leq S_\ell \left(\frac{|a_i|^{n-n_\ell} \psi_i(n_\ell)}{\psi_i(n)} + \sum_{s=n_\ell}^{n-1} \frac{|a_i|^{n-s-1}}{\psi_i(n)} (1 - |a_i|) \psi_i(s) \right) \end{aligned} \tag{4.30}$$

by the use of Proposition 4.6. This inequality is just the inequality (3.5). Consequently, using the same proof procedures as given in the corresponding part of the proof of Theorem 3.1 and taking into account Lemma 4.1, we can show the boundedness of S_ℓ as $\ell \to \infty$.

If $a_i = 0$, then

$$z_i(n) = \frac{1}{\varphi_i(n)} \sum_{j=1}^{m} b_{ij} \varphi_j(\lfloor \lambda(n-1) \rfloor) z_j(\lfloor \lambda(n-1) \rfloor), \tag{4.31}$$

and Proposition 4.6 implies

$$|z_i(n)| \le \frac{S_\ell}{\varphi_i(n)} \sum_{j=1}^{m} |b_{ij}| \varphi_j(\lfloor \lambda(n-1) \rfloor) \le S_\ell \frac{\varphi_i(n-1)}{\varphi_i(n)}. \tag{4.32}$$

The boundedness of S_ℓ as $\ell \to \infty$ now follows from the corresponding properties of $\varphi_i(n)$. □

Remark 4.8. Our final goal is to give a numerical interpretation to Theorem 4.7. Consider again the pantograph equation (4.2) with a (generally complex) diagonal matrix C. Applying the forward Euler method with a piecewise constant interpolation to such an equation (4.2) we obtain the linear difference system (4.17), where $A = I + hC$ and $B = hD$. Since $|a_i| < 1$ if and only if $2\,\mathrm{Re}\,c_i < -h|c_i|^2$, we can reformulate Theorem 4.7 as follows.

Corollary 4.9. *Consider* (4.2), *where* $C = \operatorname{diag}(c_i)$ *and* $D = (d_{ij})$ *are constant matrices such that* $2\,\mathrm{Re}\,c_i < -h|c_i|^2$, $i = 1, \dots, m$ *and D is irreducible. Let* $\mathbf{y}(n)$ *be an approximate value of* $\mathbf{x}(nh)$ *calculated via the recurrence relation* (4.17), *where* $A = I + hC$, $B = hD$, *and* $h > 0$ *is the stepsize. Then*

$$\mathbf{y}(n) = O\left(n^{-\log_\lambda \rho(Q)}\right), \quad Q = (I - |I + hC|)^{-1} h|D| \text{ as } n \longrightarrow \infty. \tag{4.33}$$

In particular, if C is a real diagonal matrix with negative diagonal elements, $I + hC$ is nonnegative (note that this is actually a stepsize restriction), and D is a nonnegative irreducible matrix, then (4.33) becomes

$$\mathbf{y}(n) = O\left(n^{-\log_\lambda \rho(C^{-1}D)}\right) \quad \text{as } n \longrightarrow \infty. \tag{4.34}$$

It should be emphasized that exactly the same (nonimprovable) asymptotic estimate holds for the solution $\mathbf{x}(t)$ of the exact pantograph equation (4.2) provided that all eigenvalues of C have negative real parts and the maximal geometric multiplicity of eigenvalues of $C^{-1}D$ with the maximal real part is equal to 1 (see [15, 16]). Consequently, the estimate (4.34) of the corresponding numerical solution is optimal in this sense.

In general, the problem of a possible mutual relationship between qualitative properties of delay differential and difference equations is not restricted only to numerical investigations and it is studied in a more general context (see, e.g., the papers [17, 18]). However, it is just the numerical analysis of delay differential equations which gives an important motivation for the joint study of both types of delay equations. Related questions concerning joint investigations of delay differential equations and its discretizations in the framework of the time-scale theory are discussed in [8, 19]. Among many papers on numerical investigations of differential equations with a proportional delay we can mention at least [9, 20–22] and the references cited therein. We emphasize that these papers deal with scalar equations of the pantograph type. The investigations of the vector case, mentioned in this paper, are just at the beginning.

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