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*Research Article*

# On the Oscillation of Second-Order Neutral Delay Differential Equations

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Some new oscillation criteria for the second-order neutral delay differential equation $(r(t)z'(t))' + q(t)x(\sigma(t)) = 0$, $t \geq t_0$ are established, where $\int_{t_0}^{\infty}(1/r(t))dt = \infty$, $z(t) = x(t) + p(t)x(\tau(t))$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) > 0$. These oscillation criteria extend and improve some known results. An example is considered to illustrate the main results.

## 1. Introduction

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [1]. In recent years, many studies have been made on the oscillatory behavior of solutions of neutral delay differential equations, and we refer to the recent papers [2–23] and the references cited therein.

This paper is concerned with the oscillatory behavior of the second-order neutral delay differential equation

$$\left(r(t)z'(t)\right)' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$.

In what follows we assume that

$(I_1)$ $p, q \in C([t_0, \infty), R)$, $0 \le p(t) \le p_0 < \infty$, $q(t) > 0$,

$(I_2)$ $r \in C([t_0, \infty), R)$, $r(t) > 0$, $\int_{t_0}^{\infty}(1/r(t))\mathrm{d}t = \infty$,

$(I_3)$ $\tau, \sigma \in C([t_0, \infty), R)$, $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) = \tau_0 > 0$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$, $\tau(\sigma(t)) = \sigma(\tau(t))$, where $\tau_0$ is a constant.

Some known results are established for (1.1) under the condition $0 \le p(t) < 1$. Grammatikopoulos et al. [6] obtained that if $0 \le p(t) \le 1$, $q(t) \ge 0$ and, $\int_{t_0}^{\infty} q(s)[1-p(s-\sigma)]\mathrm{d}s = \infty$, then the second-order neutral delay differential equation

$$\left[y(t) + p(t)y(t-\tau)\right]'' + q(t)y(t-\sigma) = 0 \tag{1.2}$$

oscillates. In [13], by employing Riccati technique and averaging functions method, Ruan established some general oscillation criteria for second-order neutral delay differential equation

$$\left[a(t)\left(x(t) + p(t)x(t-\tau)\right)'\right]' + q(t)f(x(t-\sigma)) = 0. \tag{1.3}$$

Xu and Meng [18] as well as Zhuang and Li [23] studied the oscillation of the second-order neutral delay differential equation

$$\left[r(t)\left(y(t) + p(t)y(\tau(t))\right)'\right]' + \sum_{i=1}^{n} q_i(t) f_i\left(y(\sigma_i(t))\right) = 0. \tag{1.4}$$

Motivated by [11], we will further the investigation and offer some more general new oscillation criteria for (1.1), by employing a class of function $Y$, operator $T$, and the Riccati technique and averaging technique.

Following [11], we say that a function $\phi = \phi(t, s, l)$ belongs to the function class $Y$, denoted by $\phi \in Y$ if $\phi \in C(E, R)$, where $E = \{(t, s, l) : t_0 \le l \le s \le t < \infty\}$, which satisfies $\phi(t, t, l) = 0$, $\phi(t, l, l) = 0$, and $\phi(t, s, l) > 0$, for $l < s < t$, and has the partial derivative $\partial\phi/\partial s$ on $E$ such that $\partial\phi/\partial s$ is locally integrable with respect to $s$ in $E$. By choosing the special function $\phi$, it is possible to derive several oscillation criteria for a wide range of differential equations.

Define the operator $T[\cdot; l, t]$ by

$$T\left[g; l, t\right] = \int_l^t \phi(t, s, l) g(s)\mathrm{d}s, \tag{1.5}$$

for $t \ge s \ge l \ge t_0$ and $g \in C^1[t_0, \infty)$. The function $\varphi = \varphi(t, s, l)$ is defined by

$$\frac{\partial\phi(t, s, l)}{\partial s} = \varphi(t, s, l)\phi(t, s, l). \tag{1.6}$$

It is easy to see that $T[\cdot;l,t]$ is a linear operator and that it satisfies

$$T\left[g';l,t\right] = -T\left[g\varphi;l,t\right], \quad \text{for } g(s) \in C^1[t_0,\infty). \tag{1.7}$$

## 2. Main Results

In this section, we give some new oscillation criteria for (1.1). We start with the following oscillation criteria.

**Theorem 2.1.** *If*

$$\int_{t_0}^{\infty} Q(t)dt = \infty, \tag{2.1}$$

*where* $Q(t) := \min\{q(t), q(\tau(t))\}$, *then* (1.1) *oscillates.*

*Proof.* Let $x$ be a nonoscillatory solution of (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) \neq 0$, for all $t \geq t_1$. Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \geq t_1$. From (1.1), we have

$$\left(r(t)z'(t)\right)' = -q(t)x(\sigma(t)) < 0, \quad t \geq t_1. \tag{2.2}$$

Therefore $r(t)z'(t)$ is a decreasing function. We claim that $z'(t) > 0$ for $t \geq t_1$. Otherwise, there exists $t_2 \geq t_1$ such that $z'(t_2) < 0$. Then from (2.2) we obtain

$$r(t)z'(t) \leq r(t_2)z'(t_2), \quad t \geq t_2, \tag{2.3}$$

and hence,

$$z(t) \leq z(t_2) - \left[-r(t_2)z'(t_2)\right]\int_{t_2}^{t} \frac{ds}{r(s)}. \tag{2.4}$$

Taking $t \to \infty$, we get $z(t) \to -\infty$, $t \to \infty$. This contradiction proves that $z'(t) > 0$ for $t \geq t_1$. Using definition of $z(t)$ and applying (1.1), we get for sufficiently large $t$

$$\left(r(t)z'(t)\right)' + q(t)x(\sigma(t)) + p_0 q(\tau(t))x(\sigma(\tau(t))) + \frac{p_0}{\tau'(t)}\left(r(\tau(t))z'(\tau(t))\right)' = 0, \tag{2.5}$$

and thus,

$$\left(r(t)z'(t)\right)' + Q(t)z(\sigma(t)) + \frac{p_0}{\tau'(t)}\left(r(\tau(t))z'(\tau(t))\right)' \leq 0. \tag{2.6}$$

Integrating (2.6) from $t_3$ $(\geq t_1)$ to $t$, we obtain

$$\int_{t_3}^{t}(r(s)z'(s))'\mathrm{d}s+\int_{t_3}^{t}Q(s)z(\sigma(s))\mathrm{d}s+p_0\int_{t_3}^{t}\frac{1}{\tau'(s)}(r(\tau(s))z'(\tau(s)))'\mathrm{d}s\leq 0. \tag{2.7}$$

Noting that $\tau'(t)=\tau_0>0$, we have

$$\begin{aligned}\int_{t_3}^{t}Q(s)z(\sigma(s))\mathrm{d}s&\leq-\int_{t_3}^{t}(r(s)z'(s))'\mathrm{d}s-p_0\int_{t_3}^{t}\frac{1}{(\tau'(s))^2}(r(\tau(s))z'(\tau(s)))'\mathrm{d}(\tau(s))\\&=-\int_{t_3}^{t}(r(s)z'(s))'\mathrm{d}s-\frac{p_0}{\tau_0^2}\int_{\tau(t_3)}^{\tau(t)}(r(u)z'(u))'\mathrm{d}u\\&=r(t_3)z'(t_3)-r(t)z'(t)+\frac{p_0}{\tau_0^2}r(\tau(t_3))z'(\tau(t_3))-\frac{p_0}{\tau_0^2}r(\tau(t))z'(\tau(t)).\end{aligned} \tag{2.8}$$

Since $z'(t)>0$ for $t\geq t_1$, we can find a constant $c>0$ such that $z(\sigma(t))\geq c$ for $t\geq t_3\geq t_1$. Then from (2.8) and the fact that $r(t)z'(t)$ is eventually decreasing, we have

$$\int_{t_3}^{\infty}Q(t)\mathrm{d}t<\infty, \tag{2.9}$$

which is a contradiction to (2.1). This completes the proof. □

**Theorem 2.2.** *Assume that $\sigma(t)\leq\tau(t)$, and there exist functions $\phi\in Y$ and $k\in C^1([t_0,\infty),R^+)$ such that*

$$\limsup_{t\to\infty}T\left[k(s)Q(s)-\frac{(1+(p_0/\tau_0))(\varphi+(k'(s)/k(s)))^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right]>0, \tag{2.10}$$

*where $Q(t)$ is defined as in Theorem 2.1, the operator $T$ is defined by* (1.5), *and $\varphi=\varphi(t,s,l)$ is defined by* (1.6). *Then every solution $x$ of* (1.1) *is oscillatory.*

*Proof.* Let $x$ be a nonoscillatory solution of (1.1). Then there exists $t_1\geq t_0$ such that $x(t)\neq 0$ for all $t\geq t_1$. Without loss of generality, we assume that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$, for all $t\geq t_1$. Define

$$\omega(t)=k(t)\frac{r(t)z'(t)}{z(\sigma(t))},\quad t\geq t_1. \tag{2.11}$$

Then $w(t)>0$ and

$$\omega'(t)=k'(t)\frac{r(t)z'(t)}{z(\sigma(t))}+k(t)\frac{(r(t)z'(t))'z(\sigma(t))-r(t)z'(t)z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}. \tag{2.12}$$

By (2.2) and the fact $z'(t) > 0$, we get

$$\frac{z'(\sigma(t))}{z'(t)} \geq \frac{r(t)}{r(\sigma(t))}. \tag{2.13}$$

From (2.11), (2.12), and (2.13), we have

$$\omega'(t) \leq k(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t). \tag{2.14}$$

Similarly, define

$$\nu(t) = k(t)\frac{r(\tau(t))z'(\tau(t))}{z(\sigma(t))}, \quad t \geq t_1. \tag{2.15}$$

Then $\nu(t) > 0$ and

$$\nu'(t) = k'(t)\frac{r(\tau(t))z'(\tau(t))}{z(\sigma(t))} + k(t)\frac{(r(\tau(t))z'(\tau(t)))'z(\sigma(t)) - r(\tau(t))z'(\tau(t))z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}. \tag{2.16}$$

By (2.2) and the facting $z'(t) > 0$, noting that $\sigma(t) \leq \tau(t)$, we get

$$\frac{z'(\sigma(t))}{z'(\tau(t))} \geq \frac{r(\tau(t))}{r(\sigma(t))}. \tag{2.17}$$

From (2.15), (2.16), and (2.17), we have

$$\nu'(t) \leq k(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))} + \frac{k'(t)}{k(t)}\nu(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t). \tag{2.18}$$

Therefore, from (2.14) and (2.18), we get

$$\begin{aligned}\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq{}& k(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{p_0}{\tau_0}k(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))} \\ &+ \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t) + \frac{p_0}{\tau_0}\frac{k'(t)}{k(t)}\nu(t) - \frac{p_0}{\tau_0}\frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t).\end{aligned} \tag{2.19}$$

From (2.6), we obtain

$$\begin{aligned}\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq{}& -k(t)Q(t) + \frac{k'(t)}{k(t)}\omega(t) - \frac{\sigma'(t)}{r(\sigma(t))k(t)}\omega^2(t) \\ &+ \frac{p_0}{\tau_0}\frac{k'(t)}{k(t)}\nu(t) - \frac{p_0}{\tau_0}\frac{\sigma'(t)}{r(\sigma(t))k(t)}\nu^2(t).\end{aligned} \tag{2.20}$$

Applying $T[\cdot;l,t]$ to (2.20), we get

$$T\left[\omega'(s)+\frac{p_0}{\tau_0}\nu'(s);l,t\right] \le T\left[-k(s)Q(s)+\frac{k'(s)}{k(s)}\omega(s)-\frac{\sigma'(s)}{r(\sigma(s))k(s)}\omega^2(s)+\frac{p_0}{\tau_0}\frac{k'(s)}{k(s)}\nu(s)-\frac{p_0}{\tau_0}\frac{\sigma'(s)}{r(\sigma(s))k(s)}\nu^2(s);l,t\right]. \tag{2.21}$$

By (1.7) and the above inequality, we obtain

$$T[k(s)Q(s);l,t] \le T\left[\left(\varphi+\frac{k'(s)}{k(s)}\right)\omega(s)-\frac{\sigma'(s)}{r(\sigma(s))k(s)}\omega^2(s)+\frac{p_0}{\tau_0}\left(\varphi+\frac{k'(s)}{k(s)}\right)\nu(s)-\frac{p_0}{\tau_0}\frac{\sigma'(s)}{r(\sigma(s))k(s)}\nu^2(s);l,t\right]. \tag{2.22}$$

Hence, from (2.22) we have

$$T[k(s)Q(s);l,t] \le T\left[\left(\frac{(\varphi+(k'(s)/k(s)))^2}{4}+\frac{(p_0/\tau_0)(\varphi+(k'(s)/k(s)))^2}{4}\right)\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right], \tag{2.23}$$

that is,

$$T\left[k(s)Q(s)-\frac{(1+(p_0/\tau_0))(\varphi+(k'(s)/k(s)))^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right] \le 0. \tag{2.24}$$

Taking the super limit in the above inequality, we get

$$\limsup_{t\to\infty} T\left[k(s)Q(s)-\frac{(1+(p_0/\tau_0))(\varphi+(k'(s)/k(s)))^2}{4}\frac{r(\sigma(s))k(s)}{\sigma'(s)};l,t\right] \le 0, \tag{2.25}$$

which contradicts (2.10). This completes the proof. □

*Remark 2.3.* With the different choice of $k$ and $\phi$, Theorem 2.2 can be stated with different conditions for oscillation of (1.1). For example, if we choose $\phi(t,s,l)=\rho(s)(t-s)^{\sigma}(s-l)^{\mu}$ for $\sigma>1/2$, $\mu>1/2$, $\rho\in C^1([t_0,\infty),(0,\infty))$, then

$$\varphi(t,s,l)=\frac{\rho'(s)}{\rho(s)}+\frac{\mu t-(\sigma+\mu)s+\sigma l}{(t-s)(s-l)}. \tag{2.26}$$

By Theorem 2.2 we can obtain the oscillation criterion for (1.1), the details are left to the reader.

For an application, we give the following example to illustrate the main results.

*Example 2.4.* Consider the following equation:

$$(x(t) + 2x(t-\pi))'' + x(t-\pi) = 0, \quad t \geq t_0. \tag{2.27}$$

Let $r(t) = 1$, $p(t) = 2$, $q(t) = 1$, and $\tau(t) = \sigma(t) = t - \pi$, then by Theorem 2.1 every solution of (2.27) oscillates; for example, $x(t) = \sin t$ is an oscillatory solution of (2.27).

*Remark 2.5.* The recent results cannot be applied in (2.27) since $p(t) = 2 > 1$; so our results are new ones.

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