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Research Article

Exponential Decay of Energy for Some Nonlinear Hyperbolic Equations with Strong Dissipation

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The initial boundary value problem for a class of hyperbolic equations with strong dissipative term $u_{tt} - \sum_{i=1}^{n}(\partial/\partial x_i)(|\partial u/\partial x_i|^{p-2}(\partial u/\partial x_i)) - a\Delta u_t = b|u|^{r-2}u$ in a bounded domain is studied. The existence of global solutions for this problem is proved by constructing a stable set in $W_0^{1,p}(\Omega)$ and showing the exponential decay of the energy of global solutions through the use of an important lemma of V. Komornik.

1. Introduction

We are concerned with the global solvability and exponential asymptotic stability for the following hyperbolic equation in a bounded domain:

$$u_{tt} - \Delta_p u - a\Delta u_t = b|u|^{r-2}u, \quad x \in \Omega, \quad t > 0 \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{1.2}$$

and boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \; t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $a, b > 0$ and $r, p > 2$ are real numbers, and $\Delta_p = \sum_{i=1}^{n}(\partial/\partial x_i)(|\partial/\partial x_i|^{p-2}(\partial/\partial x_i))$ is a divergence operator (degenerate Laplace operator) with $p > 2$, which is called a p-Laplace operator.

Equations of type (1.1) are used to describe longitudinal motion in viscoelasticity mechanics and can also be seen as field equations governing the longitudinal motion of a viscoelastic configuration obeying the nonlinear Voight model [1–4].

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data [4–6]. For $a = 0$, the source term causes finite time blow up of solutions with negative initial energy if $r > p$ [7].

In [8–10], Yang studied the problem (1.1)–(1.3) and obtained global existence results under the growth assumptions on the nonlinear terms and initial data. These global existence results have been improved by Liu and Zhao [11] by using a new method. As for the nonexistence of global solutions, Yang [12] obtained the blow up properties for the problem (1.1)–(1.3) with the following restriction on the initial energy $E(0) < \min\{-(rk_1 + pk_2/r - p)^{1/\delta}, -1\}$, where $r > p$ and k_1, k_2, and δ are some positive constants.

Because the p-Laplace operator Δ_p is nonlinear operator, the reasoning of proof and computation are greatly different from the Laplace operator $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$. By means of the Galerkin method and compactness criteria and a difference inequality introduced by Nakao [13], Ye [14, 15] has proved the existence and decay estimate of global solutions for the problem (1.1)–(1.3) with inhomogeneous term $f(x,t)$ and $p \geq r$.

In this paper we are going to investigate the global existence for the problem (1.1)–(1.3) by applying the potential well theory introduced by Sattinger [16], and we show the exponential asymptotic behavior of global solutions through the use of the lemma of Komornik [17].

We adopt the usual notation and convention. Let $W^{k,p}(\Omega)$ denote the Sobolev space with the norm $\|u\|_{W^{k,p}(\Omega)} = (\sum_{|\alpha|\leq k}\|D^\alpha u\|_{L^p(\Omega)}^p)^{1/p}$ and $W_0^{k,p}(\Omega)$ denote the closure in $W^{k,p}(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm, and write equivalent norm $\|\nabla\cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants, and it may be different at each appearance.

2. The Global Existence and Nonexistence

In order to state and study our main results, we first define the following functionals:

$$K(u) = \|\nabla u\|_p^p - b\|u\|_r^r,$$
$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r, \tag{2.1}$$

for $u \in W_0^{1,p}(\Omega)$. Then we define the stable set H by

$$H = \left\{u \in W_0^{1,p}(\Omega), \quad K(u) > 0, \quad J(u) < d\right\} \cup \{0\}, \tag{2.2}$$

where

$$d = \inf\left\{ \sup_{\lambda>0} J(\lambda u), \quad u \in W_0^{1,p}(\Omega)/\{0\} \right\}. \tag{2.3}$$

We denote the total energy associated with (1.1)–(1.3) by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r = \frac{1}{2}\|u_t\|^2 + J(u) \tag{2.4}$$

for $u \in W_0^{1,p}(\Omega)$, $t \geq 0$, and $E(0) = (1/2)\|u_1\|^2 + J(u_0)$ is the total energy of the initial data.

Definition 2.1. The solution $u(x,t)$ is called the weak solution of the problem (1.1)–(1.3) on $\Omega \times [0,T)$, if $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$ and $u_t \in L^\infty(0,T; L^2(\Omega))$ satisfy

$$(u_t, v) - \int_0^t (\Delta_p u, v)d\tau + a(\nabla u, \nabla v) = b\int_0^t \left(|u|^{r-2}u, v\right)d\tau + (u_1, v) + a(\nabla u_0, \nabla v) \tag{2.5}$$

for all $v \in W_0^{1,p}(\Omega)$ and $u(x,0) = u_0(x)$ in $W_0^{1,p}(\Omega)$, $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$.

We need the following local existence result, which is known as a standard one (see [14, 18, 19]).

Theorem 2.2. *Suppose that $2 < p < r < np/(n-p)$ if $p < n$ and $2 < p < r < \infty$ if $n \leq p$. If $u_0 \in W_0^{1,p}(\Omega)$, $u_1 \in L^2(\Omega)$, then there exists $T > 0$ such that the problem* (1.1)–(1.3) *has a unique local solution $u(t)$ in the class*

$$u \in L^\infty\left([0,T); W_0^{1,p}(\Omega)\right), \qquad u_t \in L^\infty\left([0,T); L^2(\Omega)\right). \tag{2.6}$$

For latter applications, we list up some lemmas.

Lemma 2.3 (see [20, 21]). *Let $u \in W_0^{1,p}(\Omega)$, then $u \in L^q(\Omega)$, and the inequality $\|u\|_q \leq C\|u\|_{W_0^{1,p}(\Omega)}$ holds with a constant $C > 0$ depending on Ω, p, and q, provided that,* (i) $2 \leq q < +\infty$ *if $2 \leq n \leq p$ and* (ii) $2 \leq q \leq np/(n-p)$, $2 < p < n$.

Lemma 2.4. *Let $u(t,x)$ be a solution to problem* (1.1)–(1.3)*. Then $E(t)$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -a\|\nabla u_t(t)\|^2. \tag{2.7}$$

Proof. By multiplying (1.1) by u_t and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p - \frac{b}{r}\frac{d}{dt}\|u\|_r^r = -a\|\nabla u_t(t)\|^2, \tag{2.8}$$

which implies from (2.4) that

$$\frac{d}{dt}E(u(t)) = -a\|\nabla u_t(t)\|^2 \leq 0. \tag{2.9}$$

Therefore, $E(t)$ is a nonincreasing function on t. □

Lemma 2.5. *Let $u \in W_0^{1,p}(\Omega)$; if the hypotheses in Theorem 2.2 hold, then $d > 0$.*

Proof. Since

$$J(\lambda u) = \frac{\lambda^p}{p}\|\nabla u\|_p^p - \frac{b\lambda^r}{r}\|u\|_r^r, \tag{2.10}$$

so, we get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda^{p-1}\|\nabla u\|_p^p - b\lambda^{r-1}\|u\|_r^r. \tag{2.11}$$

Let $(d/d\lambda)J(\lambda u) = 0$, which implies that

$$\lambda_1 = b^{-1/(r-p)}\left(\frac{\|u\|_r^r}{\|\nabla u\|_p^p}\right)^{-1/(r-p)}. \tag{2.12}$$

As $\lambda = \lambda_1$, an elementary calculation shows that

$$\frac{d^2}{d\lambda^2}J(\lambda u) < 0. \tag{2.13}$$

Hence, we have from Lemma 2.3 that

$$\begin{aligned} \sup_{\lambda\geq 0} J(\lambda u) = J(\lambda_1 u) &= \frac{r-p}{rp}b^{-p/(r-p)}\left(\frac{\|u\|_r}{\|\nabla u\|_p}\right)^{-rp/(r-p)} \\ &\geq \frac{r-p}{rp}(bC^r)^{-p/(r-p)} > 0. \end{aligned} \tag{2.14}$$

We get from the definition of d that $d > 0$. □

Lemma 2.6. *Let $u \in H$, then*

$$\frac{r-p}{rp}\|\nabla u\|_p^p < J(u). \tag{2.15}$$

Proof. By the definition of $K(u)$ and $J(u)$, we have the following identity:

$$rJ(u) = K(u) + \frac{r-p}{p}\|\nabla u\|_p^p. \tag{2.16}$$

Since $u \in H$, so we have $K(u) > 0$. Therefore, we obtain from (2.16) that

$$\frac{r-p}{rp}\|\nabla u\|_p^p \leq J(u). \tag{2.17}$$

□

In order to prove the existence of global solutions for the problem (1.1)-(1.3), we need the following lemma.

Lemma 2.7. *Suppose that* $2 < p < r < np/(n-p)$ *if* $p < n$ *and* $2 < p < r < \infty$ *if* $n \leq p$. *If* $u_0 \in H,\ u_1 \in L^2(\Omega)$, *and* $E(0) < d$, *then* $u \in H$, *for each* $t \in [0,T)$.

Proof. Assume that there exists a number $t^* \in [0,T)$ such that $u(t) \in H$ on $[0,t^*)$ and $u(t^*) \notin H$. Then, in virtue of the continuity of $u(t)$, we see that $u(t^*) \in \partial H$. From the definition of H and the continuity of $J(u(t))$ and $K(u(t))$ in t, we have either

$$J(u(t^*)) = d, \tag{2.18}$$

or

$$K(u(t^*)) = 0. \tag{2.19}$$

It follows from (2.4) that

$$J(u(t^*)) = \frac{1}{p}\|\nabla u(t^*)\|_p^p - \frac{b}{r}\|u(t^*)\|_r^r \leq E(t^*) \leq E(0) < d. \tag{2.20}$$

So, case (2.18) is impossible.

Assume that (2.19) holds, then we get that

$$\frac{d}{d\lambda}J(\lambda u(t^*)) = \lambda^{p-1}\left(1-\lambda^{r-p}\right)\|\nabla u\|_p^p. \tag{2.21}$$

We obtain from $(d/d\lambda)J(\lambda u(t^*)) = 0$ that $\lambda = 1$.

Since

$$\left.\frac{d^2}{d\lambda^2}J(\lambda u(t^*))\right|_{\lambda=1} = -(r-p)\|\nabla u(t^*)\|_p < 0, \tag{2.22}$$

consequently, we get from (2.20) that

$$\sup_{\lambda\geq 0} J(\lambda u(t^*)) = J(\lambda u(t^*))|_{\lambda=1} = J(u(t^*)) < d, \tag{2.23}$$

which contradicts the definition of d. Therefore, case (2.19) is impossible as well. Thus, we conclude that $u(t) \in H$ on $[0,T)$. □

Theorem 2.8. *Assume that* $2 < p < r < np/(n-p)$ *if* $p < n$ *and* $2 < p < r < \infty$ *if* $n \leq p$. $u(t)$ *is a local solution of problem* (1.1)–(1.3) *on* $[0,T)$. *If* $u_0 \in H$, $u_1 \in L^2(\Omega)$, *and* $E(0) < d$, *then the solution* $u(t)$ *is a global solution of the problem* (1.1)–(1.3).

Proof. It suffices to show that $\|u_t\|^2 + \|\nabla u\|_p^p$ is bounded independently of t.

Under the hypotheses in Theorem 2.8, we get from Lemma 2.7 that $u(t) \in H$ on $[0,T)$. So formula (2.15) in Lemma 2.6 holds on $[0,T)$. Therefore, we have from (2.15) and Lemma 2.4 that

$$\frac{1}{2}\|u_t\|^2 + \frac{r-p}{rp}\|\nabla u\|_p^p \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0) < d. \tag{2.24}$$

Hence, we get

$$\|u_t\|^2 + \|\nabla u\|_p^p \leq \max\left(2, \frac{rp}{r-p}\right) d < +\infty. \tag{2.25}$$

The above inequality and the continuation principle lead to the global existence of the solution, that is, $T = +\infty$. Thus, the solution $u(t)$ is a global solution of the problem (1.1)–(1.3). □

Now we employ the analysis method to discuss the blow-up solutions of the problem (1.1)–(1.3) in finite time. Our result reads as follows.

Theorem 2.9. *Suppose that* $2 < p < r < np/(n-p)$ *if* $p < n$ *and* $2 < p < r < \infty$ *if* $n \leq p$. *If* $u_0 \in H$, $u_1 \in L^2(\Omega)$, *assume that the initial value is such that*

$$E(0) < Q_0, \qquad \|u(0)\|_r > S_0, \tag{2.26}$$

where

$$Q_0 = \frac{r-p}{rp} C^{pr/(p-r)}, \qquad S_0 = C^{p/(p-r)} \tag{2.27}$$

with $C > 0$ *is a positive Sobolev constant. Then the solution of the problem* (1.1)–(1.3) *does not exist globally in time.*

Proof. On the contrary, under the conditions in Theorem 2.9, let $u(x,t)$ be a global solution of the problem (1.1)–(1.3); then by Lemma 2.3, it is well known that there exists a constant $C > 0$ depending only on n, p, and r such that $\|u\|_r \leq C\|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$.

From the above inequality, we conclude that

$$\|\nabla u\|_p^p \geq C^{-p}\|u\|_r^p. \tag{2.28}$$

By using (2.28), it follows from the definition of $E(t)$ that

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + J(u(t)) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r \\ &\geq \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r \geq \frac{1}{pC^p}\|u\|_r^p - \frac{b}{r}\|u\|_r^r. \end{aligned} \tag{2.29}$$

Setting

$$s = s(t) = \|u(t)\|_r = \left\{\int_\Omega |u(x,t)|^r dx\right\}^{1/r}, \tag{2.30}$$

we denote the right side of (2.29) by $Q(s) = Q(\|u(t)\|_r)$, then

$$Q(s) = \frac{1}{pC^p}s^p - \frac{b}{r}s^r, \qquad s \geq 0. \tag{2.31}$$

We have

$$Q'(s) = C^{-p}s^{p-1} - bs^{r-1}. \tag{2.32}$$

Letting $Q'(t) = 0$, we obtain $S_0 = (bC^p)^{1/(p-r)}$.

As $s = S_0$, we have

$$Q''(s)\big|_{s=S_0} = \left(\frac{p-1}{C^p}s^{p-2} - b(r-1)s^{r-2}\right)\Big|_{s=S_0} = (p-r)\left(b^{p-2}C^{(r-2)p}\right)^{1/(p-r)} < 0. \tag{2.33}$$

Consequently, the function $Q(s)$ has a single maximum value Q_0 at S_0, where

$$Q_0 = Q(S_0) = \frac{1}{pC^p}(bC^p)^{p/(p-r)} - \frac{b}{r}(bC^p)^{r/(p-r)} = \frac{r-p}{rp}(b^pC^{pr})^{1/(p-r)}. \tag{2.34}$$

Since the initial data is such that $E(0),\ s(0)$ satisfies

$$E(0) < Q_0, \quad \|u(0)\|_r > S_0. \tag{2.35}$$

Therefore, from Lemma 2.4 we get

$$E(u(t)) \le E(0) < Q_0, \quad \forall t > 0. \tag{2.36}$$

At the same time, by (2.29) and (2.31), it is clear that there can be no time $t > 0$ for which

$$E(u(t)) < Q_0, \qquad s(t) = S_0. \tag{2.37}$$

Hence we have also $s(t) > S_0$ for all $t > 0$ from the continuity of $E(u(t))$ and $s(t)$.

According to the above contradiction, we know that the global solution of the problem (1.1)–(1.3) does not exist, that is, the solution blows up in some finite time.

This completes the proof of Theorem 2.9. □

3. The Exponential Asymptotic Behavior

Lemma 3.1 (see [17]). *Let $y(t) : R^+ \to R^+$ be a nonincreasing function, and assume that there is a constant $A > 0$ such that*

$$\int_s^{+\infty} y(t)dt \le Ay(s), \quad 0 \le s < +\infty, \tag{3.1}$$

then $y(t) \le y(0)e^{1-(t/A)}$, for all $t \ge 0$.

The following theorem shows the exponential asymptotic behavior of global solutions of problem (1.1)–(1.3).

Theorem 3.2. *If the hypotheses in Theorem 2.8 are valid, then the global solutions of problem* (1.1)–(1.3) *have the following exponential asymptotic behavior:*

$$\frac{1}{2}\|u_t\|^2 + \frac{r-p}{rp}\|\nabla u\|_p^p \le E(0)e^{1-(t/M)}, \qquad \forall t \ge 0. \tag{3.2}$$

Proof. Multiplying by u on both sides of (1.1) and integrating over $\Omega \times [S, T]$ gives

$$0 = \int_S^T \int_\Omega u\Big[u_{tt} - \Delta_p u - a\Delta u_t - bu|u|^{r-2}\Big]dx\,dt, \tag{3.3}$$

where $0 \le S < T < +\infty$.

Since

$$\int_S^T \int_\Omega uu_{tt}dx\,dt = \int_\Omega uu_t dx\Big|_S^T - \int_S^T \int_\Omega |u_t|^2 dx\,dt, \tag{3.4}$$

so, substituting the formula (3.4) into the right-hand side of (3.3) gives

$$\begin{aligned}0 &= \int_S^T \int_\Omega \left(|u_t|^2 + \frac{2}{p}|\nabla u|_p^p - \frac{2b}{r}|u|^r \right) dx\, dt \\ &\quad - \int_S^T \int_\Omega \left[2|u_t|^2 - a\nabla u_t \nabla u \right] dx\, dt + \int_\Omega u u_t dx \Big|_S^T \\ &\quad + b\left(\frac{2}{r} - 1\right) \int_S^T \|u\|_r^r dt + \frac{p-2}{p} \int_S^T \|\nabla u\|_p^p dt.\end{aligned} \tag{3.5}$$

By exploiting Lemma 2.3 and (2.24), we easily arrive at

$$\begin{aligned}b\|u(t)\|_r^r &\le bC^r \|\nabla u(t)\|_p^r = bC^r \|\nabla u(t)\|_p^{r-p} \|\nabla u(t)\|_p^p \\ &< bC^r \left(\frac{rpd}{r-p}\right)^{(r-p)/p} \left\| \nabla u(t)_p^p \right\|.\end{aligned} \tag{3.6}$$

We obtain from (3.6) and (2.24) that

$$\begin{aligned}b\left(1 - \frac{2}{r}\right)\|u\|_r^r &\le bC^r \left(\frac{rpd}{r-p}\right)^{(r-p)/p} \frac{r-2}{r} \|\nabla u(t)\|_p^p \\ &\le bC^r \left(\frac{rpd}{r-p}\right)^{(r-p)/p} \frac{r-2}{r} \cdot \frac{rp}{r-p} E(t) \\ &= \frac{bp(r-2)C^r}{r-p} \left(\frac{rpd}{r-p}\right)^{(r-p)/p} E(t), \\ \frac{p-2}{p} \int_S^T \|\nabla u\|_p^p dx\, dt &\le \frac{r(p-2)}{r-p} \int_S^T E(t) dt.\end{aligned} \tag{3.7}$$

It follows from (3.7) and (3.5) that

$$\begin{aligned}&\left[2 - \frac{bp(r-2)C^r}{r-p} \left(\frac{rpd}{r-p}\right)^{(r-p)/p} - \frac{r(p-2)}{r-p} \right] \int_S^T E(t) dt \\ &\quad \le \int_S^T \int_\Omega \left[2|u_t|^2 - a\nabla u_t \nabla u \right] dx\, dt - \int_\Omega u u_t\, dx \Big|_S^T.\end{aligned} \tag{3.8}$$

We have from Hölder inequality, Lemma 2.3 and (2.24) that

$$\begin{aligned}\left| - \int_\Omega u u_t\, dx \Big|_S^T \right| &\le \left| \left(\frac{C^p rp}{r-p} \cdot \frac{r-p}{rp} \|\nabla u\|_p^p + \frac{1}{2}\|u_t\|^2 \right) \Big|_S^T \right| \\ &\le \max\left(\frac{C^p rp}{r-p}, 1\right) \left| \; E(t)\big|_S^T \; \right| \le ME(S).\end{aligned} \tag{3.9}$$

Substituting the estimates of (3.9) into (3.8), we conclude that

$$\begin{aligned}
&\left[2-\frac{bp(r-2)C^r}{r-p}\left(\frac{rpd}{r-p}\right)^{(r-p)/p}-\frac{r(p-2)}{r-p}\right]\int_S^T E(t)dt \\
&\quad\leq \int_S^T\int_\Omega\left[2|u_t|^2-a\nabla u_t\nabla u\right]dx\,dt+ME(S).
\end{aligned} \tag{3.10}$$

We get from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
2\int_S^T\int_\Omega |u_t|^2dx\,dt &= 2\int_S^T \|u_t\|^2dt \leq 2C^2\int_S^T\|\nabla u_t\|^2dt \\
&= -\frac{2C^2}{a}(E(T)-E(S)) \leq \frac{2C^2}{a}E(S).
\end{aligned} \tag{3.11}$$

From Young inequality, Lemmas 2.3 and 2.4, and (2.24), it follows that

$$\begin{aligned}
-a\int_S^T\int_\Omega \nabla u\nabla u_t dx\,dt &\leq a\int_S^T\left(\varepsilon C^2\|\nabla u\|_p^2+M(\varepsilon)\|\nabla u_t\|^2\right)dt \\
&\leq \frac{aC^2rp\varepsilon}{r-p}\int_S^T E(t)dt+M(\varepsilon)(E(S)-E(T)) \\
&\leq \frac{aC^2rp\varepsilon}{r-p}\int_S^T E(t)dt+M(\varepsilon)E(S).
\end{aligned} \tag{3.12}$$

Choosing ε small enough, such that

$$\frac{1}{2}\left[\frac{bp(r-2)C^r}{r-p}\left(\frac{rpd}{r-p}\right)^{(r-p)/p}+\frac{r(p-2)}{r-p}+\frac{aC^2rp\varepsilon}{r-p}\right]<1, \tag{3.13}$$

and, substituting (3.11) and (3.12) into (3.10), we get

$$\int_S^T E(t)dt \leq ME(S). \tag{3.14}$$

We let $T \to +\infty$ in (3.14) to get

$$\int_S^{+\infty} E(t)dt \leq ME(S). \tag{3.15}$$

Therefore, we have from (3.15) and Lemma 3.1 that

$$E(t) \leq E(0)e^{1-(t/M)}, \quad t\in[0,+\infty). \tag{3.16}$$

We conclude from $u \in H$, (2.4) and (3.16) that

$$\frac{1}{2}\|u_t\|^2 + \frac{r-p}{rp}\|\nabla u\|_p^p \le E(0)e^{1-(t/M)}, \quad \forall t \ge 0. \tag{3.17}$$

The proof of Theorem 3.2 is thus finished. □

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