Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 406231, 18 pages
doi:10.1155/2010/406231

Research Article

On Linear Combinations of Two Orthogonal Polynomial Sequences on the Unit Circle

C. Suárez

Departamento de Matemática Aplicada I, E.T.S.I.I., Universidad de Vigo, Campus Lagoas-Marcosende, 36200 Vigo, Spain

Correspondence should be addressed to C. Suárez, csuarez@uvigo.es

Received 1 August 2009; Revised 1 December 2009; Accepted 5 March 2010

Academic Editor: Panayiotis Siafarikas

Let $\{\Phi_n\}$ be a monic orthogonal polynomial sequence on the unit circle. We define recursively a new sequence $\{\Psi_n\}$ of polynomials by the following linear combination: $\Psi_n(z) + p_n\Psi_{n-1}(z) = \Phi_n(z) + q_n\Phi_{n-1}(z)$, $p_n, q_n \in \mathbb{C}$, $p_n q_n \neq 0$. In this paper, we give necessary and sufficient conditions in order to make $\{\Psi_n\}$ be an orthogonal polynomial sequence too. Moreover, we obtain an explicit representation for the Verblunsky coefficients $\{\Phi_n(0)\}$ and $\{\Psi_n(0)\}$ in terms of p_n and q_n. Finally, we show the relation between their corresponding Carathéodory functions and their associated linear functionals.

1. Notation and Preliminary Results

We recall some definitions and general results about orthogonal polynomials on the unit circle (OPUC). They can be seen in [1–3].

Along this paper, we will use the following notations. We denote by $\Lambda = \text{span}\{z^k, k \in \mathbb{Z}\}$ the linear space of Laurent polynomials with complex coefficients and by Λ' the dual algebraic space of Λ. Let $\mathbb{P} = \text{span}\{z^k, k \in \mathbb{N}\}$ be the space of complex polynomials.

Definition 1.1. Let $u \in \Lambda'$. Denoting by $u_n = u(z^n)$, $n \in \mathbb{Z}$, we say that

(i) u is Hermitian if for all $n \geq 0$, $u_{-n} = \overline{u_n}$;

(ii) u is regular or quasidefinite (positive definite) if the principal minors of the moment matrix are nonsingular (positive), that is,

$$\forall n \geq 0, \quad \Delta_n = \det\left(u\left(z^{i-j}\right)\right)_{i=0\cdots n;\ j=0\cdots n} \neq 0 \quad (> 0). \tag{1.1}$$

In any case we denote for all $n \geq 0$, $e_n = \Delta_n / \Delta_{n-1}$ with $\Delta_{-1} = 1$.

The sequence $\{u_n\}$ is said to be the sequence of the moments associated with u.

Furthermore, if u is a positive definite linear functional then a finite nontrivial positive Borel measure μ supported on the unit circle exists such that

$$u(P(z)) = \frac{1}{2\pi}\int_0^{2\pi} P\left(e^{i\theta}\right)d\mu, \quad P \in \Lambda. \tag{1.2}$$

Definition 1.2. Let $\{\Phi_n(z)\}_0^{+\infty}$ be a complex polynomial sequence with $\deg \Phi_n(z) = n$. We say that $\{\Phi_n(z)\}_0^{+\infty}$ is a sequence of orthogonal polynomials (OPSs) with respect to the linear and Hermitian functional u if

$$\forall n, m \geq 0, \quad u\left(\Phi_n(z)\overline{\Phi_m}\left(\frac{1}{z}\right)\right) = e_n\delta_{nm} \quad \text{with } e_n \neq 0. \tag{1.3}$$

In the sequel, we denote by $\{\Phi_n\}$ the monic orthogonal polynomial sequence (MOPS) associated with u.

For simplicity, along this paper we also assume that u is normalized (i.e., $u_0 = 1$). It is well known that the regularity of u is a necessary and sufficient condition for the existence of a sequence of orthogonal polynomials on the unit circle. On the other hand, the polynomials Φ_n satisfy the so-called *Szegö recurrence relations*

$$\forall n \geq 1, \quad \Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \tag{1.4}$$

$$\forall n \geq 1, \quad \Phi_n^*(z) = \Phi_{n-1}^*(z) + \overline{\Phi_n(0)}z\Phi_{n-1}(z), \tag{1.5}$$

$$\forall n \geq 1, \quad \Phi_n(z) = \left(1 - |\Phi_n(0)|^2\right)z\Phi_{n-1}(z) + \Phi_n(0)\Phi_n^*(z), \tag{1.6}$$

$$\forall n \geq 1, \quad \Phi_n^*(z) = \left(1 - |\Phi_n(0)|^2\right)\Phi_{n-1}^*(z) + \overline{\Phi_n(0)}\Phi_n(z), \tag{1.7}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n}(1/z)$ is the reversed polynomial of $\Phi_n(z)$, $n \geq 0$.

Definition 1.3. Given an MOPS $\{\Phi_n\}$, the sequence of kernels of parameter $y \in \mathbb{C}$ associated with the linear functional u is defined by

$$\forall n \geq 0, \quad K_n(z, y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}}{e_j}\Phi_j(z). \tag{1.8}$$

This sequence verifies the following properties:

$$\forall n \geq 0, \quad K_n(z,y) = \frac{1}{e_n}\left(\frac{\Phi_n^*(z)\overline{\Phi_n^*(y)} - z\overline{y}\Phi_n(z)\overline{\Phi_n(y)}}{1 - z\overline{y}}\right), \tag{1.9}$$

$$\forall n \geq 0, \quad K_n(z,y) = \frac{1}{e_{n+1}}\left(\frac{\Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(y)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(y)}}{1 - z\overline{y}}\right), \tag{1.10}$$

$$\forall n \geq 0, \quad K_n(z,y) = \overline{K_n(y,z)} \quad \Phi_n^*(z) = e_n K_n(z,0), \tag{1.11}$$

$$\forall n \geq 1, \quad K_n(z,y) = z\overline{y}K_{n-1}(z,y) + \frac{\overline{\Phi_n^*(y)}}{e_n}\Phi_n^*(z). \tag{1.12}$$

To the linear functional u we can associate a formal series F_u as follows:

$$F_u(z) = 1 + 2\sum_{n=1}^{+\infty}\overline{u_n}z^n. \tag{1.13}$$

In the positive definite case, F_u is called the Carathéodory function associated with u. In this case, F_u can be written as

$$F_u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta), \quad |z| < 1. \tag{1.14}$$

The measure $d\mu$ can be reconstructed from F_u by means of the inversion formula. The aim of this paper is the analysis of the following problem. Given an MOPS on the unit circle $\{\Phi_n\}$, orthogonal with respect to a linear functional u, to find necessary and sufficient conditions in order to make a sequence of monic polynomials $\{\Psi_n\}$ defined by

$$\forall n \geq 1, \quad \Phi_n(z) + q_n\Phi_{n-1}(z) = \Psi_n(z) + p_n\Psi_{n-1}(z) \quad \text{with } p_n q_n \neq 0 \tag{1.15}$$

an MOPS with respect to a linear functional $\mathfrak{L}$. Further more, to find the relation between the linear functionals $\mathfrak{L}$ and u and their corresponding Carathéodory functions.

Many authors have dealt with this kind of problems. In the constructive theory of orthogonal polynomials they have been called *inverse problems*. Concretely, an inverse problem for linear functionals can be stated as follows. Given two sequences of monic polynomials $\{\Phi_n\}$ and $\{\Psi_n\}$, to find necessary and sufficient conditions in order to make $\{\Psi_n\}$ an MOPS when $\{\Phi_n\}$ is a MOPS and they are related by

$$F(\Phi_n,\ldots,\Phi_{n-l}) = G(\Psi_n,\ldots,\Psi_{n-k}), \tag{1.16}$$

where F and G are fixed functions. As a next step, to find the relation between the functionals.

For instance, this subject has been treated in [4–6] in the context of the theory of orthogonal polynomials on the real line. For orthogonal polynomials with respect to measures supported on the unit circle, in [7] there have been relevant results.

The structure of this paper is the following. In Section 2 we give the necessary conditions in order to be sure that the problem (1.15) admits a nontrivial solution. In Section 3, we prove a sufficient condition and we obtain the explicit solution in terms of p_n and q_n. Section 4 is devoted to find the functional relation between $\mathfrak{L}$ and u. Finally, Section 5 contains the rational relation between the corresponding Carathéodory functions.

2. Necessary Conditions

Let $\{\Phi_n\}_{n\geq 0}$ be a monic orthogonal polynomial sequence and let $\{\Psi_n\}_{n\geq 0}$ be a monic polynomial sequence. We assume that there exist sequences of complex numbers $\{q_n\}_{n\geq 2}$ and $\{p_n\}_{n\geq 2}$ such that the following relation holds:

$$\forall n \geq 2, \quad \Phi_n(z) + q_n \Phi_{n-1}(z) = \Psi_n(z) + p_n \Psi_{n-1}(z). \tag{2.1}$$

Also, we assume $\Phi_0(z) = \Psi_0(z) = 1$ and $\Phi_1(z) = z - q_1$ and $\Psi_1(z) = z - p_1$, with $|q_1| \neq 1$ and $|p_1| \neq 1$.

In this section, we find some necessary conditions in order to make the sequence $\{\Psi_n\}$ defined recursively from $\{\Phi_n\}$ by relation (2.1) an MOPS.

With this aim, we define the complex numbers N_{n+1} and D_{n+1} as follows:

$$\forall n \geq 1, \quad N_{n+1} = \Phi_{n+1}(0) + q_{n+1}\Phi_n(0), \tag{2.2}$$

$$\forall n \geq 1, \quad D_{n+1} = q_{n+1} - q_n + \overline{\Phi_n(0)}\Phi_{n+1}(0). \tag{2.3}$$

The following proposition justifies this choice.

Proposition 2.1. *Let $\{\Psi_n\}_{n\geq 0}$ be the monic sequence given as in* (2.1). *If $\{\Psi_n\}_{n\geq 0}$ is an MOPS, then the following relations hold:*

$$\forall n \geq 1, \quad N_{n+1} = \Psi_{n+1}(0) + p_{n+1}\Psi_n(0), \tag{2.4}$$

$$\forall n \geq 1, \quad D_{n+1} = p_{n+1} - p_n + \overline{\Psi_n(0)}\Psi_{n+1}(0). \tag{2.5}$$

Moreover,

$$\forall n \geq 1, \quad zD_{n+1}(\Phi_{n-1}(z) - \Psi_{n-1}(z)) = N_{n+1}\left(\Psi^*_{n-1}(z) - \Phi^*_{n-1}(z)\right), \tag{2.6}$$

$$\forall n \geq 1, \quad D_{n+2}\left(q_n\Phi_{n-1}(z) - p_n\Psi_{n-1}(z)\right) = N_{n+2}\left(\overline{p_n}\Psi^*_{n-1}(z) - \overline{q_n}\Phi^*_{n-1}(z)\right). \tag{2.7}$$

Proof. From (1.4) together with the definition of Ψ_n (2.1), we have

$$
\begin{aligned}
\forall n \geq 1, \quad \Psi_{n+1}(z) &= \Phi_{n+1}(z) + q_{n+1}\Phi_n(z) - p_{n+1}\Psi_n(z) \\
&= \left(z + q_{n+1}\right)\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z) - p_{n+1}\Psi_n(z) \\
&= z\Psi_n(z) - zq_n\Phi_{n-1}(z) + zp_n\Psi_{n-1}(z) + q_{n+1}\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z) - p_{n+1}\Psi_n(z).
\end{aligned}
\tag{2.8}
$$

Since $\{\Psi_n\}$ is a MOPS, using (1.4) we have

$$
\Psi_{n+1}(0)\Psi_n^*(z) = -zq_n\Phi_{n-1}(z) + zp_n\Psi_{n-1}(z) + q_{n+1}\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z) - p_{n+1}\Psi_n(z).
\tag{2.9}
$$

That is,

$$
p_{n+1}\Psi_n(z) + \Psi_{n+1}(0)\Psi_n^*(z) - zp_n\Psi_{n-1}(z) = q_{n+1}\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z) - zq_n\Phi_{n-1}(z).
\tag{2.10}
$$

Using (1.4) and (1.5) in both sequences, we get

$$
\begin{aligned}
\forall n \geq 1, \quad &\left(p_{n+1} - p_n + \Psi_{n+1}(0)\overline{\Psi_n(0)}\right)z\Psi_{n-1}(z) + \left(\Psi_{n+1}(0) + p_{n+1}\Psi_n(0)\right)\Psi_{n-1}^*(z) \\
&= \left(q_{n+1} - q_n + \Phi_{n+1}(0)\overline{\Phi_n(0)}\right)z\Phi_{n-1}(z) + \left(\Phi_{n+1}(0) + q_{n+1}\Phi_n(0)\right)\Phi_{n-1}^*(z).
\end{aligned}
\tag{2.11}
$$

Taking $z = 0$ we have (2.4). Observe that, this is as same as (2.1) for $z = 0$.

Identifying the coefficients of degree n, then (2.5) holds. Therefore, we can rewrite (2.11) as (2.6).

On the other hand, applying the $*$-operator in (2.1) we have $\Psi_n^*(z) + \overline{p_n}z\Psi_{n-1}^*(z) = \Phi_n^*(z) + \overline{q_n}z\Phi_{n-1}^*(z)$. Substituting in (2.6) and using (2.1), we obtain (2.7). □

In the sequel, we denote by u the linear regular functional associated with $\{\Phi_n\}$ and by $\mathfrak{L}$ the linear regular functional associated with $\{\Psi_n\}$. Besides, we denote by E_n the real number such that $E_n = \mathfrak{L}(\Psi_n(z)z^{-n})$ with $E_0 = 1$. Therefore, $E_n/E_{n-1} = 1 - |\Psi_n(0)|^2$.

Corollary 2.2. *Under the same conditions as in Proposition 2.1, the following assertions hold:*

(i) *If $p_1 = q_1$, then $p_n = q_n$ and $\Phi_n(z) = \Psi_n(z)$, for all $n \geq 1$,*

(ii) *If $p_1 \neq q_1$ and $p_nq_n \neq 0$, for all $n \geq 2$, then $\Phi_n(z) \neq \Psi_n(z)$, for all $n \geq 1$,*

(iii) *Assume $p_1 \neq q_1$ and $p_nq_n \neq 0$, for all $n \geq 2$, then $N_{n+1} \neq 0$ if and only if $D_{n+1} \neq 0$, for all $n \geq 2$. Moreover, $|N_{n+1}| = |D_{n+1}|$ for all $n \geq 2$.*

Proof. (i) We eliminate $\Psi_{n+1}(0)$ using equalities (2.2)–(2.4) and (2.3)–(2.5). By doing this, we get

$$\forall n \geq 1, \quad q_{n+1} - q_n - \left(p_{n+1}\frac{E_n}{E_{n-1}} - p_n\right) = \Phi_{n+1}(0)\left(\overline{\Psi_n(0)} - \overline{\Phi_n(0)}\right) + q_{n+1}\Phi_n(0)\overline{\Psi_n(0)}. \tag{2.12}$$

Taking $n = 1$, we obtain $q_2 - q_1 - (p_2E_1 - p_1) = \Phi_2(0)(\overline{q_1} - \overline{p_1}) + q_2q_1\overline{p_1}$.

If $p_1 = q_1$, then $q_2 - p_2 = |q_1|^2(q_2 - p_2)$ and thus $(q_2 - p_2)e_1 = 0$. Wherefrom $q_2 = p_2$. Now, using (2.1) we have $\Psi_2(z) = \Phi_2(z)$.

Proceeding in the same way for $n = 2$ we obtain $(q_3 - p_3)e_2/e_1 = 0$, hence $q_3 = p_3$ and $\Psi_3(z) = \Phi_3(z)$, and thus successively.

(ii) Assume that there exits $n_0 \geq 2$ such that $\Phi_{n_0} = \Psi_{n_0}$. From (2.1), written for $n = n_0$, it holds that $q_{n_0} = p_{n_0}$ and then $\Phi_{n_0-1} = \Psi_{n_0-1}$, and thus successively.

Hence, $\Psi_1 = \Phi_1$, in contradiction with the hypothesis.

(iii) The result follows from (2.6) and the above item.

On the other hand, applying the $*n$-operator in (2.6), we obtain $|N_{n+1}| = |D_{n+1}|$ for $n \geq 2$. □

Remark 2.3. The situation $p_1 = q_1$ is the trivial case, that is, $\Psi_n = \Phi_n$, for all $n \geq 1$. For this reason, in the sequel, it will be excluded.

The next result will be used later.

Lemma 2.4. *Under the same conditions as in Proposition 2.1 together with $p_1 \neq q_1$, the following assertions hold:*

$$\forall n \geq 2, \quad u(\Psi_n(z)) = (-1)^{n+1}p_n \cdots p_2(q_1 - p_1), \tag{2.13}$$

$$\forall n \geq 2, \quad \mathfrak{L}(\Phi_n(z)) = (-1)^{n+1}q_n \cdots q_2(p_1 - q_1). \tag{2.14}$$

Proof. Using (2.1) we obtain

$$\forall n \geq 2, \quad u(\Psi_n) = -p_n u(\Psi_{n-1}). \tag{2.15}$$

Since that $u(\Psi_1(z)) = u(z - p_1) = q_1 - p_1$, then (2.13) follows.

We obtain (2.14) changing u by $\mathfrak{L}$. □

For all $n \geq 1$ such that $D_{n+1} \neq 0$, we define the following complex number:

$$T_{n+1} = \frac{N_{n+1}}{D_{n+1}} = \frac{\Phi_{n+1}(0) + q_{n+1}\Phi_n(0)}{q_{n+1} - q_n + \Phi_{n+1}(0)\overline{\Phi_n(0)}}. \tag{2.16}$$

This number plays a very important role in the solution of our problem.

Proposition 2.5. *Assume that* $\{\Phi_n\}$ *and* $\{\Psi_n\}$ *are two MOPSs that verify* (2.1) *with* $p_1 \neq q_1$ *and* $p_n q_n \neq 0$*, for all* $n \geq 2$*. If moreover* $D_{n+2} \neq 0$ *for all* $n \geq 1$*, then the following relation linking* $\Psi_n(z)$ *and* $\Phi_n(z)$ *holds:*

$$\forall n \geq 1, \quad \left(z - \frac{p_{n+1} T_{n+2}}{\overline{p_{n+1}} T_{n+3}}\right)\Psi_n(z) = \left(z - \frac{q_{n+1} T_{n+2}}{\overline{p_{n+1}} T_{n+3}}\right)\Phi_n(z) + \frac{T_{n+2}}{\overline{p_{n+1}}}\left(\overline{p_{n+1}} - \overline{q_{n+1}}\right)\Phi_n^*(z). \tag{2.17}$$

Proof. From (2.6) and (2.7), for all $n \geq 1$, we have the system

$$\begin{gathered} p_{n+1} D_{n+3} \Psi_n(z) + \overline{p_{n+1}} N_{n+3} \Psi_n^*(z) = q_{n+1} D_{n+3} \Phi_n(z) + \overline{q_{n+1}} N_{n+3} \Phi_n^*(z), \\ z D_{n+2} \Psi_n(z) + N_{n+2} \Psi_n^*(z) = z D_{n+2} \Phi_n(z) + N_{n+2} \Phi_n^*(z). \end{gathered} \tag{2.18}$$

The corresponding determinant

$$\forall n \geq 1, \quad \begin{vmatrix} p_{n+1} D_{n+3} & \overline{p_{n+1}} N_{n+3} \\ z D_{n+2} & N_{n+2} \end{vmatrix} \tag{2.19}$$

is not null, since $D_{n+2} \neq 0$ together with Corollary 2.2(iii). Wherefrom, it has a unique solution for Ψ_n.

By solving this, we get (2.17). □

In the sequel, we denote by $\widetilde{K_n}(z, y)$ the sequence of the kernels corresponding to $\{\Psi_n\}$. For the sequence $\{\Phi_n\}$ we keep the same notations as in Section 1.

Proposition 2.6. *Assume that* $\{\Psi_n\}$ *and* $\{\Phi_n\}$ *are two MOPS that verify* (2.1) *with* $p_1 \neq q_1$ *and* $p_n q_n \neq 0$*, for all* $n \geq 2$*. Also assume* $D_{n+2} \neq 0$*, for all* $n \geq 1$*. Under these conditions, then the following assertions hold*

(i) $p_n \neq q_n$*, for all* $n \geq 2$*,*

(ii) *There exist two complex numbers* α *and* β *with* $|\alpha| = |\beta| = 1$ *such that*

$$\forall n \geq 2, \quad \alpha = \frac{p_n}{\overline{p_n}} \frac{T_{n+1}}{T_{n+2}}, \tag{2.20}$$

$$\forall n \geq 2, \quad \beta = \frac{q_n}{\overline{q_n}} \frac{T_{n+1}}{T_{n+2}}. \tag{2.21}$$

Here, the initial parameters T_3 *and* T_4 *are given by* $T_3 = -(q_1 - p_1)/(\overline{q_1} - \overline{p_1})$ *and* $T_4 = (q_2 - p_2)/(\overline{q_2 q_1} - \overline{p_2 p_1})$*,*

(iii) *The sequences* $\{\Phi_n\}$ *and* $\{\Psi_n\}$ *are connected by the following formulas:*

$$\forall n \geq 1, \quad \Psi_n(z) = \Phi_n(z) + \frac{e_n}{\overline{\Phi_n(\alpha)}} \frac{(p_{n+1} - q_{n+1})}{p_{n+1}} K_{n-1}(z, \alpha), \tag{2.22}$$

$$\forall n \geq 1, \quad \Phi_n(z) = \Psi_n(z) + \frac{E_n}{\overline{\Psi_n(\beta)}} \frac{(q_{n+1} - p_{n+1})}{q_{n+1}} \widetilde{K_{n-1}}(z, \beta). \tag{2.23}$$

Proof. Item (i) follows immediately from (2.17). Indeed, if we take $p_{n+1} = q_{n+1}$, we obtain $\Phi_n = \Psi_n$.

Let us proceed with the proof of (ii). Inserting

$$\Psi_n(z) = \frac{(\Phi_{n+1}(z) + q_{n+1}\Phi_n(z) - \Psi_{n+1}(z))}{p_{n+1}}, \tag{2.24}$$

in (2.17), we have

$$\begin{aligned} \forall n \geq 1, \quad & \left(z - \frac{p_{n+1}T_{n+2}}{\overline{p_{n+1}}T_{n+3}}\right)\Psi_{n+1}(z) \\ &= \left(z - \frac{p_{n+1}T_{n+2}}{\overline{p_{n+1}}T_{n+3}}\right)\Phi_{n+1}(z) + z(q_{n+1} - p_{n+1})\Phi_n(z) - \frac{p_{n+1}}{\overline{p_{n+1}}}T_{n+2}(\overline{p_{n+1}} - \overline{q_{n+1}})\Phi_n^*(z). \end{aligned} \tag{2.25}$$

Using the recurrences (1.6) and (1.7) in the right-hand side we deduce

$$\begin{aligned} \forall n \geq 1, \quad & \left(z - \frac{p_{n+1}T_{n+2}}{\overline{p_{n+1}}T_{n+3}}\right)\Psi_{n+1}(z) \\ &= \left(z - \frac{p_{n+1}T_{n+2}}{\overline{p_{n+1}}T_{n+3}} + \left(q_{n+1} - p_{n+1} + \frac{p_{n+1}(\overline{p_{n+1}} - \overline{q_{n+1}})}{\overline{p_{n+1}}}T_{n+2}\overline{\Phi_{n+1}(0)}\right)\frac{e_n}{e_{n+1}}\right)\Phi_{n+1}(z) \\ &\quad - \left((q_{n+1} - p_{n+1})\Phi_{n+1}(0) + \frac{p_{n+1}(\overline{p_{n+1}} - \overline{q_{n+1}})}{\overline{p_{n+1}}}T_{n+2}\right)\frac{e_n}{e_{n+1}}\Phi_{n+1}^*(z). \end{aligned} \tag{2.26}$$

In order to eliminate the polynomial Ψ_n, we write (2.26) for Ψ_n and we combine it with (2.17). Concretely, we multiply (2.17) by $(z - p_nT_{n+1}/\overline{p_n}T_{n+2})$ and (2.26) by $(z - p_{n+1}T_{n+2}/\overline{p_{n+1}}T_{n+3})$. By doing this, we obtain

$$\forall n \geq 2, \quad (d_nz + f_n)\Phi_n(z) = (g_nz + h_n)\Phi_n^*(z), \tag{2.27}$$

where

$$d_n = (p_{n+1} - q_{n+1}) \frac{T_{n+2}}{\overline{p_{n+1}} T_{n+3}} - \left((q_n - p_n) + \frac{p_n}{\overline{p_n}} T_{n+1} (\overline{p_n} - \overline{q_n}) \overline{\Phi_n(0)} \right) \frac{e_{n-1}}{e_n},$$
$$f_n = -(p_{n+1} - q_{n+1}) \frac{p_n T_{n+1}}{\overline{p_n p_{n+1}} T_{n+3}} + \frac{p_{n+1} T_{n+2}}{\overline{p_{n+1}} T_{n+3}} \left((q_n - p_n) + \frac{p_n}{\overline{p_n}} T_{n+1} (\overline{p_n} - \overline{q_n}) \overline{\Phi_n(0)} \right) \frac{e_{n-1}}{e_n}. \tag{2.28}$$

Given that Φ_n and Φ_n^* have not common roots, then $d_n = f_n = g_n = h_n = 0$, for all $n \geq 2$.

Using (2.28) we obtain

$$(p_{n+1} - q_{n+1}) \frac{T_{n+2}}{\overline{p_{n+1}} T_{n+3}} = \left((q_n - p_n) + \frac{p_n}{\overline{p_n}} T_{n+1} (\overline{p_n} - \overline{q_n}) \overline{\Phi_n(0)} \right) \frac{e_{n-1}}{e_n},$$
$$(p_{n+1} - q_{n+1}) \frac{p_n T_{n+1}}{\overline{p_n p_{n+1}} T_{n+3}} = \frac{p_{n+1} T_{n+2}}{\overline{p_{n+1}} T_{n+3}} \left((q_n - p_n) + \frac{p_n}{\overline{p_n}} T_{n+1} (\overline{p_n} - \overline{q_n}) \overline{\Phi_n(0)} \right) \frac{e_{n-1}}{e_n}. \tag{2.29}$$

Combining these relations, we deduce

$$\forall n \geq 2, \quad \frac{p_n}{\overline{p_n}} \frac{T_{n+1}}{T_{n+2}} = \frac{p_{n+1}}{\overline{p_{n+1}}} \frac{T_{n+2}}{T_{n+3}}, \tag{2.30}$$

since $p_{n+1} \neq q_{n+1}$.

This complex constant is denoted in the statement by α. The property $|\alpha| = 1$ is a consequence of Corollary 2.2(iii). On the other hand, the explicit expressions of T_3 and T_4 follow from (2.7) for $n = 1$ and $n = 2$, respectively.

This completes the proof of (ii) because the complex number β exits by the symmetry of the problem.

Finally, we show (iii). Using again (2.17), we have

$$\forall n \geq 1, \quad (z - \alpha) \Psi_n(z) = \left(z - \frac{q_{n+1}}{p_{n+1}} \alpha \right) \Phi_n(z) + \frac{T_{n+2}}{\overline{p_{n+1}}} (\overline{p_{n+1}} - \overline{q_{n+1}}) \Phi_n^*(z). \tag{2.31}$$

Putting $z = \alpha$, we get

$$\alpha \frac{(p_{n+1} - q_{n+1})}{p_{n+1}} \Phi_n(\alpha) = \frac{(\overline{q_{n+1}} - \overline{p_{n+1}})}{\overline{p_{n+1}}} \Phi_n^*(\alpha) T_{n+2}. \tag{2.32}$$

Substituting this relation in (2.31) and using the recurrences of the kernels (1.9) and (1.10), (2.22) holds. □

In order to state the converse we need the following assertions.

Proposition 2.7. *Under the hypothesis of Proposition 2.6,*

$$\forall n \geq 2, \quad \Phi_n(0) = T_{n+1} + T_{n+2}\overline{q_n}, \tag{2.33}$$

$$\forall n \geq 2, \quad \Psi_n(0) = T_{n+1} + T_{n+2}\overline{p_n} \tag{2.34}$$

$$\forall n \geq 2, \quad \frac{p_n}{q_n}\frac{e_n}{e_{n-1}} = \frac{p_{n+1}}{q_{n+1}}\frac{E_n}{E_{n-1}}. \tag{2.35}$$

Proof. From (1.4),

$$\forall n \geq 1, \quad (z-\alpha)\Psi_n = z(z-\alpha)\Psi_{n-1}(z) + \Psi_n(0)(z-\alpha)\Psi^*_{n-1}(z). \tag{2.36}$$

Here, we use (2.31) to substitute the terms $(z-\alpha)\Psi_{n-1}(z)$ and $(z-\alpha)\Psi^*_{n-1}(z)$ as function of $\{\Phi_n\}$. By doing this, we deduce

$$\begin{aligned}\forall n \geq 2, \quad (z-\alpha)\Psi_n(z) = {}& \left(z - \frac{q_n}{p_n}\alpha - \Psi_n(0)\frac{\overline{T_{n+1}}}{p_n}(p_n - q_n)\alpha\right) z\Phi_{n-1} \\ &+ \left(z\frac{T_{n+1}}{\overline{p_n}}(\overline{p_n} - \overline{q_n}) + \Psi_n(0)\frac{\overline{q_n}}{\overline{p_n}}\left(z - \frac{\overline{p_n}}{\overline{q_n}}\alpha\right)\right)\Phi^*_{n-1}(z).\end{aligned} \tag{2.37}$$

Equating this formula with (2.25), previously written for Ψ_n, and applying (1.4) we get

$$\begin{aligned}\forall n \geq 2, \quad & \left(-\alpha + q_n - p_n + \frac{q_n}{p_n}\alpha + \Psi_n(0)\frac{\overline{T_{n+1}}}{p_n}(p_n - q_n)\alpha\right) z\Phi_{n-1}(z) \\ &= \left(-(z-\alpha)\Phi_n(0) + (z+p_n)\frac{T_{n+1}}{\overline{p_n}}(\overline{p_n} - \overline{q_n}) + \Psi_n(0)\frac{\overline{q_n}}{\overline{p_n}}\left(z - \frac{\overline{p_n}}{\overline{q_n}}\alpha\right)\right)\Phi^*_{n-1}(z).\end{aligned} \tag{2.38}$$

Putting $z = 0$, then the independent term vanishes and the previous relation becomes

$$\frac{(q_n - p_n)}{p_n}\left(\alpha + p_n - \frac{\Psi_n(0)}{T_{n+1}}\alpha\right)\Phi_{n-1}(z) = \left(-\Phi_n(0) + \frac{T_{n+1}}{\overline{p_n}}(\overline{p_n} - \overline{q_n}) + \Psi_n(0)\frac{\overline{q_n}}{\overline{p_n}}\right)\Phi^*_{n-1}(z). \tag{2.39}$$

Using again the fact that Φ_{n-1} and Φ^*_{n-1} have no common roots and $q_n \neq p_n$, it follows that the coefficients in the last relation are zero and this implies (2.33) and (2.34).

Let us proceed with (2.35). From (2.33) and (2.34), for all $n \geq 2$, we have

$$\begin{aligned}\forall n \geq 2, \quad \frac{e_n}{e_{n-1}} &= 1 - |\Phi_n(0)|^2 = -\frac{T_{n+2}}{T_{n+1}}\overline{q_n} - \frac{T_{n+1}}{T_{n+2}}q_n - |q_n|^2, \\ \forall n \geq 2, \quad \frac{E_n}{E_{n-1}} &= -\frac{T_{n+2}}{T_{n+1}}\overline{p_n} - \frac{T_{n+1}}{T_{n+2}}p_n - |p_n|^2.\end{aligned} \tag{2.40}$$

On the other hand, substituting in $N_{n+1} = \Psi_{n+1}(0) + p_{n+1}\Psi_n(0) = \Phi_{n+1}(0) + q_{n+1}\Phi_n(0)$ the relations (2.33) and (2.34), we obtain

$$\forall n \geq 2, \quad \left(p_{n+1}\overline{p_n} - q_{n+1}\overline{q_n}\right) = \left(q_{n+1} - p_{n+1}\right)\frac{T_{n+1}}{T_{n+2}} + \left(\overline{q_{n+1}} - \overline{p_{n+1}}\right)\frac{T_{n+3}}{T_{n+2}}. \tag{2.41}$$

We can eliminate T_{n+3}/T_{n+2} using $(\overline{p_{n+1}}/p_{n+1})(T_{n+3}/T_{n+2}) = (\overline{p_n}/p_n)(T_{n+2}/T_{n+1}) = \alpha$. Moreover, multiplying by p_n/q_{n+1} we find

$$\left(p_{n+1}\overline{p_n} - q_{n+1}\overline{q_n}\right)\frac{p_n}{q_{n+1}} = \left(q_{n+1} - p_{n+1}\right)\frac{p_n}{q_{n+1}}\frac{T_{n+1}}{T_{n+2}} + \left(\overline{q_{n+1}} - \overline{p_{n+1}}\right)\frac{p_{n+1}\overline{p_n}}{\overline{p_{n+1}}q_{n+1}}\frac{T_{n+2}}{T_{n+1}}. \tag{2.42}$$

Therefore,

$$\begin{aligned}\forall n \geq 2, \quad & -\frac{p_n}{q_n}\frac{e_n}{e_{n-1}} + \frac{p_{n+1}}{q_{n+1}}\frac{E_n}{E_{n-1}} \\ & = \left(\frac{p_n\overline{q_n}}{q_n} - \frac{\overline{p_n}p_{n+1}}{q_{n+1}}\right)\frac{T_{n+2}}{T_{n+1}} + \left(q_{n+1} - p_{n+1}\right)\frac{p_n}{q_{n+1}}\frac{T_{n+1}}{T_{n+2}} - \left(\overline{p_n}\frac{p_{n+1}}{q_{n+1}} - \overline{q_n}\right)p_n.\end{aligned} \tag{2.43}$$

Finally, we use (2.42) in order to calculate the right-hand side

$$\forall n \geq 2, \quad -\frac{p_n}{q_n}\frac{e_n}{e_{n-1}} + \frac{p_{n+1}}{q_{n+1}}\frac{E_n}{E_{n-1}} = \overline{p_n}\left(\frac{p_n\overline{q_n}}{\overline{p_n}q_n} - \frac{p_{n+1}\overline{q_{n+1}}}{\overline{p_{n+1}}q_{n+1}}\right)\frac{T_{n+2}}{T_{n+1}} = 0, \tag{2.44}$$

since $\alpha/\beta = p_n\overline{q_n}/\overline{p_n}q_n$ is a constant. □

Corollary 2.8. *Under the hypothesis of Proposition 2.6,*

$$\forall n \geq 1, \quad \frac{\left(\overline{p_{n+2}} - \overline{q_{n+2}}\right)}{\overline{p_{n+2}}}\frac{e_{n+1}}{e_n} = \left(\overline{q_{n+1}} - \overline{p_{n+1}}\right)\frac{\Phi_{n+1}(\alpha)}{\Phi_n(\alpha)}. \tag{2.45}$$

Proof. From (1.6) it follows that the formula in the statement is equivalent to

$$\left(\frac{\left(\overline{p_{n+2}} - \overline{q_{n+2}}\right)}{\overline{p_{n+2}}} - \alpha\left(\overline{q_{n+1}} - \overline{p_{n+1}}\right)\right)\Phi_{n+1}(\alpha) = \left(\frac{\left(\overline{p_{n+2}} - \overline{q_{n+2}}\right)}{\overline{p_{n+2}}}\right)\Phi_{n+1}(0)\Phi^*_{n+1}(\alpha). \tag{2.46}$$

We prove this last relation. Substituting (2.32) and (2.33), it suffices to show that

$$\left(\overline{q_{n+2}} - \overline{p_{n+2}}\right)\overline{\alpha}\frac{p_{n+2}}{\overline{p_{n+2}}} + p_{n+2}\left(\overline{q_{n+1}} - \overline{p_{n+1}}\right) = \left(\frac{T_{n+2}}{T_{n+3}} + \overline{q_{n+1}}\right)\left(p_{n+2} - q_{n+2}\right). \tag{2.47}$$

Now, written T_{n+2}/T_{n+3} in terms of α as in (2.20), the previous relation becomes

$$(\overline{q_{n+2}} - \overline{p_{n+2}})\overline{\alpha}\frac{p_{n+2}}{\overline{p_{n+2}}} + p_{n+2}(\overline{q_{n+1}} - \overline{p_{n+1}}) = \left(\frac{\overline{p_{n+1}}}{p_{n+1}}\alpha + \overline{q_{n+1}}\right)(p_{n+2} - q_{n+2}), \tag{2.48}$$

and it is true according to (2.42). □

3. Some Solutions

We state a necessary and sufficient condition in terms of the data $\{\Phi_n\}$.

Theorem 3.1. *Let $\{\Phi_n\}_{n\geq 0}$ be a MOPS such that $\Phi_1(z) = z - q_1$, $q_1 \in \mathbb{C}$ and $|q_1| \neq 1$. Also assume $D_{n+2} \neq 0$, for all $n \geq 1$. We define recursively a sequence $\{\Psi_n\}_{n\geq 0}$ of monic polynomials by the relations*

$$\forall n \geq 2, \quad \Phi_n(z) + q_n\Phi_{n-1}(z) = \Psi_n(z) + p_n\Psi_{n-1}(z), \quad p_n, q_n \in \mathbb{C}, \ p_nq_n \neq 0, \tag{3.1}$$

and $\Psi_1(z) = z - p_1$ with $p_1 \in \mathbb{C}$, $|p_1| \neq 1$, $p_1 \neq q_1$. Then, $\{\Psi_n(z)\}$ is a MOPS different from $\{\Phi_n(z)\}$ if and only if the following formulas hold:

(i) *For all $n \geq 2$, $p_n \neq q_n$,*

(ii) *For all $n \geq 2$, $|T_{n+1}| = 1$, where T_n is defined by* (2.16)*,*

(iii) *there exist two complex numbers α, β such that*

$$\forall n \geq 2, \quad \alpha = \frac{p_n}{\overline{p_n}}\frac{T_{n+1}}{T_{n+2}}, \qquad \beta = \frac{q_n}{\overline{q_n}}\frac{T_{n+1}}{T_{n+2}}, \tag{3.2}$$

(iv)

$$\forall n \geq 1, \quad \alpha\frac{(p_{n+1} - q_{n+1})}{p_{n+1}}\Phi_n(\alpha) = \frac{(\overline{q_{n+1}} - \overline{p_{n+1}})}{\overline{p_{n+1}}}\Phi_n^*(\alpha)T_{n+2}, \tag{3.3}$$

(v)

$$\forall n \geq 2, \quad \Phi_n(0) = T_{n+1} + T_{n+2}\overline{q_n}, \tag{3.4}$$

(vi)

$$\forall n \geq 1, \quad \frac{(p_{n+2} - q_{n+2})}{p_{n+2}}\frac{e_{n+1}}{e_n} = (q_{n+1} - p_{n+1})\frac{\overline{\Phi_{n+1}(\alpha)}}{\overline{\Phi_n(\alpha)}}. \tag{3.5}$$

Moreover, the sequences $\{\Phi_n\}$ and $\{\Psi_n\}$ are connected by

$$\forall n \geq 1, \quad \Psi_n(z) = \Phi_n(z) + \frac{e_n}{\overline{\Phi_n(\alpha)}}\frac{(p_{n+1} - q_{n+1})}{p_{n+1}}K_{n-1}(z, \alpha). \tag{3.6}$$

Proof. It only remains to establish the sufficient condition.

We first show that (3.6) implies (3.1)

$$\begin{aligned}\forall n \geq 1, \quad & \Psi_{n+1}(z) + p_{n+1}\Psi_n(z) \\ &= \Phi_{n+1}(z) + q_{n+1}\Phi_n(z) \\ &\quad + \left((p_{n+1} - q_{n+1})\Phi_n(z) + \frac{e_{n+1}}{\overline{\Phi_{n+1}(\alpha)}} \frac{(p_{n+2} - q_{n+2})}{p_{n+2}} K_n(z,\alpha) \right. \\ &\quad \left. + \frac{e_n}{\overline{\Phi_n(\alpha)}}(p_{n+1} - q_{n+1})K_{n-1}(z,\alpha) \right).\end{aligned} \tag{3.7}$$

The task is now to obtain that the expression in the brackets is null. Using (1.8), this expression becomes

$$\begin{aligned}&\left((p_{n+1} - q_{n+1}) + \frac{e_{n+1}}{e_n} \frac{\overline{\Phi_n(\alpha)}}{\overline{\Phi_{n+1}(\alpha)}} \frac{(p_{n+2} - q_{n+2})}{p_{n+2}} \right)\Phi_n(z) \\ &\quad + \left(\frac{e_{n+1}}{\overline{\Phi_{n+1}(\alpha)}} \frac{(p_{n+2} - q_{n+2})}{p_{n+2}} + \frac{e_n}{\overline{\Phi_n(\alpha)}}(p_{n+1} - q_{n+1}) \right) K_{n-1}(z,\alpha).\end{aligned} \tag{3.8}$$

Therefore, the result follows immediately from (3.5).

In order to obtain $\Psi_{n+1}(0)$, we take $z = 0$ in (3.6)

$$\forall n \geq 1, \quad \Psi_{n+1}(0) = \Phi_{n+1}(0) + \frac{(p_{n+2} - q_{n+2})}{p_{n+2}} \frac{e_{n+1}}{e_n} \frac{\Phi_n(\alpha)}{\overline{\Phi_{n+1}(\alpha)}} \overline{\alpha}^n. \tag{3.9}$$

Substituting (3.5) and (3.3), we get

$$\forall n \geq 1, \quad \Psi_{n+1}(0) - \Phi_{n+1}(0) = (q_{n+1} - p_{n+1}) \frac{\Phi_n(\alpha)}{\Phi_n^*(\alpha)} = (\overline{p_{n+1}} - \overline{q_{n+1}})\overline{\alpha} \frac{p_{n+1}}{\overline{p_{n+1}}} T_{n+2}. \tag{3.10}$$

Using (3.2) and (3.4), it is easy to check that

$$\forall n \geq 1, \quad \Psi_{n+1}(0) = T_{n+2} + T_{n+3}\overline{p_{n+1}}. \tag{3.11}$$

Now, we show that the sequence given by (3.6) satisfies (1.4) with $|\Psi_n(0)| \neq 1$, then it is a MOPS.

We will apply (1.12) as well as $K_n^*(z,\alpha) = \alpha^n K_n(z,\alpha)$, since $|\alpha| = 1$

$$\begin{aligned}
\forall n \geq 1, \quad & z\Psi_n(z) + \Psi_{n+1}(0)\Psi_n^*(z) \\
&= \Phi_{n+1}(z) - (\Phi_{n+1}(0) - \Psi_{n+1}(0))\Phi_n^*(z) \\
&\quad + \left(\frac{(p_{n+1} - q_{n+1})}{\overline{\Phi_n(\alpha)} p_{n+1}} + \Psi_{n+1}(0) \frac{(\overline{p_{n+1}} - \overline{q_{n+1}})}{\Phi_n(\alpha)\overline{p_{n+1}}} \alpha^{n-1} \right) z K_{n-1}(z,\alpha) e_n \\
&= \Phi_{n+1}(z) - (\Phi_{n+1}(0) - \Psi_{n+1}(0))\Phi_n^*(z) \\
&\quad + \left(\frac{\alpha(p_{n+1} - q_{n+1})}{\overline{\Phi_n(\alpha)} p_{n+1}} + \frac{\Psi_{n+1}(0)(\overline{p_{n+1}} - \overline{q_{n+1}})}{\Phi_n(\alpha)\overline{p_{n+1}}} \alpha^{n} \right) \left(K_n(z,\alpha) - \frac{\overline{\Phi_n^*(\alpha)}}{e_n} \Phi_n^*(z) \right) e_n.
\end{aligned} \tag{3.12}$$

If we show that the coefficient of $\Phi_n^*(z)$ is null and the coefficient of $K_n(z,\alpha)$ is $(e_{n+1}/\overline{\Phi_{n+1}(\alpha)})((p_{n+2} - q_{n+2})/p_{n+2})$, then (1.4) is true.

At first, we compute the coefficient of $\Phi_n^*(z)$.

$$\begin{aligned}
&\Psi_{n+1}(0) - \Phi_{n+1}(0) - \overline{\Phi_n^*(\alpha)} \left(\frac{\alpha(p_{n+1} - q_{n+1})}{\overline{\Phi_n(\alpha)} p_{n+1}} + \frac{\Psi_{n+1}(0)(\overline{p_{n+1}} - \overline{q_{n+1}})}{\Phi_n(\alpha)\overline{p_{n+1}}} \alpha^{n} \right) \\
&\quad = \frac{\overline{q_{n+1}}}{\overline{p_{n+1}}} \Psi_{n+1}(0) - \Phi_{n+1}(0) - \frac{\Phi_n(\alpha)}{\overline{\Phi_n(\alpha)}} \overline{\alpha}^{n-1} \frac{(p_{n+1} - q_{n+1})}{p_{n+1}} \\
&\quad = \frac{(\overline{q_{n+1}} - \overline{p_{n+1}})}{\overline{p_{n+1}}} T_{n+2} - \frac{\Phi_n(\alpha)}{\overline{\Phi_n(\alpha)}} \overline{\alpha}^{n-1} \frac{(p_{n+1} - q_{n+1})}{p_{n+1}},
\end{aligned} \tag{3.13}$$

and this is equal to zero from (3.3).

We can obtain the coefficient of $K_n(z,\alpha)$ by observation of (3.13). It is easy to see that this coefficient is $((\Psi_{n+1}(0) - \Phi_{n+1}(0))/\overline{\Phi_n^*(\alpha)})e_n$. By virtue of (3.9) it is equal to $e_{n+1}(p_{n+2} - q_{n+2})/\overline{\Phi_{n+1}(\alpha)} p_{n+2}$, and then the required result follows.

Finally, the condition $1 - |\Psi_n(0)|^2 \neq 0$ follows from (3.11) by using the same method as in the proof of (2.35).

Observe that condition (i) together with (3.6) gives $\Psi_n \neq \Phi_n$. □

Corollary 3.2. *Under the same conditions as in the previous theorem, the following relations hold*

(i)

$$\alpha = q_1 + \frac{e_1}{\overline{p_2}} \left(\frac{\overline{p_2} - \overline{q_2}}{\overline{q_1} - \overline{p_1}} \right), \tag{3.14}$$

where $e_1 = 1 - |q_1|^2$.

(ii)

$$\beta = p_1 + \frac{E_1}{\overline{q_2}}\left(\frac{\overline{p_2} - \overline{q_2}}{\overline{q_1} - \overline{p_1}}\right), \tag{3.15}$$

where $E_1 = 1 - |p_1|^2$.

(iii)

$$\forall n \geq 2, \quad T_{n+2} = \frac{p_n \cdots p_2}{\overline{p_n} \cdots \overline{p_2}}\left(\overline{q_1} + \frac{e_1}{p_2}\frac{(p_2 - q_2)}{(q_1 - p_1)}\right)^{n-1} T_3, \tag{3.16}$$

(iv)

$$\forall n \geq 2, \quad T_{n+2} = \frac{q_n \cdots q_2}{\overline{q_n} \cdots \overline{q_2}}\left(\overline{p_1} + \frac{E_1}{q_2}\frac{(p_2 - q_2)}{(q_1 - p_1)}\right)^{n-1} T_3. \tag{3.17}$$

Proof. We obtain α and β from (2.22) and (2.23) for $n = 1$, respectively. The items (iii) and (iv) are straightforward from (2.20) and (2.21). □

Now, we are going to express the Verblunsky coefficients for the solutions in terms of $\{p_n\}$ and $\{q_n\}$. We remember that to give a MOPS $\{\Phi_n\}$ on the unit circle is equivalent to know the sequence of complex numbers $\{\Phi_n(0)\}$ with $|\Phi_n(0)| \neq 1$.

Theorem 3.3. *Let* $\{p_n\}_{n\geq 1}$ *and* $\{q_n\}_{n\geq 1}$ *be two sequences of complex numbers such that*

(i) $p_n q_n \neq 0$ *and* $p_n \neq q_n$, *for all* $n \geq 1$,

(ii) $|q_1| \neq 1$ *and* $|\beta + q_n| \neq 1$, *for all* $n \geq 2$, *where* β *is given by* (3.15) *and* $|\beta| = 1$,

(iii) $|p_1| \neq 1$ *and* $|\alpha + p_n| \neq 1$, *for all* $n \geq 2$, *where* α *is given by* (3.14),

(iv) $\alpha/\beta = p_n\overline{q_n}/\overline{p_n}q_n$, *for all* $n \geq 2$.

Then, the only MOPS solutions of (3.1), such that $D_{n+2} \neq 0$ for all $n \geq 1$, verify

$$\Phi_n(0) = \begin{cases} -q_1, & \text{if } n = 1, \\ \dfrac{p_n \cdots p_2}{\overline{p_n} \cdots \overline{p_2}}\dfrac{\overline{q_n}}{q_n}\overline{\alpha}^{n-1}(\beta + q_n)T_3, \quad T_3 = \dfrac{p_1 - q_1}{\overline{q_1} - \overline{p_1}} & \text{if } n \geq 2, \end{cases}$$
$$\Psi_n(0) = \begin{cases} -p_1, & \text{if } n = 1, \\ \dfrac{q_n \cdots q_2}{\overline{q_n} \cdots \overline{q_2}}\dfrac{\overline{p_n}}{p_n}\overline{\beta}^{n-1}(\alpha + p_n)T_3, \quad T_3 = \dfrac{p_1 - q_1}{\overline{q_1} - \overline{p_1}} & \text{if } n \geq 2. \end{cases} \tag{3.18}$$

Moreover, the sequences $\{\Phi_n(z)\}$ and $\{\Psi_n(z)\}$ are connected by

$$\forall n \geq 2, \quad \Psi_n(z) = \Phi_n(z) + (-1)^{n-1} p_2 \cdots p_n (q_1 - p_1) K_{n-1}(z, \alpha),$$
$$\forall n \geq 2, \quad \Phi_n(z) = \Psi_n(z) + (-1)^{n-1} q_2 \cdots q_n (p_1 - q_1) \widetilde{K_{n-1}}(z, \beta). \tag{3.19}$$

Proof. In order to obtain $\Phi_n(0)$ and $\Psi_n(0)$ we use the hypothesis (iv) as well as (2.33)–(3.16) and (2.34)–(3.17), respectively. The conditions $|\Phi_n(0)|, |\Psi_n(0)| \neq 1$, follow from (ii) and (iii), respectively.

Applying u in (3.6), it holds that

$$\forall n \geq 1, \quad u(\Psi_n(z)) = \frac{e_n}{\overline{\Phi_n(\alpha)}} \frac{(p_{n+1} - q_{n+1})}{p_{n+1}}. \tag{3.20}$$

Combining with (2.13), we get

$$e_n = (-1)^{n+1} p_n \cdots p_2 (q_1 - p_1) \frac{p_{n+1}}{(p_{n+1} - q_{n+1})} \overline{\Phi_n(\alpha)}. \tag{3.21}$$

Again from (2.22) we get

$$\forall n \geq 2, \quad \Psi_n(z) = \Phi_n(z) + (-1)^{n-1} p_2 \cdots p_n (q_1 - p_1) K_{n-1}(z, \alpha). \tag{3.22}$$

This completes the proof because of the symmetry of the problem. □

Remark 3.4. Notice that the restrictions given for p_n and q_n in the previous theorem ensure that the sequences generated by $\Phi_n(0)$ and $\Psi_n(0)$ are MPOS, but they do not ensure that $\Phi_n(z)$ and $\Psi_n(z)$ fulfill (3.1). In fact, other similar conditions to (2.42) seem to be necessary in order to obtain a characterization of the Verblunsky coefficients in terms of p_n and q_n.

4. Linear Functionals

In this section we establish the relation between the regular functionals associated with the sequences $\{\Phi_n\}$ and $\{\Psi_n\}$.

Proposition 4.1. *Let u and $\mathfrak{L}$ be the regular functionals normalized by $u(1) = u_0 = 1$ and $\mathfrak{L}(1) = v_0 = 1$ associated with $\{\Phi_n\}$ and $\{\Psi_n\}$, respectively. Then, the following relation holds*

$$\lambda(z - \beta)\mathfrak{L} = (z - \alpha)u, \quad \text{where } \lambda = \frac{\Phi_1(\alpha)}{\Psi_1(\beta)}. \tag{4.1}$$

Proof. We will show that

$$\forall n \geq 0, \quad (\lambda(z - \beta)\mathfrak{L})(\Psi_n) = ((z - \alpha)u)(\Psi_n), \tag{4.2}$$

wherefrom the result follows because $\{\Psi_n\}$ is a basis in $\mathbb{P}$.

If $n = 0$ the equality is trivial by definition of λ.

If $n \geq 1$, the left-hand side in (4.1) is

$$\lambda\mathfrak{L}((z - \beta)\Psi_n(z)) = -\lambda\Psi_{n+1}(0)E_n = -\lambda(T_{n+2} + \overline{p_{n+1}}T_{n+3})E_n, \tag{4.3}$$

where the last equality follows from (2.34).

We compute the right-hand side using(2.31)

$$\forall n \geq 2, \quad u((z-\alpha)\Psi_n(z)) = -\Phi_{n+1}(0)e_n + \frac{T_{n+2}}{p_{n+1}}\left(\overline{p_{n+1}} - \overline{q_{n+1}}\right)e_n. \tag{4.4}$$

In the same way, by virtue of (2.33), the right-hand side is equal to $-(\overline{q_{n+1}}/\overline{p_{n+1}})(T_{n+2} + \overline{p_{n+1}}T_{n+3})e_n$. Therefore, it only remains to check the equality $\lambda E_n = \overline{q_{n+1}}/\overline{p_{n+1}}e_n$. In order to do this we take the conjugate in (2.35), obtaining

$$\forall n \geq 2, \quad \frac{E_n}{e_n} = \frac{\overline{q_{n+1}}}{\overline{p_{n+1}}}\frac{\overline{p_2}}{\overline{q_2}}\frac{E_1}{e_1}. \tag{4.5}$$

Finally, we see that the equality $\lambda = (\alpha - q_1)/(\beta - p_1) = (\overline{q_2}/\overline{p_2})(e_1/E_1)$ is true due to (3.14) and (3.15). □

Remark 4.2. The opposite question has been proved in [8]. That is, if u and $\mathfrak{L}$ are regular functionals related by (4.1) with $|\alpha| = |\beta| = 1$, then the corresponding orthogonal polynomials satisfy (2.1).

5. Carathéodory's Functions

In this section we obtain the relation between the Carathéodory functions associated with the sequences $\{\Phi_n\}$ and $\{\Psi_n\}$. We denote by $\{v_n\}$ the sequence of the moments corresponding to $\mathfrak{L}$, that is, $\mathfrak{L}(z^n) = v_n$, for all $n \geq 0$.

Proposition 5.1. *Let F_u and $F_{\mathfrak{L}}$ be the Carathéodory functions associated with $\{\Phi_n\}$ and $\{\Psi_n\}$, respectively. Then, $F_{\mathfrak{L}}$ is the following rational transformation of F_u:*

$$F_{\mathfrak{L}}(z) = \frac{(1/\lambda)((z-\alpha)F_u(z) + (z+\alpha)) - (z+\beta)}{(z-\beta)}. \tag{5.1}$$

Proof. Indeed, from (1.13) $F_{\mathfrak{L}}(z) = v_0 + 2\sum_{n=1}^{+\infty} \overline{v_k} z^k$, thus

$$\left(1 - z\overline{\beta}\right)F_{\mathfrak{L}}(z) = \left(1 - z\overline{\beta}\right)v_0 + 2\overline{v_1}z + 2\sum_{n=1}^{+\infty}\left(\overline{v_{n+1}} - \overline{\beta}\overline{v_n}\right)z^{n+1}. \tag{5.2}$$

Using (4.1), it holds that $\lambda(v_{n+1} - \beta v_n) = u_{n+1} - \alpha u_n$. Therefore,

$$\sum_{n=1}^{+\infty}\left(\overline{v_{n+1}} - \overline{\beta}\overline{v_n}\right)z^{n+1} = \frac{1}{\overline{\lambda}}\sum_{n=1}^{+\infty}\left(\overline{u_{n+1}} - \overline{\alpha}\overline{u_n}\right)z^{n+1}, \tag{5.3}$$

where from

$$F_{\mathfrak{L}}(z) = \frac{\left(1/\overline{\lambda}\right)((1 - z\overline{\alpha})F_u - (1 + z\overline{\alpha})) + \left(1 + z\overline{\beta}\right)v_0}{\left(1 - z\overline{\beta}\right)}. \tag{5.4}$$

Putting $\overline{\alpha}\beta = \overline{\lambda}/\lambda$ and $v_0 = 1$, we find (5.1). □

References

[1] Y. L. Geronimus, "Polynomials orthogonal on a circle and their applications," *American Mathematical Society Translations, Series 1*, vol. 3, pp. 1–78, 1962.

[2] B. Simon, *Orthogonal Polynomials on the Unit Circle*, vol. 54 of *AMS Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2005.

[3] G. Szegö, *Orthogonal Polynomials*, vol. 23 of *AMS Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 4th edition, 1975.

[4] M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola, "On linearly related orthogonal polynomials and their functionals," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 307–319, 2003.

[5] K. H. Kwon, J. H. Lee, and F. Marcellán, "Orthogonality of linear combinations of two orthogonal polynomial sequences," *Journal of Computational and Applied Mathematics*, vol. 137, no. 1, pp. 109–122, 2001.

[6] P. Maroni, "Prolégomènes à l'étude des polynômes orthogonaux semi-classiques," *Annali di Matematica Pura ed Applicata*, vol. 149, no. 4, pp. 165–184, 1987.

[7] F. Marcellán, F. Peherstorfer, and R. Steinbauer, "Orthogonality properties of linear combinations of orthogonal polynomials. II," *Advances in Computational Mathematics*, vol. 7, no. 3, pp. 401–428, 1997.

[8] C. Suárez, "On second kind polynomials associated with rational transformations of linear functionals," *Journal of Mathematical Analysis and Applications*, vol. 358, no. 1, pp. 148–158, 2009.