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Research Article

On the Global Character of the System of Piecewise Linear Difference Equations $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$

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We consider the system in the title where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. We show that the system has exactly two prime period-5 solutions and a unique equilibrium point $(0, -1)$. We also show that every solution of the system is eventually one of the two prime period-5 solutions or else the unique equilibrium point.

1. Introduction

In this paper, we consider the system of piecewise linear difference equations

$$\begin{aligned} x_{n+1} &= |x_n| - y_n - 1, \\ y_{n+1} &= x_n - |y_n|, \end{aligned} \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. We show that every solution of System (1.1) is eventually either one of two prime period-5 solutions or else the unique equilibrium point $(0, -1)$.

System (1.1) was motivated by Devaney's Gingerbread man map [1, 2]

$$x_{n+1} = |x_n| - x_{n-1} + 1 \tag{1.2}$$

or its equivalent system of piecewise linear difference equations [3, 4]

$$\begin{aligned} x_{n+1} &= |x_n| - y_n + 1, \\ y_{n+1} &= x_n, \end{aligned} \qquad n = 0, 1, 2, \ldots. \tag{1.3}$$

We believe that the methods and techniques used in this paper will be useful in discovering the global character of solutions of similar systems, including the Gingerbread man map.

2. The Global Behavior of the Solutions of System (1.1)

System (1.1) has the equilibrium point $(\overline{x}, \overline{y}) \in \mathbf{R}^2$ given by

$$(\overline{x}, \overline{y}) = (0, -1). \tag{2.1}$$

System (1.1) has two prime period-5 solutions,

$$P_5^1 = \begin{pmatrix} x_0 = 0, & y_0 = 1 \\ x_1 = -2, & y_1 = -1 \\ x_2 = 2, & y_2 = -3 \\ x_3 = 4, & y_3 = -1 \\ x_4 = 4, & y_4 = 3 \end{pmatrix},$$
$$P_5^2 = \begin{pmatrix} x_0 = 0, & y_0 = \dfrac{1}{7} \\ x_1 = -\dfrac{8}{7}, & y_1 = -\dfrac{1}{7} \\ x_2 = \dfrac{2}{7}, & y_2 = -\dfrac{9}{7} \\ x_3 = \dfrac{4}{7}, & y_3 = -1 \\ x_4 = \dfrac{4}{7}, & y_4 = -\dfrac{3}{7}. \end{pmatrix}. \tag{2.2}$$

Set

$$\begin{aligned}
l_1 &= \{(x, y) : x \geq 0,\ y = 0\},\\
l_2 &= \{(x, y) : x = 0,\ y \geq 0\},\\
l_3 &= \{(x, y) : x < 0,\ y = 0\},\\
l_4 &= \{(x, y) : x = 0,\ y < 0\},\\
Q_1 &= \{(x, y) : x > 0,\ y > 0\},\\
Q_2 &= \{(x, y) : x < 0,\ y > 0\},\\
Q_3 &= \{(x, y) : x < 0,\ y < 0\},\\
Q_4 &= \{(x, y) : x > 0,\ y < 0\}.
\end{aligned} \tag{2.3}$$

Theorem 2.1. *Let $(x_0, y_0) \in \mathbf{R}^2$. Then there exists an integer $\mathcal{N} \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=\mathcal{N}}^{\infty}$ is eventually either the prime period-5 solution P_5^1, the prime period-5 solution P_5^2, or else the unique equilibrium point $(0, -1)$.*

The proof is a direct consequence of the following lemmas.

Lemma 2.2. *Suppose there exists an integer $M \geq 0$ such that $-1 \leq x_M \leq 0$ and $y_M = -x_M - 1$. Then $(x_{M+1}, y_{M+1}) = (0, -1)$, and so $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is the equilibrium solution.*

Proof. Note that

$$\begin{aligned}
x_{M+1} &= |x_M| - y_M - 1 = -x_M - (-x_M - 1) - 1 = 0,\\
y_{M+1} &= x_M - |y_M| = x_M - (x_M + 1) = -1,
\end{aligned} \tag{2.4}$$

and so the proof is complete. □

Lemma 2.3. *Suppose there exists an integer $M \geq 0$ such that $x_M \geq 1$ and $y_M = x_M - 1$. Then $(x_{M+1}, y_{M+1}) = (0, 1)$, and so $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^1.*

Proof. We have

$$\begin{aligned}
x_{M+1} &= |x_M| - y_M - 1 = x_M - (x_M - 1) - 1 = 0,\\
y_{M+1} &= x_M - |y_M| = x_M - (x_M - 1) = 1,
\end{aligned} \tag{2.5}$$

and so the proof is complete. □

Lemma 2.4. *Suppose there exists an integer $M \geq 0$ such that $x_M = 0$ and $y_M \geq 0$. Then the following statements are true.*

(1) $x_{M+5} = 0$.

(2) *If $y_M > 1/4$, then $\{(x_n, y_n)\}_{n=M+5}^{\infty}$ is P_5^1.*

(3) *If $0 \leq y_M \leq 1/4$, then $y_{M+5} = 8y_M - 1$.*

Proof. We have $x_M = 0$ and $y_M \geq 0$. Then

$$\begin{aligned}
x_{M+1} &= |x_M| - y_M - 1 = -y_M - 1 < 0,\\
y_{M+1} &= x_M - |y_M| = -y_M \leq 0,\\
x_{M+2} &= |x_{M+1}| - y_{M+1} - 1 = 2y_M \geq 0,\\
y_{M+2} &= x_{M+1} - |y_{M+1}| = -2y_M - 1 < 0,\\
x_{M+3} &= |x_{M+2}| - y_{M+2} - 1 = 4y_M \geq 0,\\
y_{M+3} &= x_{M+2} - |y_{M+2}| = -1,\\
x_{M+4} &= |x_{M+3}| - y_{M+3} - 1 = 4y_M \geq 0,\\
y_{M+4} &= x_{M+3} - |y_{M+3}| = 4y_M - 1,\\
x_{M+5} &= |x_{M+4}| - y_{M+4} - 1 = 0,
\end{aligned} \tag{2.6}$$

and so statement (1) is true.

If $y_M > 1/4$, then $y_{M+5} = x_{M+4} - |y_{M+4}| = 1$. That is, $(x_{M+5}, y_{M+5}) = (0, 1)$ and so statement (2) is true.

If $0 \leq y_M \leq 1/4$, then $y_{M+5} = x_{M+4} - |y_{M+4}| = 8y_M - 1$, and so statement (3) is true. □

Lemma 2.5. *Suppose there exists an integer $M \geq 0$ such that $x_M = 0$ and $y_M < -1$. Then the following statements are true.*

(1) $x_{M+4} = 0$.

(2) *If $-3/2 < y_M < -1$, then $y_{M+4} = -4y_M - 5$.*

(3) *If $y_M \leq -3/2$, then $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1.*

Proof. We have $x_M = 0$ and $y_M < -1$. Then

$$\begin{aligned}
x_{M+1} &= |x_M| - y_M - 1 = -y_M - 1 > 0,\\
y_{M+1} &= x_M - |y_M| = y_M < 0,\\
x_{M+2} &= |x_{M+1}| - y_{M+1} - 1 = -2y_M - 2 > 0,\\
y_{M+2} &= x_{M+1} - |y_{M+1}| = -1,\\
x_{M+3} &= |x_{M+2}| - y_{M+2} - 1 = -2y_M - 2 > 0,\\
y_{M+3} &= x_{M+2} - |y_{M+2}| = -2y_M - 3,\\
x_{M+4} &= |x_{M+3}| - y_{M+3} - 1 = 0,
\end{aligned} \tag{2.7}$$

and so statement (1) is true.

Now if $-3/2 < y_M < -1$, then $y_{M+3} = -2y_M - 3 < 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = -4y_M - 5$, and so statement (2) is true.

Lastly, if $y_M \leq -3/2$, then $y_{M+3} = -2y_M - 3 \geq 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = 1$; that is, $(x_{M+4}, y_{M+4}) = (0, 1)$ and so statement (3) is true. □

Lemma 2.6. *Suppose there exists an integer $M \geq 0$ such that $x_M \geq 0$ and $y_M = 0$. Then the following statements are true.*

(1) *If $x_M \geq 1$, then $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1.*

(2) *If $1/4 < x_M < 1$, then $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is P_5^1.*

(3) *If $0 \leq x_M \leq 1/4$, then $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$.*

Proof. First consider the case $x_M \geq 1$ and $y_M = 0$. Then

$$\begin{gathered} x_{M+1} = |x_M| - y_M - 1 = x_M - 1 \geq 0, \\ y_{M+1} = x_M - \left|y_M\right| = x_M > 0, \\ x_{M+2} = |x_{M+1}| - y_{M+1} - 1 = -2, \\ y_{M+2} = x_{M+1} - \left|y_{M+1}\right| = -1, \end{gathered} \tag{2.8}$$

and so statement (1) is true.

Next consider the case $0 \leq x_M < 1$ and $y_M = 0$. Then

$$\begin{gathered} x_{M+1} = |x_M| - y_M - 1 = x_M - 1 < 0, \\ y_{M+1} = x_M - \left|y_M\right| = x_M \geq 0, \\ x_{M+2} = |x_{M+1}| - y_{M+1} - 1 = -2x_M \leq 0, \\ y_{M+2} = x_{M+1} - \left|y_{M+1}\right| = -1, \\ x_{M+3} = |x_{M+2}| - y_{M+2} - 1 = 2x_M \geq 0, \\ y_{M+3} = x_{M+2} - \left|y_{M+2}\right| = -2x_M - 1 < 0, \\ x_{M+4} = |x_{M+3}| - y_{M+3} - 1 = 4x_M \geq 0, \\ y_{M+4} = x_{M+3} - \left|y_{M+3}\right| = -1, \\ x_{M+5} = |x_{M+4}| - y_{M+4} - 1 = 4x_M \geq 0, \\ y_{M+5} = x_{M+4} - \left|y_{M+4}\right| = 4x_M - 1, \\ x_{M+6} = |x_{M+5}| - y_{M+5} - 1 = 0. \end{gathered} \tag{2.9}$$

If $1/4 < x_M < 1$, then $y_{M+5} = 4x_M - 1 > 0$ and so $y_{M+6} = x_{M+5} - |y_{M+5}| = 1$. That is, $(x_{M+6}, y_{M+6}) = (0, 1)$ and so statement (2) is true.

If $0 \leq x_M \leq 1/4$, then $y_{M+5} = 4x_M - 1 \leq 0$. Thus $y_{M+6} = x_{M+5} - |y_{M+5}| = 8x_M - 1$, and so statement (3) is true. □

Lemma 2.7. *Suppose there exists an integer $M \geq 0$ such that $x_M < -1$ and $y_M = 0$. Then the following statements are true.*

(1) $x_{M+4} = 0$.

(2) *If $-3/2 \leq x_M < -1$, then $y_{M+4} = -4x_M - 5$.*

(3) *If $x_M < -3/2$, then $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1.*

Proof. Let $x_M < -1$ and $y_M = 0$. Then

$$\begin{aligned}
x_{M+1} &= |x_M| - y_M - 1 = -x_M - 1 > 0, \\
y_{M+1} &= x_M - |y_M| = x_M < 0, \\
x_{M+2} &= |x_{M+1}| - y_{M+1} - 1 = -2x_M - 2 > 0, \\
y_{M+2} &= x_{M+1} - |y_{M+1}| = -1, \\
x_{M+3} &= |x_{M+2}| - y_{M+2} - 1 = -2x_M - 2 > 0, \\
y_{M+3} &= x_{M+2} - |y_{M+2}| = -2x_M - 3, \\
x_{M+4} &= |x_{M+3}| - y_{M+3} - 1 = 0,
\end{aligned} \tag{2.10}$$

and so statement (1) is true.

If $-3/2 \le x_M < -1$, then $y_{M+3} = -2x_M - 3 \le 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = -4x_M - 5$, and so statement (2) is true.

If $x_M < -3/2$, then $y_{M+3} = -2x_M - 3 > 0$ and $y_{M+4} = x_{M+3} - |y_{M+3}| = 1$. That is, $(x_{M+4}, y_{M+4}) = (0, 1)$ and so $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1 and the proof is complete. □

We now give the proof of Theorem 2.1 when (x_M, y_M) is in $l_2 = \{(x, y) : x = 0, y \ge 0\}$.

Lemma 2.8. *Suppose there exists an integer $M \ge 0$ such that $(x_M, y_M) \in l_2$. Then the following statements are true.*

(1) *If $0 \le y_M < 1/7$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*

(2) *If $y_M = 1/7$, then the solution $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^2.*

(3) *If $y_M > 1/7$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.*

Proof. (1) We will first show that statement (1) is true. Suppose $0 \le y_M < 1/7$; for each $n \ge 0$, let

$$a_n = \frac{2^{3n} - 1}{7 \cdot 2^{3n}}. \tag{2.11}$$

Observe that

$$0 = a_0 < a_1 < a_2 < \cdots < \frac{1}{7}, \quad \lim_{n \to \infty} a_n = \frac{1}{7}. \tag{2.12}$$

Thus there exists a unique integer $K \ge 0$ such that $y_M \in [a_K, a_{K+1})$.

We first consider the case $K = 0$; that is, $y_M \in [0, 1/8)$. By statements (1) and (3) of Lemma 2.4, $x_{M+5} = 0$ and $y_{M+5} = 8y_M - 1$. Clearly $y_{M+5} < 0$, and so

$$\begin{aligned}
x_{M+6} &= |x_{M+5}| - y_{M+5} - 1 = -8y_M \le 0, \\
y_{M+6} &= x_{M+5} - |y_{M+5}| = 8y_M - 1.
\end{aligned} \tag{2.13}$$

Now $-1 < x_{M+6} \leq 0$ and $y_{M+6} = -x_{M+6} - 1$, and so by Lemma 2.2, $\{(x_n, y_n)\}_{n=M+7}^{\infty}$ is the equilibrium solution.

Without loss of generality, we may assume $K \geq 1$.

For each integer n such that $n \geq 0$, let $\mathcal{P}(n)$ be the following statement:

$$\begin{aligned} x_{M+5n+5} &= 0, \\ y_{M+5n+5} &= 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right) \geq 0. \end{aligned} \tag{2.14}$$

Claim 1. $\mathcal{P}(n)$ is true for $0 \leq n \leq K-1$.

The proof Claim 1 will be by induction on n. We will first show that $\mathcal{P}(0)$ is true.

Recall that $x_M = 0$ and $y_M \in [a_K, a_{K+1}) \subset [1/8, 1/7)$. Then by statements (1) and (3) of Lemma 2.4, we have $x_{M+5(0)+5} = 0$ and $y_{M+5(0)+5} = 8y_M - 1$.

Note that,

$$y_{M+5(0)+5} = 8y_M - 1 = 2^{3(0+1)} y_M - \left(\frac{2^{3(0+1)} - 1}{7}\right) \geq 0 \tag{2.15}$$

and so $\mathcal{P}(0)$ is true. Thus if $K = 1$, then we have shown that for $0 \leq n \leq K-1$, $\mathcal{P}(n)$ is true. It remains to consider the case $K \geq 2$. So assume that $K \geq 2$. Let n be an integer such that $0 \leq n \leq K-2$ and suppose $\mathcal{P}(n)$ is true. We will show that $\mathcal{P}(n+1)$ is true.

Since $\mathcal{P}(n)$ is true, we know

$$x_{M+5n+5} = 0, \qquad y_{M+5n+5} = 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right) \geq 0. \tag{2.16}$$

It is easy to verify that for $y_M \in [1/8, 1/7)$,

$$y_{M+5n+5} = 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right) < \frac{1}{4}. \tag{2.17}$$

Thus by statements (1) and (3) of Lemma 2.4,

$$\begin{aligned} x_{M+5(n+1)+5} &= 0, \\ y_{M+5(n+1)+5} &= 8\left(y_{M+5n+5}\right) - 1 \\ &= 2^3\left[2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right)\right] - 1 \\ &= 2^{3n+6} y_M - \frac{2^{3n+6}}{7} + \frac{2^3}{7} - 1 \\ &= 2^{3(n+2)} y_M - \left(\frac{2^{3(n+2)} - 1}{7}\right). \end{aligned} \tag{2.18}$$

Recall that $y_M \in [a_K, a_{K+1}) = [(2^{3K}-1)/(7\cdot 2^{3K}), (2^{3(K+1)}-1)/(7\cdot 2^{3(K+1)}))$.
In particular,

$$\begin{aligned}
y_{M+5(n+1)+5} &= 2^{3(n+2)}y_M - \left(\frac{2^{3(n+2)}-1}{7}\right)\\
&\geq 2^{3(n+2)}\left(\frac{2^{3K}-1}{7\cdot 2^{3K}}\right) - \left(\frac{2^{3(n+2)}-1}{7}\right)\\
&= \frac{2^{3n+3K+6}}{7\cdot 2^{3K}} - \frac{2^{3n+6}}{7\cdot 2^{3K}} - \frac{2^{3n+6}}{7} + \frac{1}{7}\\
&= \frac{1}{7}\left(1-2^{3[n-(K-2)]}\right) \geq \frac{1}{7}(1-1)\\
&= 0,
\end{aligned} \tag{2.19}$$

and so $\mathcal{P}(n+1)$ is true. Thus the proof of the claim is complete. That is, $\mathcal{P}(n)$ is true for $0 \leq n \leq K-1$. Specifically, $\mathcal{P}(K-1)$ is true, and so

$$x_{M+5(K-1)+5} = 0, \qquad y_{M+5(K-1)+5} = 2^{3K}y_M - \left(\frac{2^{3K}-1}{7}\right) \geq 0. \tag{2.20}$$

In particular,

$$2^{3K}\left(\frac{2^{3K}-1}{7\cdot 2^{3K}}\right) - \left(\frac{2^{3K}-1}{7}\right) \leq y_{M+5(K-1)+5} < 2^{3K}\left(\frac{2^{3K+3}-1}{7\cdot 2^{3K+3}}\right) - \left(\frac{2^{3K}-1}{7}\right). \tag{2.21}$$

That is, $0 \leq y_{M+5(K-1)+5} < 1/8$, and so by case $K=0$, $\{(x_n, y_n)\}_{n=M+5K+7}^{\infty}$ is the equilibrium solution, and the proof of statement (1) is complete.

(2) We will next show that statement (2) is true. Suppose $(x_M, y_M) = (0, 1/7)$. Note that $(0, 1/7) \in P_5^2$. Thus the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is P_5^2.

(3) Finally, we will show that statement (3) is true. Suppose $y_M > 1/7$.

First consider $y_M > 1/4$. By statement (2) of Lemma 2.4, the solution $\{(x_n, y_n)\}_{n=M+5}^{\infty}$ is P_5^1.

Next consider the case $y_M \in (1/7, 1/4]$. For each $n \geq 1$, let

$$b_n = \frac{2^{3n-1}+3}{7\cdot 2^{3n-1}}. \tag{2.22}$$

Observe that

$$\frac{1}{4} = b_1 > b_2 > b_3 > \cdots > \frac{1}{7}, \quad \lim_{n\to\infty} b_n = \frac{1}{7}. \tag{2.23}$$

Thus there exists a unique integer $K \geq 1$ such that $y_M \in (b_{K+1}, b_K]$.

Note that the statement $\wp(n)$ which we stated and proved in the proof of statement (1) of this lemma still holds. Specifically $\wp(K-1)$ is true, and so

$$x_{M+5(K-1)+5} = 0, \qquad y_{M+5(K-1)+5} = 2^{3K} y_M - \left(\frac{2^{3K}-1}{7}\right) \geq 0. \tag{2.24}$$

Recall that for $y_M \in (b_{K+1}, b_K]$.

In particular,

$$y_{M+5K} = 2^{3K} y_M - \left(\frac{2^{3K}-1}{7}\right) > 2^{3K}\left(\frac{2^{3K+2}+3}{7\cdot 2^{3K+2}}\right) - \left(\frac{2^{3K}-1}{7}\right) = \frac{1}{4}. \tag{2.25}$$

By statement (2) of Lemma 2.4, the solution $\{(x_n, y_n)\}_{n=M+5K+5}^{\infty}$ is P_5^1. □

We now give the proof of Theorem 2.1 when (x_M, y_M) is in $l_4 = \{(x,y) : x = 0, y < 0\}$.

Lemma 2.9. *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_4$. Then the following statements are true.*

(1) *If $-9/7 < y_M < 0$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*

(2) *If $y_M = -9/7$, then the solution $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.*

(3) *If $y_M < -9/7$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.*

Proof. (1) We will first show that statement (1) is true. So suppose $-9/7 < y_M < 0$.

Case 1. Suppose $-1 \leq y_M < 0$. Then

$$\begin{gathered} x_{M+1} = |x_M| - y_M - 1 = -y_M - 1 \leq 0, \\ y_{M+1} = x_M - |y_M| = y_M. \end{gathered} \tag{2.26}$$

In particular, $-1 < x_{M+1} \leq 0$ and $y_{M+1} = -x_{M+1} - 1$, and so by Lemma 2.2, $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.

Case 2. Suppose $-5/4 \leq y_M < -1$. By statements (1) and (2) of Lemma 2.5, $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Then

$$\begin{gathered} x_{M+5} = |x_{M+4}| - y_{M+4} - 1 = 4y_M + 4 < 0, \\ y_{M+5} = x_{M+4} - |y_{M+4}| = -4y_M - 5. \end{gathered} \tag{2.27}$$

Thus $-1 \leq x_{M+5} < 0$ and $y_{M+5} = -x_{M+5} - 1$, and so by Lemma 2.2, $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is the equilibrium solution.

Case 3. Suppose $-9/7 < y_M < -5/4$. By statements (1) and (2) of Lemma 2.5, $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Note that $0 < y_{M+4} < 1/7$ and so by statement (1) of Lemma 2.8, $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually equilibrium solution.

(2) We will next show that statement (2) is true. Suppose $y_M = -9/7$. By direct calculations we have $(x_{M+1}, y_{M+1}) = (2/7, -9/7)$. So the solution $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.

(3) Finally, we will show that statement (3) is true. Suppose $x_M = 0$ and $y_M < -9/7$.

Case 1. Suppose $-3/2 < y_M < -9/7$. By statements (1) and (2) of Lemma 2.5, we have $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Note that $1/7 < y_{M+4} < 1$ and so by statement (3) of Lemma 2.8, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually P_5^1.

Case 2. Suppose $y_M \leq -3/2$. By statement (3) of Lemma 2.5, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1. □

We now give the proof of Theorem 2.1 when (x_M, y_M) is in $l_1 = \{(x, y) : x \geq 0, y = 0\}$.

Lemma 2.10. *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_1$. Then the following statements are true.*

(1) *If $0 \leq x_M < 1/7$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*

(2) *If $x_M = 1/7$, then the solution $\{(x_n, y_n)\}_{n=M+3}^{\infty}$ is P_5^2.*

(3) *If $x_M > 1/7$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.*

Proof. (1) We will first show that statement (1) is true. So suppose $0 \leq x_M < 1/7$ and $y_M = 0$. By statement (3) of Lemma 2.6, $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$. In particular, $-1 < y_{M+6} < 1/7$ and so by statement (1) of Lemma 2.8 and statement (1) of Lemma 2.9, $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is eventually the equilibrium solution.

(2) We will next show that statement (2) is true. Suppose $x_M = 1/7$. By direct calculations we have $(x_{M+3}, y_{M+3}) = (2/7, -9/7)$. Thus the solution $\{(x_n, y_n)\}_{n=M+3}^{\infty}$ is P_5^2.

(3) Finally, we will show statement (3) is true.

First consider the case $1/7 < x_M \leq 1/4$. By statement (3) of Lemma 2.6, $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$. Now, $1/7 < y_{M+6} \leq 1$ and so by statement (3) of Lemma 2.8, the solution $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is eventually P_5^1.

Next consider the case $x_M > 1/4$. Then by statements (1) and (2) of Lemma 2.6, if $x_M \geq 1$ then $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1, and if $1/4 < x_M < 1$ then $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is P_5^1. □

We next give the proof of Theorem 2.1 when (x_M, y_M) is in $l_3 = \{(x, y) : x < 0, y = 0\}$.

Lemma 2.11. *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_3$. Then the following statements are true.*

(1) *If $-9/7 < x_M < 0$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*

(2) *If $x_M = -9/7$, then the solution $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.*

(3) *If $x_M < -9/7$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.*

Proof. (1) We will first prove statement (1) is true. Suppose $-9/7 < x_M < 0$.

First consider the case $-1 \leq x_M < 0$. Then

$$\begin{aligned} x_{M+1} &= |x_M| - y_M - 1 = -x_M - 1, \\ y_{M+1} &= x_M - |y_M| = x_M. \end{aligned} \tag{2.28}$$

In particular, $-1 < x_{M+1} \le 0$ and $y_{M+1} = -x_M - 1$ and so by Lemma 2.2, $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.

Next consider the case $-9/7 < x_M < -1$. By statements (1) and (2) of Lemma 2.7, x_{M+4}=0 and $y_{M+4} = -4x_M - 5$. In particular, $-1 < y_{M+4} < 1/7$ and so by statement (1) of Lemma 2.8 and statement (1) of Lemma 2.9, $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually the equilibrium solution.

(2) We will next show that statement (2) is true. Suppose $x_M = -9/7$. By direct calculations, we have $(x_{M+1}, y_{M+1}) = (2/7, -9/7)$. That is, $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.

(3) Lastly, we will show that statement (3) is true. Suppose $x_M < -9/7$.

First consider the case $-3/2 \le x_M < -9/7$. By statements (1) and (2) of Lemma 2.7, $x_{M+4} = 0$ and $y_{M+4} = -4x_M - 5$. In particular, $1/7 < y_{M+4} \le 1$ and so by statement (3) of Lemma 2.8, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually P_5^1.

Next consider the case $x_M < -3/2$. By statement (3) of Lemma 2.7, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1. □

We next give the proof of Theorem 2.1 when (x_M, y_M) is in $Q_1 = \{(x, y) : x > 0, y > 0\}$.

Lemma 2.12. *Suppose there exists an integer $M \ge 0$ such that $(x_M, y_M) \in Q_1$. Then the following statements are true.*

(1) *If $y_M \le x_M - 1$, then the solution $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1.*

(2) *If $y_M > x_M - 1$, then there exists an integer N such that $(x_{M+N}, y_{M+N}) \in l_2 \cup l_4$.*

Proof. Suppose $x_M > 0$ and $y_M > 0$.

Then

$$x_{M+1} = |x_M| - y_M - 1 = x_M - y_M - 1,$$
$$y_{M+1} = x_M - |y_M| = x_M - y_M. \tag{2.29}$$

Case 1. Suppose $y_M \le x_M - 1$. Then, in particular, $x_{M+1} = x_M - y_M - 1 \ge 0$ and $y_{M+1} = x_M - y_M > 0$. Thus

$$x_{M+2} = |x_{M+1}| - y_{M+1} - 1 = -2,$$
$$y_{M+2} = x_{M+1} - |y_{M+1}| = -1, \tag{2.30}$$

and so statement (1) is true.

Case 2. Suppose $y_M > x_M - 1$. Then, in particular, $x_{M+1} = x_M - y_M - 1 < 0$.

Subcase 1. Suppose $x_M - y_M < 0$.

Then $y_{M+1} = x_M - y_M < 0$. It follows by a straight forward computation, which will be omitted, that $x_{M+5} = 0$. Hence $(x_{M+5}, y_{M+5}) \in l_2 \cup l_4$.

Subcase 2. Suppose $x_M - y_M \ge 0$.

Then $y_{M+1} = x_M - y_M \ge 0$. It follows by a straight forward computation, which will be omitted, that $x_{M+6} = 0$. Hence $(x_{M+6}, y_{M+6}) \in l_2 \cup l_4$, and the proof is complete. □

We next give the proof of Theorem 2.1 when (x_M, y_M) is in $Q_3 = \{(x, y) : x < 0, y < 0\}$.

Lemma 2.13. *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_3$. Then the following statements are true.*

(1) *If $y_M \geq -x_M - 1$, then the solution $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.*

(2) *If $y_M < -x_M - 1$, then $(x_{M+4}, y_{M+4}) \in l_2 \cup l_4$.*

Proof. By assumption, we have $x_M < 0$ and $y_M < 0$.

If $y_M \geq -x_M - 1$, then

$$\begin{aligned} x_{M+1} &= |x_M| - y_M - 1 = -x_M - y_M - 1 \leq 0, \\ y_{M+1} &= x_M - |y_M| = x_M + y_M < 0, \\ x_{M+2} &= |x_{M+1}| - y_{M+1} - 1 = 0, \\ y_{M+2} &= x_{M+1} - |y_{M+1}| = -1. \end{aligned} \tag{2.31}$$

Hence $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution and statement (1) is true.

If $y_M < -x_M - 1$, then it follows by a straight forward computation, which will be omitted, that $x_{M+4} = 0$. Thus $(x_{M+4}, y_{M+4}) \in l_2 \cup l_4$ and statement (2) is true. □

We next give the proof of Theorem 2.1 when (x_M, y_M) is in $Q_2 = \{(x, y) : x < 0,\ y > 0\}$.

Lemma 2.14. *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_2$. Then the following statements are true.*

(1) *If $y_M \geq -x_M - 1$, then $(x_{M+1}, y_{M+1}) \in Q_3 \cup l_4$.*

(2) *If $y_M \leq -x_M - 3/2$, then $(x_{M+3}, y_{M+3}) \in Q_1 \cup l_1$.*

(3) *If $y_M < -x_M - 1$, $y_M > -x_M - 3/2$ and $x_M \leq -5/4$, then $(x_{M+4}, y_{M+4}) \in Q_1 \cup l_1$.*

(4) *If $y_M < -x_M - 1$, $y_M > -x_M - 3/2$, $x_M > -5/4$ and $y_M \leq x_M + 5/4$, then $(x_{M+5}, y_{M+5}) \in Q_3 \cup l_4$.*

(5) *If $y_M < -x_M - 1$, $y_M > -x_M - 3/2$, $x_M > -5/4$ and $y_M > x_M + 5/4$, then $(x_{M+6}, y_{M+6}) \in Q_3 \cup l_4$.*

Proof. Now $x_M < 0$ and $y_M > 0$.

(1) If $y_M \geq -x_M - 1$, then

$$\begin{aligned} x_{M+1} &= -x_M - y_M - 1 \leq 0, \\ y_{M+1} &= x_M - y_M < 0. \end{aligned} \tag{2.32}$$

Thus $(x_{M+1}, y_{M+1}) \in Q_3 \cup l_4$.

(2) If $y_M \leq -x_M - 3/2$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+3} &= -2x_M + 2y_M - 2 > 0, \\ y_{M+3} &= -2x_M - 2y_M - 3 \geq 0. \end{aligned} \tag{2.33}$$

Hence $(x_{M+3}, y_{M+3}) \in Q_1 \cup l_1$.

(3) If $y_M < -x_M - 1$, $y_M > -x_M - 3/2$, and $x_M \le -5/4$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+4} &= 4y_M > 0, \\ y_{M+4} &= -4x_M - 5 \ge 0. \end{aligned} \tag{2.34}$$

Thus $(x_4, y_4) \in Q_1 \cup l_1$.

(4) If $y_M < -x_M - 1$, $y_M > -x_M - 3/2$, $x_M > -5/4$, and $y_M \le x_M + 5/4$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+5} &= 4x_M + 4y_M + 4 < 0, \\ y_{M+5} &= -4x_M + 4y_M - 5 \le 0. \end{aligned} \tag{2.35}$$

Thus $(x_{M+5}, y_{M+5}) \in Q_3 \cup l_4$.

(5) Finally, suppose that $y_M < -x_M - 1$, $y_M > -x_M - 3/2$, $x_M > -5/4$, and $y_M > x_M + 5/4$. Then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+5} &= 4x_M + 4y_M + 4 < 0, \\ y_{M+5} &= -4x_M + 4y_M - 5 > 0. \end{aligned} \tag{2.36}$$

Note that

$$y_{M+5} = -4x_M + 4y_M - 5 > -4x_M - 4y_M - 5 = -x_{M+5} - 1 \tag{2.37}$$

and so by the first statement of this Lemma, $(x_{M+6}, y_{M+6}) \in Q_3 \cup l_4$. □

Thus we see that if there exists an integer $N \ge 0$ such that $(x_N, y_N) \notin Q_4$, then the proof of Theorem 2.1 is complete. Finally, we consider the case where the initial condition $(x_M, y_M) \in Q_4 = \{(x, y) : x > 0,\ y < 0\}$.

Lemma 2.15. *Suppose there exists an integer $M \ge 0$ such that $(x_M, y_M) \in Q_4$. Then there exists a positive integer $N \le 4$ such that $(x_{M+N}, y_{M+N}) \notin Q_4$.*

Proof. Without loss of generality, it suffices to consider the case where

$$(x_{M+n}, y_{M+n}) \in Q_4 \quad \text{for } 0 \le n \le 3. \tag{2.38}$$

Now $(x_M, y_M) \in Q_4$, and hence $x_M > 0$ and $y_M < 0$.

Thus

$$\begin{aligned} x_{M+1} &= |x_M| - y_M - 1 = x_M - y_M - 1, \\ y_{M+1} &= x_M - |y_M| = x_M + y_M. \end{aligned} \tag{2.39}$$

We have $(x_{M+1}, y_{M+1}) \in Q_4$, and thus

$$\begin{aligned} x_{M+2} &= |x_{M+1}| - y_{M+1} - 1 = -2y_M - 2, \\ y_{M+2} &= x_{M+1} - |y_{M+1}| = 2x_M - 1. \end{aligned} \tag{2.40}$$

We also have $(x_2, y_2) \in Q_4$, and hence

$$\begin{aligned} x_{M+3} &= |x_{M+2}| - y_{M+2} - 1 = -2x_M - 2y_M - 2, \\ y_{M+3} &= x_{M+2} - |y_{M+2}| = 2x_M - 2y_M - 3. \end{aligned} \tag{2.41}$$

Finally, we have $(x_{M+3}, y_{M+3}) \in Q_4$, and so

$$\begin{aligned} x_{M+4} &= |x_{M+3}| - y_{M+3} - 1 = -4x_M < 0, \\ y_{M+4} &= x_{M+3} - |y_{M+3}| = -4y_M - 5. \end{aligned} \tag{2.42}$$

In particular, $x_{M+4} < 0$ and hence $(x_{M+4}, y_{M+4}) \notin Q_4$. □

3. Conclusion

We have presented the complete results concerning the global character of the solutions to System (1.1). We divided the real plane into 8 sections and utilized mathematical induction, proof by iteration, and direct computations to show that every solution of System (1.1) is eventually either the prime period-5 solution P_5^1, the prime period-5 solution P_5^2, or else the unique equilibrium point $(0, -1)$. The proofs involve careful consideration of the various cases and subcases.

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