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Research Article

Nonlinear Delay Discrete Inequalities and Their Applications to Volterra Type Difference Equations

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Delay discrete inequalities with more than one nonlinear term are discussed, which generalize some known results and can be used in the analysis of various problems in the theory of certain classes of discrete equations. Application examples to show boundedness and uniqueness of solutions of a Volterra type difference equation are also given.

1. Introduction

Gronwall-Bellman inequalities and their various linear and nonlinear generalizations play very important roles in the discussion of existence, uniqueness, continuation, boundedness, and stability properties of solutions of differential equations and difference equations. The literature on such inequalities and their applications is vast. For example, see [1–12] for continuous cases, and [13–20] for discrete cases. In particular, the book [21] written by Pachpatte considered three types of discrete inequalities:

$$\begin{aligned} u(n) &\le a(n) + \sum_{s=0}^{n-1} f(s) w(u(s)), \\ u^2(n) &\le a(n) + 2\sum_{s=0}^{n-1} f(s) u(s), \\ u^2(n) &\le a(n) + \sum_{s=0}^{n-1} f(s) w(u(s)). \end{aligned} \tag{1.1}$$

In this paper, we consider a delay discrete inequality

$$u(n) \le a(n) + \sum_{i=1}^{m} \sum_{s=b_i(0)}^{b_i(n-1)} f_i(n,s) w_i(u(s)), \quad n \in \mathbf{N}_0 \tag{1.2}$$

which has m nonlinear terms where $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$. We will show that many discrete inequalities like (1.1) can be reduced to this form. Our main result can be applied to analyze properties of solutions of discrete equations. We also give examples to show boundedness and uniqueness of solutions of a Volterra type difference equation.

2. Main Results

Assume that

(C_1) $a(n)$ is nonnegative for $n \in \mathbf{N}_0$ and $a(0) > 0$;

(C_2) $b_i(n)$ $(i = 1, \ldots, m)$ are nondecreasing for $n \in \mathbf{N}_0$, the range of each b_i belongs to $\mathbf{N}_0$, and $b_i(n) \le n$;

(C_3) all $f_i(n, j)$ $(i = 1, \ldots, m)$ are nonnegative for $n, j \in \mathbf{N}_0$;

(C_4) all w_i $(i = 1, \ldots, m)$ are continuous and nondecreasing functions on $[0, \infty)$ and are positive on $(0, \infty)$. They satisfy the relationship $w_1 \propto w_2 \propto \cdots \propto w_m$ where $w_i \propto w_{i+1}$ means that $(w_{i+1})/w_i$ is nondecreasing on $(0, \infty)$ (see [10]).

Let $W_i(u) = \int_{u_i}^{u} (dz/w_i(z))$ for $u \ge u_i$ where $u_i > 0$ is a given constant. Then, W_i is strictly increasing so its inverse W_i^{-1} is well defined, continuous, and increasing in its corresponding domain. Define $b_i(-1) = -1$, $\Delta u(n) = u(n+1) - u(n)$ and $\Delta_2 r(n, j) = r(n, j+1) - r(n, j)$.

Theorem 2.1. *Suppose that (C_1)–(C_4) hold and $u(n)$ is a nonnegative function for $n \in \mathbf{N}_0$ satisfying* (1.2). *Then*

$$u(n) \le W_m^{-1}\left[W_m(\tilde{a}(0)) + \sum_{s=b_m(0)}^{b_m(n-1)} \tilde{f}_m(n,s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_m(n,s)}{\phi_m\left(W_{m-1}^{-1}(r_m(0,s))\right)} \right], \quad n \le N_1, \tag{2.1}$$

where $\tilde{a}(n) = \max_{0 \le \tau \le n, \tau \in \mathbf{N}_0} a(\tau)$, $\tilde{f}_i(n, j) = \max_{0 \le \tau \le n, \tau \in N_0} f_i(\tau, j)$, $r_m(n, j)$ *is determined recursively by*

$$\begin{aligned} r_1(n,j) &= \tilde{a}(j), \\ r_{i+1}(n,j) &= W_i(r_1(n,0)) + \sum_{s=b_i(0)}^{b_i(j-1)} \tilde{f}_i(n,s) + \sum_{s=0}^{j-1} \frac{\Delta_2 r_i(n,s)}{\phi_i\left(W_{i-1}^{-1}(r_i(0,s))\right)}, \quad i = 1, \ldots, m-1, \end{aligned} \tag{2.2}$$

$\phi_i(u) = w_i(u)/w_{i-1}(u)$, $\phi_1(u) = w_1(u)$, $W_0 = I$ *(Identity), and* N_1 *is the largest positive integer such that*

$$W_i(\tilde{a}(0)) + \sum_{s=b_i(0)}^{b_i(N_1-1)} \tilde{f}_i(N_1,s) + \sum_{s=0}^{N_1-1} \frac{\Delta_2 r_i(N_1,s)}{\phi_i\left(W_{i-1}^{-1}(r_i(0,s))\right)} \le \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \quad i = 1, \ldots, m. \tag{2.3}$$

Remark 2.2. (1) N_1 is defined by (2.3) and $N_1 = \infty$ when all w_i $(i = 1,\dots,m)$ satisfy $\int_{u_i}^{\infty}(dz/w_i(z)) = \infty$. Different choices of u_i in W_i do not affect our results (see [2]).

(2) If $b_i(n) = n$ for $i = 1,\dots,m$, then (2.1) gives the estimate of the following inequality:

$$u(n) \leq a(n) + \sum_{i=1}^{m}\sum_{s=0}^{n-1} f_i(n,s)w_i(u(s)), \qquad n \in \mathbf{N}_0 \tag{2.4}$$

by replacing $b_m(n-1)$, $b_m(0)$, $b_i(j-1)$, $b_i(0)$, and $b_i(N_1-1)$ with $n-1$, 0, $j-1$, 0 and N_1-1, respectively. Especially, if $b_1(n) = n$ and $f_1(n,s) = f(s)$, then (1.2) for $m = 1$ becomes the first inequality of (1.1). Equation (2.1) shows the same estimate given by (b_1) of Theorem 4.2.3 in the book [21].

Lemma 2.3. *$\Delta_2 r_i(n,j)$ is nonnegative and nondecreasing in n, and $r_i(n,j)$ is nonnegative and nondecreasing in n and j for $i = 1,\dots,m$.*

Proof. By the definitions of $\tilde{a}(n)$ and $\tilde{f}_i(n,j)$, it is easy to check that they are nonnegative and nondecreasing in n, and $\tilde{a}(n) \geq a(n)$ and $\tilde{f}_i(n,j) \geq f_i(n,j)$ for each fixed j where $i = 1,\dots,m$. $a(0) > 0$ in (C_1) implies that $\tilde{a}(n) > 0$ for all $n \leq N_1$. Clearly,

$$\begin{gathered}\Delta_2 r_1(n+1,j) - \Delta_2 r_1(n,j) = 0,\\ \Delta_2 r_2(n+1,j) - \Delta_2 r_2(n,j) = \tilde{f}_1(n+1,b_1(j)) - \tilde{f}_1(n,b_1(j)) + \frac{\Delta_2 r_1(n+1,j) - \Delta_2 r_1(n,j)}{w_1(r_1(0,j))} \geq 0,\end{gathered} \tag{2.5}$$

where $r_1(0,j) = \tilde{a}(j) > 0$ is used, which yields that $\Delta_2 r_1(n,j)$ and $\Delta_2 r_2(n,j)$ are nondecreasing in n. Assume that $\Delta_2 r_l(n,j)$ is nondecreasing in n. Then

$$\Delta_2 r_{l+1}(n+1,j) - \Delta_2 r_{l+1}(n,j) = \tilde{f}_l(n+1,b_l(j)) - \tilde{f}_l(n,b_l(j)) + \frac{\Delta_2 r_l(n+1,j) - \Delta_2 r_l(n,j)}{\phi_l(W_{l-1}^{-1}(r_l(0,j)))} \geq 0, \tag{2.6}$$

which implies that $\Delta_2 r_{l+1}(n,j)$ is nondecreasing in n. By induction, $\Delta_2 r_i(n,j)$ $(i = 1,\dots,m)$ are nondecreasing in n. Similarly, we can prove that they are nonnegative by induction again. Then $r_i(n,j)$ $(i = 1,\dots,m)$ are nonnegative and nondecreasing in n and j. □

Proof of Theorem 2.1. Take any arbitrary positive integer $\tilde{n} \leq N_1$ and consider the auxiliary inequality

$$u(n) \leq r_1(\tilde{n},n) + \sum_{i=1}^{m}\sum_{s=b_i(0)}^{b_i(n-1)} \tilde{f}_i(\tilde{n},s)w_i(u(s)), \quad n \leq \tilde{n}. \tag{2.7}$$

Claim that $u(n)$ in (2.7) satisfies

$$u(n) \leq W_m^{-1}\left[W_m(r_1(\tilde{n},0)) + \sum_{s=b_m(0)}^{b_m(n-1)} \tilde{f}_{\mathrm{m}}(\tilde{n},s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_m(\tilde{n},s)}{\phi_m(W_{m-1}^{-1}(r_m(0,s)))}\right] \tag{2.8}$$

for $n \leq \{\tilde{n}, N_2\}$ where N_2 is the largest positive integer such that

$$W_i(r_1(\tilde{n},0)) + \sum_{s=b_i(0)}^{b_i(N_2-1)} \tilde{f}_i(\tilde{n},s) + \sum_{s=0}^{N_2-1} \frac{\Delta_2 r_i(\tilde{n},s)}{\phi_i(W_{i-1}^{-1}(r_i(0,s)))} \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \tag{2.9}$$

$i = 1, \ldots, m$.

Before we prove (2.8), notice that $N_1 \leq N_2$. In fact, $r_i(\tilde{n},n)$, $\Delta_2 r_i(\tilde{n},n)$, and $\tilde{f}_i(\tilde{n},n)$ are nondecreasing in $\tilde{n}$ by Lemma 2.3. Thus, N_2 satisfying (2.9) gets smaller as $\tilde{n}$ is chosen larger. In particular, N_2 satisfies the same (2.3) as N_1 for $\tilde{n} = N_1$ if $r_1(\tilde{n},0) = \tilde{a}(0)$ is applied.

We divide the proof of (2.8) into two steps by using induction.

Step 1 ($m = 1$). Let $z(n) = \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s) w_1(u(s))$ for $n \leq \tilde{n}$ and $z(0) = 0$. It is clear that $z(n)$ is nonnegative and nondecreasing. Observe that (2.7) is equivalent to $u(n) \leq r_1(\tilde{n},n) + z(n)$ for $n \leq \tilde{n}$ and by assumptions (C_2) and (C_4) and Lemma 2.3,

$$\begin{aligned}\Delta z(n) &= \tilde{f}_1(\tilde{n},b_1(n)) w_1(u(b_1(n))) \leq \tilde{f}_1(\tilde{n},b_1(n)) w_1(r_1(\tilde{n},b_1(n)) + z(b_1(n))) \\ &\leq \tilde{f}_1(\tilde{n},b_1(n)) w_1(r_1(\tilde{n},n) + z(n)).\end{aligned} \tag{2.10}$$

Since w_1 is nondecreasing and $r_1(\tilde{n},n) = \tilde{a}(n) > 0$, we have

$$\begin{aligned}\frac{\Delta z(n) + \Delta_2 r_1(\tilde{n},n)}{w_1(r_1(\tilde{n},n) + z(n))} &\leq \tilde{f}_1(\tilde{n},b_1(n)) + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(\tilde{n},n) + z(n))} \\ &\leq \tilde{f}_1(\tilde{n},b_1(n)) + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(0,n))}.\end{aligned} \tag{2.11}$$

Then

$$\begin{aligned}\int_{z(n)+r_1(\tilde{n},n)}^{z(n+1)+r_1(\tilde{n},n+1)} \frac{d\tau}{w_1(\tau)} &\leq \int_{z(n)+r_1(\tilde{n},n)}^{z(n+1)+r_1(\tilde{n},n+1)} \frac{d\tau}{w_1(z(n) + r_1(\tilde{n},n))} \\ &\leq \frac{\Delta z(n) + \Delta_2 r_1(\tilde{n},n)}{w_1(z(n) + r_1(\tilde{n},n))} \\ &\leq \tilde{f}_1(\tilde{n},b_1(n)) + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(0,n))},\end{aligned} \tag{2.12}$$

and so

$$\begin{aligned}\int_{z(0)+r_1(\tilde{n},0)}^{z(n)+r_1(\tilde{n},n)} \frac{d\tau}{w_1(\tau)} &= \sum_{s=0}^{n-1} \int_{z(s)+r_1(\tilde{n},s)}^{z(s+1)+r_1(\tilde{n},s+1)} \frac{d\tau}{w_1(\tau)} \\ &\leq \sum_{s=0}^{n-1} \tilde{f}_1(\tilde{n}, b_1(s)) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n}, s)}{w_1(r_1(0,s))} \\ &= \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n}, s)}{w_1(r_1(0,s))}.\end{aligned} \tag{2.13}$$

The definition of W_1 in Theorem 2.1 and $z(0) = 0$ show

$$W_1(z(n) + r_1(\tilde{n}, n)) \leq W_1(r_1(\tilde{n}, 0)) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n}, s)}{w_1(r_1(0,s))}, \quad n \leq \tilde{n}. \tag{2.14}$$

Equation (2.9) shows that the right side of (2.14) is in the domain of W_1^{-1} for all $n \leq \tilde{n}$. Thus the monotonicity of W_1^{-1} implies

$$u(n) \leq z(n) + r_1(\tilde{n}, n) \leq W_1^{-1}\left[W_1(r_1(\tilde{n}, 0)) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n}, s)}{w_1(r_1(0,s))}\right] \tag{2.15}$$

for $n \leq \tilde{n}$; that is, (2.8) is true for $m = 1$.

Step 2 ($m = k + 1$). Assume that (2.8) is true for $m = k$. Consider

$$u(n) \leq r_1(\tilde{n}, n) + \sum_{i=1}^{k+1} \sum_{s=b_i(0)}^{b_i(n-1)} \tilde{f}_i(\tilde{n}, s) w_i(u(s)), \quad n \leq \tilde{n}. \tag{2.16}$$

Let $z(n) = \sum_{i=1}^{k+1} \sum_{s=b_i(0)}^{b_i(n-1)} \tilde{f}_i(\tilde{n}, s) w_i(u(s))$ and $z(0) = 0$. Then $z(n)$ is nonnegative and nondecreasing and satisfies $u(n) \leq r_1(\tilde{n}, n) + z(n)$ for $n \leq \tilde{n}$. Moreover, we have

$$\Delta z(n) = \sum_{i=1}^{k+1} \tilde{f}_i(\tilde{n}, b_i(n)) w_i(u(b_i(n))) \leq \sum_{i=1}^{k+1} \tilde{f}_i(\tilde{n}, b_i(n)) w_i(r_1(\tilde{n}, b_i(n)) + z(b_i(n))). \tag{2.17}$$

Since w_i and r_1 are nondecreasing in their arguments and $r_1(\tilde{n},n) > 0$, we have by the assumption $b_i(n) \le n$

$$\begin{aligned}
\frac{\Delta z(n) + \Delta_2 r_1(\tilde{n},n)}{w_1(z(n)+r_1(\tilde{n},\mathrm{n}))} &\le \frac{\sum_{i=1}^{k+1} \tilde{f}_i(\tilde{n},b_i(n)) w_i(z(b_i(n)) + r_1(\tilde{n},b_i(n)))}{w_1(z(n)+r_1(\tilde{n},n))} + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(\tilde{n},n))} \\
&\le \tilde{f}_1(\tilde{n},b_1(n)) + \sum_{i=2}^{k+1} \tilde{f}_i(\tilde{n},b_i(n)) \frac{w_i(z(b_i(n)) + r_1(\tilde{n},b_i(n)))}{w_1(z(b_i(n)) + r_1(\tilde{n},b_i(n)))} + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(0,n))} \\
&\le \tilde{f}_1(\tilde{n},b_1(n)) + \sum_{i=1}^{k} \tilde{f}_{i+1}(\tilde{n},b_{i+1}(n)) \tilde{\phi}_{i+1}(z(b_{i+1}(n)) + r_1(\tilde{n},b_{i+1}(n))) \\
&\quad + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(0,n))}
\end{aligned} \tag{2.18}$$

for $n \le \tilde{n}$ where $\tilde{\phi}_{i+1}(u) = w_{i+1}(u)/w_1(u)$ for $i = 1,\ldots,k$, which gives

$$\begin{aligned}
\int_{z(n)+r_1(\tilde{n},n)}^{z(n+1)+r_1(\tilde{n},n+1)} \frac{d\tau}{w_1(\tau)} &\le \int_{z(n)+r_1(\tilde{n},n)}^{z(n+1)+r_1(\tilde{n},n+1)} \frac{d\tau}{w_1(z(n)+r_1(\tilde{n},n))} \\
&\le \frac{\Delta z(n) + \Delta_2 r_1(\tilde{n},n)}{w_1(r_1(\tilde{n},n)+z(n))} \\
&\le \tilde{f}_1(\tilde{n},b_1(n)) + \frac{\Delta_2 r_1(\tilde{n},n)}{w_1(r_1(0,n))} \\
&\quad + \sum_{i=1}^{k} \tilde{f}_{i+1}(\tilde{n},b_{i+1}(n)) \tilde{\phi}_{i+1}(z(b_{i+1}(n)) + r_1(\tilde{n},b_{i+1}(n))).
\end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned}
\int_{z(0)+r_1(\tilde{n},0)}^{z(n)+r_1(\tilde{n},n)} \frac{d\tau}{w_1(\tau)} &\le \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n},s)}{w_1(r_1(0,s))} \\
&\quad + \sum_{i=1}^{k} \sum_{s=0}^{n-1} \tilde{f}_{i+1}(\tilde{n},b_{i+1}(s)) \tilde{\phi}_{i+1}(z(b_{i+1}(s)) + r_1(\tilde{n},b_{i+1}(s))),
\end{aligned} \tag{2.20}$$

that is,

$$\begin{aligned}
W_1(z(n)+r_1(\tilde{n},n)) &\le W_1(r_1(\tilde{n},0)) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n},s)}{w_1(r_1(0,s))} \\
&\quad + \sum_{i=1}^{k} \sum_{s=b_{i+1}(0)}^{b_{i+1}(n-1)} \tilde{f}_{i+1}(\tilde{n},s) \tilde{\phi}_{i+1}(z(s) + r_1(\tilde{n},s)),
\end{aligned} \tag{2.21}$$

or equivalently

$$\xi(n) \le c_1(\tilde{n}, n) + \sum_{i=1}^{k} \sum_{s=b_{i+1}(0)}^{b_{i+1}(n-1)} \tilde{f}_{i+1}(\tilde{n}, s)\tilde{\phi}_{i+1}\left(W_1^{-1}(\xi(s))\right), \quad n \le \tilde{n}, \tag{2.22}$$

the same as (2.7) for $m = k$ where $\xi(n) = W_1(z(n) + r_1(\tilde{n}, n))$ and

$$c_1(\tilde{n}, n) = W_1(r_1(\tilde{n}, 0)) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_1(\tilde{n}, s)}{w_1(r_1(0, s))}. \tag{2.23}$$

From the assumption (C_4), each $\tilde{\phi}_{i+1}(W_1^{-1})$, $i = 1, \dots, k$, is continuous and nondecreasing on $[0, \infty)$ and is positive on $(0, \infty)$ since W_1^{-1} is continuous and nondecreasing on $[0, \infty)$. Moreover, $\tilde{\phi}_2(W_1^{-1}) \propto \tilde{\phi}_3(W_1^{-1}) \propto \cdots \propto \tilde{\phi}_{k+1}(W_1^{-1})$. By the inductive assumption, we have

$$\xi(n) \le \Phi_{k+1}^{-1}\left[\Phi_{k+1}(c_1(\tilde{n}, 0)) + \sum_{s=b_{k+1}(0)}^{b_{k+1}(n-1)} \tilde{f}_{k+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 c_k(\tilde{n}, s)}{\varphi_{k+1}(\Phi_k^{-1}(c_k(0, s)))}\right] \tag{2.24}$$

for $n \le \min\{\tilde{n}, N_3\}$ where $\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^{u} (dz/\tilde{\phi}_{i+1}(W_1^{-1}(z)))$, $u > 0$, $\Phi_1 = I$ (Identity), $\tilde{u}_{i+1} = W_1(u_{i+1})$, Φ_{i+1}^{-1} is the inverse of Φ_{i+1}, $\varphi_{i+1}(u) = \tilde{\phi}_{i+1}(W_1^{-1}(u))/\tilde{\phi}_i(W_1^{-1}(u)) = w_{i+1}(W_1^{-1}(u))/w_i(W_1^{-1}(u))$, $i = 1, \dots, k$,

$$c_{i+1}(\tilde{n}, n) = \Phi_{i+1}(c_1(\tilde{n}, 0)) + \sum_{s=b_{i+1}(0)}^{b_{i+1}(n-1)} \tilde{f}_{i+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 c_i(\tilde{n}, s)}{\varphi_{i+1}(\Phi_i^{-1}(c_i(0, s)))}, \tag{2.25}$$

$i = 1, \dots, k-1$, and N_3 is the largest positive integer such that

$$\begin{aligned} &\Phi_{i+1}(c_1(\tilde{n}, 0)) + \sum_{s=b_{i+1}(0)}^{b_{i+1}(N_3-1)} \tilde{f}_{i+1}(\tilde{n}, s) + \sum_{s=0}^{N_3-1} \frac{\Delta_2 c_i(\tilde{n}, s)}{\varphi_{i+1}(\Phi_i^{-1}(c_i(0, s)))} \\ &\quad \le \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\tilde{\phi}_{i+1}(W_1^{-1}(z))}, \quad i = 1, \dots, k. \end{aligned} \tag{2.26}$$

Note that

$$\begin{aligned}
\Phi_i(u) &= \int_{\tilde{u}_i}^{u} \frac{dz}{\tilde{\phi}_i(W_1^{-1}(z))} = \int_{W_1(u_i)}^{u} \frac{w_1(W_1^{-1}(z))dz}{w_i(W_1^{-1}(z))} \\
&= \int_{u_i}^{W_1^{-1}(u)} \frac{dz}{w_i(z)} = W_i \circ W_1^{-1}(u), \quad i = 2, \dots, k+1, \\
\varphi_{i+1}\left(\Phi_i^{-1}(u)\right) &= \frac{w_{i+1}(W_1^{-1}(\Phi_i^{-1}(u)))}{w_i(W_1^{-1}(\Phi_i^{-1}(u)))} = \frac{w_{i+1}(W_1^{-1}(W_1(W_i^{-1}(u))))}{w_i(W_1^{-1}(W_1(W_i^{-1}(u))))} \\
&= \frac{w_{i+1}(W_i^{-1}(u))}{w_i(W_i^{-1}(u))} = \phi_{i+1}\left(W_i^{-1}(u)\right), \quad i = 1, \dots, k+1.
\end{aligned} \tag{2.27}$$

Thus, we have from (2.24) that

$$\begin{aligned}
u(n) &\le r_1(\tilde{n}, n) + z(n) = W_1^{-1}(\xi(n)) \\
&\le W_{k+1}^{-1}\left[W_{k+1}\left(W_1^{-1}(c_1(\tilde{n}, 0))\right) + \sum_{s=b_{k+1}(0)}^{b_{k+1}(n-1)} \tilde{f}_{k+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 c_k(\tilde{n}, s)}{\phi_{k+1}(W_k^{-1}(c_k(0, s)))}\right] \\
&\le W_{k+1}^{-1}\left[W_{k+1}(r_1(\tilde{n}, 0)) + \sum_{s=b_{k+1}(0)}^{b_{k+1}(n-1)} \tilde{f}_{k+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 c_k(\tilde{n}, s)}{\phi_{k+1}(W_k^{-1}(c_k(0, s)))}\right]
\end{aligned} \tag{2.28}$$

for $n \le \min\{\tilde{n}, N_3\}$ since $c_1(\tilde{n}, 0) = W_1(r_1(\tilde{n}, 0))$.

In the following, we prove that $c_i(\tilde{n}, n) = r_{i+1}(\tilde{n}, n)$ by induction again.

It is clear that $c_1(\tilde{n}, n) = r_2(\tilde{n}, n)$ for $i = 1$. Suppose that $c_l(\tilde{n}, n) = r_{l+1}(\tilde{n}, n)$ for $i = l$. We have

$$\begin{aligned}
c_{l+1}(\tilde{n}, n) &= \Phi_{l+1}(c_1(\tilde{n}, 0)) + \sum_{s=b_{l+1}(0)}^{b_{l+1}(n-1)} \tilde{f}_{l+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 c_l(\tilde{n}, s)}{\varphi_{l+1}(\Phi_l^{-1}(c_l(0, s)))} \\
&= W_{l+1}(r_1(\tilde{n}, 0)) + \sum_{s=b_{l+1}(0)}^{b_{l+1}(n-1)} \tilde{f}_{l+1}(\tilde{n}, s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_{l+1}(\tilde{n}, s)}{\phi_{l+1}(W_l^{-1}(r_{l+1}(0, s)))} \\
&= r_{l+2}(\tilde{n}, n),
\end{aligned} \tag{2.29}$$

where $c_1(\tilde{n}, 0) = W_1(r_1(\tilde{n}, 0))$ is applied. It implies that it is true for $i = l + 1$. Thus, $c_i(\tilde{n}, n) = r_{i+1}(\tilde{n}, n)$ for $i = 1, \dots, k$.

Equation (2.26) becomes

$$\begin{aligned} &W_{i+1}(r_1(\tilde{n},0)) + \sum_{s=b_{i+1}(0)}^{b_{i+1}(N_3-1)} \tilde{f}_{i+1}(\tilde{n},s) + \sum_{s=0}^{N_3-1} \frac{\Delta_2 r_{i+1}(\tilde{n},s)}{\phi_{i+1}(W_i^{-1}(r_{i+1}(0,s)))} \\ &\le \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\tilde{\phi}_{i+1}(W_1^{-1}(z))} = \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{w_1(W_1^{-1}(z))}{w_{i+1}(W_1^{-1}(z))} dz = \int_{u_{i+1}}^{\infty} \frac{dz}{w_{i+1}(z)} \end{aligned} \tag{2.30}$$

for $i = 1, \dots, k$. It implies that $N_2 = N_3$. Thus, (2.28) becomes

$$u(n) \le W_{k+1}^{-1}\left[W_{k+1}(r_1(\tilde{n},0)) + \sum_{s=b_{k+1}(0)}^{b_{k+1}(n-1)} \tilde{f}_{k+1}(\tilde{n},s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_{k+1}(\tilde{n},s)}{\phi_{k+1}(W_k^{-1}(r_{k+1}(0,s)))}\right] \tag{2.31}$$

for $n \le \tilde{n}$. It shows that (2.8) is true for $m = k+1$. Thus, the claim is proved.

Now we prove (2.1). Replacing n by $\tilde{n}$ in (2.8), we have

$$u(\tilde{n}) \le W_m^{-1}\left[W_m(r_1(\tilde{n},0)) + \sum_{s=b_m(0)}^{b_m(\tilde{n}-1)} \tilde{f}_m(\tilde{n},s) + \sum_{s=0}^{\tilde{n}-1} \frac{\Delta_2 r_m(\tilde{n},s)}{\phi_m(W_{m-1}^{-1}(r_m(0,s)))}\right]. \tag{2.32}$$

Since (2.8) is true for any $\tilde{n} \le N_1$, we replace $\tilde{n}$ by n and get

$$u(n) \le W_m^{-1}\left[W_m(r_1(n,0)) + \sum_{s=b_m(0)}^{b_m(n-1)} \tilde{f}_m(n,s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_m(n,s)}{\phi_m(W_{m-1}^{-1}(r_m(0,s)))}\right]. \tag{2.33}$$

This is exactly (2.1) since $r_1(n,0) = \tilde{a}(0)$. This proves Theorem 2.1. □

Remark 2.4. If $a(n) = 0$ for all $n \in \mathbf{N}_0$, then $\tilde{a}(0) = 0$. Let $r_{1,u_1}(n,j) := r_1(n,j) + u_1$ where $u_1 > 0$ is given in $W_1(u) = \int_{u_1}^{u} (dz/w_1(z))$. Using the same arguments as in (2.11) where $r_1(n,j)$ is replaced with the positive $r_{1,u_1}(n,j)$, we have $\Delta_2 r_{1,u_1}(\tilde{n},s) = 0$ and (2.14) becomes

$$\begin{aligned} W_1(z(n) + r_{1,u_1}(\tilde{n},n)) &\le W_1(r_{1,u_1}(\tilde{n},0)) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s) \\ &= W_1(u_1) + \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s) = \sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s), \end{aligned} \tag{2.34}$$

that is,

$$\begin{aligned} u(n) &\le z(n) + r_{1,u_1}(\tilde{n},n) = z(n) + u_1 \\ &\le W_1^{-1}\left[\sum_{s=b_1(0)}^{b_1(n-1)} \tilde{f}_1(\tilde{n},s)\right], \quad n \le \tilde{n}, \end{aligned} \tag{2.35}$$

which is the same as (2.15) with a complementary definition that $W_1(0) = 0$. From (1) of Remark 2.2, the estimate of (2.35) is independent of u_1. Then we similarly obtain (2.1) and all r_i are defined by the same formula (2.2) where we define $W_i(0) = 0$ for $i = 1, \ldots, m$.

3. Some Corollaries

In this section, we apply Theorem 2.1 and obtain some corollaries.

Assume that $\varphi \in C(\mathbf{R}_+, \mathbf{R}_+)$ is a strictly increasing function with $\varphi(\infty) = \infty$ where $\mathbf{R}_+ = [0, \infty)$. Consider the inequality

$$\varphi(u(n)) \le a(n) + \sum_{i=1}^{m} \sum_{s=b_i(0)}^{b_i(n-1)} f_i(n,s) w_i(u(s)), \quad n \in N_0. \tag{3.1}$$

Corollary 3.1. *Suppose that* (C_1)*–*(C_4) *hold. If* $u(n)$ *in*(3.1)*is nonnegative for* $n \in \mathbf{N}_0$, *then*

$$u(n) \le \varphi^{-1}\left[\widetilde{W}_m^{-1}\left(\widetilde{W}_m(\tilde{a}(0)) + \sum_{s=b_m(0)}^{b_m(n-1)} \widetilde{f}_m(n,s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_m(n,s)}{\widehat{\phi}_m\left(\widetilde{W}_{m-1}^{-1}(r_m(0,s))\right)}\right)\right] \tag{3.2}$$

for $n \le N_1$ *where* $\widetilde{W}_i(u) = \int_{u_i}^{u}(dz/w_i(\varphi^{-1}(z)))$, $\widetilde{W}_i^{-1}$ *is the inverse of* $\widetilde{W}_i$, $\widetilde{W}_0 = I$*(Identity),* $\widehat{\phi}_i(u) = (w_i(\varphi^{-1}(u)))/(w_{i-1}(\varphi^{-1}(u)))$, $\widehat{\phi}_1(u) = w_1(\varphi^{-1}(u))$, *and other related functions are defined as in Theorem 2.1 by replacing* $w_i(u)$ *with* $w_i(\varphi^{-1}(u))$.

Proof. Let $\xi(n) = \varphi(u(n))$. Then (3.1) becomes

$$\xi(n) \le a(n) + \sum_{i=1}^{m} \sum_{s=b_i(0)}^{b_i(n-1)} f_i(n,s) w_i\left(\varphi^{-1}(\xi(s))\right), \quad n \in \mathbf{N}_0. \tag{3.3}$$

Note that $w_i(\varphi^{-1}(u))$ satisfy (C_4) for $i = 1, \ldots, m$. Using Theorem 2.1, we obtain the estimate about $\xi(n)$ by replacing $w_i(u)$ with $w_i(\varphi^{-1}(u))$. Then use the fact that $u(n) = \varphi^{-1}(\xi(n))$ and we get Corollary 3.1. □

If $\varphi(u) = u^p$ where $p > 0$, then (3.1) reads

$$u^p(n) \le a(n) + \sum_{i=1}^{m} \sum_{s=b_i(0)}^{b_i(n-1)} f_i(n,s) w_i(u(s)), \quad n \in \mathbf{N}_0. \tag{3.4}$$

Directly using Corollary 3.1, we have the following result.

Corollary 3.2. *Suppose that* (C_1)*–*(C_4) *hold. If* $u(n)$ *in* (3.4) *is nonnegative for* $n \in \mathbf{N}_0$, *then*

$$u(n) \le \left[\widetilde{W}_m^{-1}\left(\widetilde{W}_m(\tilde{a}(0)) + \sum_{s=b_m(0)}^{b_m(n-1)} \widetilde{f}_m(n,s) + \sum_{s=0}^{n-1} \frac{\Delta_2 r_m(n,s)}{\widehat{\phi}_m\left(\widetilde{W}_{m-1}^{-1}(r_m(0,s))\right)}\right)\right]^{1/p} \tag{3.5}$$

for $n \le N_1$ *where* $\widetilde{W}_i(u) = \int_{u_i}^{u}(dz/w_i(z^{1/p}))$, $\widetilde{W}_i$ *is the inverse of* $\widetilde{W}_i$, $\widetilde{W}_0 = I(\text{Identity})$, $\widehat{\phi}_i(u) = w_i(u^{1/p})/w_{i-1}(u^{1/p})$, $\widehat{\phi}_1(u) = w_1(u^{1/p})$, *and other related functions are defined as in Theorem 2.1 by replacing* $w_i(u)$ *with* $w_i(u^{1/p})$.

If $m = 1$, $p = 2$, $b_1(n) = n$, (3.4) becomes the second inequality of (1.1) with $f_1(n,s) = 2f(s)$ and $w_1(u) = u$, and the third inequality of (1.1) with $f_1(n,s) = f(s)$ and $w_1(u) = w(u)$, which are discussed in the book [21]. Equation (3.5) yields the same estimates of Theorem 4.2.4 in the book [21].

4. Applications to Volterra Type Difference Equations

In this section, we apply Theorem 2.1 to study boundedness and uniqueness of solutions of a nonlinear delay difference equation of the form

$$y(n) = \beta(n) + \sum_{s=b_1(0)}^{b_1(n-1)} F(n,s,y(s)) + \sum_{s=b_2(0)}^{b_2(n-1)} H(n,s,y(s)), \quad n \in \mathbf{N}_0, \tag{4.1}$$

where $y : \mathbf{N}_0 \to \mathbf{R}$ is an unknown function, β maps from $\mathbf{N}_0$ to $\mathbf{R}$, F and H map from $\mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{R}$ to $\mathbf{R}$, and b_i satisfies the assumption (C_2) for $i = 1,2$.

Theorem 4.1. *Suppose that* $\beta(0) \ne 0$ *and the functions* F *and* H *in* (4.1) *satisfy the conditions*

$$\begin{aligned} \left|F(n,s,y)\right| &\le f_1(n,s)\sqrt{|y|}, \\ \left|H(n,s,y)\right| &\le f_2(n,s)|y|, \end{aligned} \tag{4.2}$$

where $f_1, f_2 : \mathbf{N}_0 \times \mathbf{N}_0 \to [0,\infty)$. *If* $y(n)$ *is a solution of* (4.1) *on* $\mathbf{N}_0$, *then*

$$\left|y(n)\right| \le \widetilde{a}(0)\exp\left[\sum_{s=b_2(0)}^{b_2(n-1)} \widetilde{f}_2(n,s) + \sum_{s=0}^{n-1} \frac{\widetilde{f}_1(n,b_1(s)) + \left(\Delta\widetilde{a}(s)/\sqrt{\widetilde{a}(s)}\right)}{h(s)}\right], \tag{4.3}$$

where

$$\begin{gathered} \widetilde{a}(s) = \max_{0\le\tau\le s,\tau\in\mathbf{N}_0} \left|\beta(\tau)\right|, \qquad \widetilde{f}_1(n,s) = \max_{0\le\tau\le n,\tau\in N_0} f_1(\tau,s), \\ \widetilde{f}_2(n,s) = \max_{0\le\tau\le n,\tau\in\mathbf{N}_0} f_2(\tau,s), \\ h(n) = \sqrt{\widetilde{a}(0)} + \frac{1}{2}\sum_{s=b_1(0)}^{b_1(n-1)} \widetilde{f}_1(0,s) + \frac{1}{2}\sum_{s=0}^{n-1}\frac{\Delta\widetilde{a}(s)}{\sqrt{\widetilde{a}(s)}}. \end{gathered} \tag{4.4}$$

Proof. Using (4.1) and (4.2), the solution $y(n)$ satisfies

$$u(n) \leq a(n) + \sum_{s=b_1(0)}^{b_1(n-1)} f_1(n,s)w_1(u(s)) + \sum_{s=b_2(0)}^{b_2(n-1)} f_2(n,s)w_2(u(s)), \quad n \in \mathbf{N}_0, \tag{4.5}$$

where

$$u(n) = |y(n)|, \quad a(n) = |\beta(n)|, \quad w_1(u) = \sqrt{u}, \quad w_2(u) = u. \tag{4.6}$$

Clearly, $\tilde{a}(n) > 0$ for all $n \in N_0$ since $\beta(0) \neq 0$. For positive constants u_1, u_2, we have

$$\begin{gathered} W_1(u) = \int_{u_1}^{u} \frac{dz}{w_1(z)} = 2\left(\sqrt{u} - \sqrt{u_1}\right), \quad W_1^{-1}(u) = \left(\frac{u}{2} + \sqrt{u_1}\right)^2, \\ W_2(u) = \int_{u_2}^{u} \frac{dz}{w_2(z)} = \ln\frac{u}{u_2}, \quad W_2^{-1}(u) = u_2\exp(u), \\ r_1(n,j) = \tilde{a}(j) > 0, \quad r_1(n,0) = \tilde{a}(0), \\ r_2(n,j) = 2\left(\sqrt{\tilde{a}(0)} - \sqrt{u_1}\right) + \sum_{s=b_1(0)}^{b_1(j-1)} \tilde{f}_1(n,s) + \sum_{s=0}^{j-1}\frac{\Delta\tilde{a}(s)}{\sqrt{\tilde{a}(s)}}, \\ \Delta_2 r_2(n,j) = \tilde{f}_1(n,b_1(j)) + \frac{\Delta\tilde{a}(j)}{\sqrt{\tilde{a}(j)}}, \quad \phi_2(u) = \frac{w_2(u)}{w_1(u)} = \sqrt{u}. \end{gathered} \tag{4.7}$$

It is obvious that w_1 and w_2 satisfy (C_4). Applying Theorem 2.1 gives

$$u(n) \leq \tilde{a}(0)\exp\left[\sum_{s=b_2(0)}^{b_2(n-1)} \tilde{f}_2(n,s) + \sum_{s=0}^{n-1}\frac{\tilde{f}_1(n,b_1(s)) + \left(\Delta\tilde{a}(s)/\sqrt{\tilde{a}(s)}\right)}{h(s)}\right] \tag{4.8}$$

which implies (4.3). □

Theorem 4.2. *Suppose that $\beta(0) \neq 0$ and the functions F and H in* (4.1) *satisfy the conditions*

$$\begin{gathered} |F(n,s,y_1) - F(n,s,y_2)| \leq f_1(n,s)\sqrt{|y_1 - y_2|}, \\ |H(n,s,y_1) - H(n,s,y_2)| \leq f_2(n,s)|y_1 - y_2|, \end{gathered} \tag{4.9}$$

where $f_1, f_2 : \mathbf{N}_0 \times \mathbf{N}_0 \to [0,\infty)$. Then (4.1) *has at most one solution on $\mathbf{N}_0$.*

Proof. Let $y_1(n)$ and $y_2(n)$ be two solutions of (4.1) on $\mathbf{N}_0$. From (4.9), we have

$$|u(n)| \leq \sum_{s=b_1(0)}^{b_1(n-1)} f_1(n,s)w_1(u(s)) + \sum_{s=b_2(0)}^{b_2(n-1)} f_2(n,s)w_2(u(s)), \quad n \in \mathbf{N}_0, \tag{4.10}$$

where $u(n) = |y_1(n) - y_2(n)|$, $a(n) = 0$, $w_1(u) = \sqrt{u}$ and $w_2(u) = u$. Appling Theorem 2.1, Remark 2.4, and the notation $W_i(0) = 0$ for $i = 1, 2$, we obtain that $u(n) = 0$ which implies that the solution is unique. □

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