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Research Article

Notes on the Propagators of Evolution Equations

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We consider the propagator of an evolution equation, which is a semigroup of linear operators. Questions related to its operator norm function and its behavior at the *critical point* for norm continuity or compactness or differentiability are studied.

1. Introduction

As it is well known, each well-posed Cauchy problem for first-order evolution equation in Banach spaces

$$\begin{gathered} u'(t) = Au(t), \quad t \geq 0, \\ u(0) = u_0 \end{gathered} \tag{1.1}$$

gives rise to a well-defined propagator, which is a semigroup of linear operators, and the theory of semigroups of linear operators on Banach spaces has developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1948. By now, it is a rich theory with substantial applications to many fields (cf., e.g., [1–6]).

In this paper, we pay attention to some basic problems on the semigroups of linear operators and reveal some essential properties of theirs.

Let X be a Banach space.

Definition 1.1 (see [1–6]). *A one-parameter family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a strongly continuous semigroup (or simply C_0-semigroup) if it satisfies the following conditions:*

(i) $T(0) = I$, with (I being the identity operator on X),

(ii) $T(t+s) = T(t)T(s)$ for $t, s \geq 0$,

(iii) the map $t \to T(t)x$ is continuous on $[0,\infty)$ for every $x \in X$.

The *infinitesimal generator* A of $(T(t))_{t\geq 0}$ is defined as

$$Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t} \tag{1.2}$$

with domain

$$D(A) = \left\{ x \in X;\ \lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists} \right\}. \tag{1.3}$$

For a comprehensive theory of C_0-semigroups we refer to [2].

2. Properties of the Function $t \mapsto \|T(t)\|$

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X and define $g(t) := \|T(t)\|$ for $t \geq 0$. Clearly, from Definition 1.1 we see that

(I) $g(0) = 1$, $g(t) \geq 0$ for $t \geq 0$;

(II) $g(t+s) \leq g(t)g(s)$ for $t, s \geq 0$.

Furthermore, we can infer from the strong continuity of $(T(t))_{t\geq 0}$ that

(III) $g(t)$ is lower-semicontinuous, that is,

$$g(t) \leq \liminf_{s\to t} g(s). \tag{2.1}$$

In fact,

$$\|T(t)x\| = \liminf_{s\to t}\|T(s)x\| \leq \liminf_{s\to t} \sup_{\|x\|=1} \|T(s)x\| = \liminf_{s\to t} g(s) \tag{2.2}$$

holds for all $x \in X$ with $\|x\| = 1$. Thus, taking the supremum for all $x \in X$ with $\|x\| = 1$ on the left-hand side leads to (2.1).

We ask the following question

For every function $g(\cdot)$ satisfying (I), (II), and (III), does there exist a C_0 semigroup $(T(t))_{t\geq 0}$ on some Banach space X such that $\|T(t)\| = g(t)$ for all $t \geq 0$?

We show that this is not true even if X is a finite-dimensional space.

Theorem 2.1. *Let X be an n-dimensional Banach space with $2 \leq n < \infty$. Let*

$$g(t) = \begin{cases} 1+t & \text{if } 0 \leq t \leq 2^n - 1, \\ \left(\dfrac{2}{2^n - 1}\right)^n t^n & \text{if } t > 2^n - 1. \end{cases} \tag{2.3}$$

Then $g(\cdot)$ satisfies (I), (II), and (III), and there exists no C_0 semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| = g(t)$ for all $t \geq 0$.

Proof. First, we show that $g(\cdot)$ satisfies (I), (II), and (III).

(I) is clearly satisfied.

To show (III) and (II), we write

$$\gamma := 2^n - 1. \tag{2.4}$$

Then

$$\lim_{t\to\gamma-} g(t) = g(\gamma) = 1 + \gamma = 2^n = \lim_{t\to\gamma+} g(t), \tag{2.5}$$

hence $g(\cdot)$ satisfies (III).

For (II), suppose $t, s \geqslant 0$, and consider the following four cases.

Case 1 ($t \geqslant \gamma$ and $s \geqslant \gamma$). In this case

$$\frac{g(t+s)}{g(t)g(s)} = \left(\frac{\gamma}{2}\right)^n \left(\frac{1}{t} + \frac{1}{s}\right)^n \leqslant \left(\frac{\gamma}{2}\right)^n \left(\frac{2}{\gamma}\right)^n = 1, \tag{2.6}$$

that is,

$$g(t+s) \leqslant g(t)g(s). \tag{2.7}$$

Case 2 ($t \geqslant \gamma$ and $s \leqslant \gamma$). Let

$$h(s) = \left(1 + \frac{s}{\gamma}\right)^n - 1 - s. \tag{2.8}$$

Then

$$h''(s) = \frac{n(n-1)}{\gamma^2}\left(1 + \frac{s}{\gamma}\right)^{n-2} \geqslant 0, \tag{2.9}$$

and $h(\cdot)$ is a convex function on $[0, \gamma]$. So by Jensen's inequality, we have

$$h(s) \leqslant \frac{s}{\gamma}h(0) + \frac{\gamma - s}{\gamma}h(\gamma) = \frac{\gamma - s}{\gamma}(2^n - 1 - \gamma) = 0, \tag{2.10}$$

that is,

$$\left(1 + \frac{s}{\gamma}\right)^n \leqslant 1 + s \quad \text{for } 0 \leqslant s \leqslant \gamma. \tag{2.11}$$

Therefore

$$\frac{g(t+s)}{g(t)} = \left(1+\frac{s}{t}\right)^n \leqslant \left(1+\frac{s}{\gamma}\right)^n \leqslant 1+s = g(s), \tag{2.12}$$

that is, $g(t+s) \leqslant g(t)g(s)$.

Case 3 ($t+s>\gamma$, but $t \leqslant \gamma$ and $s \leqslant \gamma$). It follows from Case 2 that

$$\begin{aligned} g(t+s) &\leqslant g(\gamma)g(t+s-\gamma) \\ &= (1+\gamma)(1+t+s-\gamma) \\ &= (1+t)(1+s) - (\gamma-s)(\gamma-t) \\ &\leqslant (1+t)(1+s) \\ &= g(t)g(s). \end{aligned} \tag{2.13}$$

Case 4 ($t+s \leqslant \gamma$). Again we have

$$g(t+s) = 1+t+s \leqslant 1+t+s+st = (1+t)(1+s) = g(t)g(s). \tag{2.14}$$

Next, we prove that there does not exist any C_0 semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| = g(t)$. Suppose $g(t) = \|T(t)\|$ $(t \geq 0)$ for some C_0 semigroup $(T(t))_{t\geq 0}$ on X and let A be its infinitesimal generator.

First we note from (2.3) that

$$\lim_{t\to+\infty} \left\|e^{-\varepsilon t}T(t)\right\| = \lim_{t\to+\infty} e^{-\varepsilon t}g(t) = 0 \tag{2.15}$$

for every $\varepsilon > 0$, while

$$\|T(t)\| = O(t^n) \quad (t \longrightarrow +\infty). \tag{2.16}$$

By the well-known Lyapunov theorem [2, Chapter I, Theorem 2.10], all eigenvalues of $A-\varepsilon I$ (the infinitesimal generator of $e^{-\varepsilon t}T(t)$) have negative parts for every $\varepsilon > 0$. Letting $\lambda_1,\ldots,\lambda_r$ be the eigenvalues of A, we then have

$$\operatorname{Re}\lambda_j - \varepsilon < 0 \quad \text{for every } \varepsilon > 0,\ j = 1,\ldots,r, \tag{2.17}$$

and this implies that

$$\operatorname{Re}\lambda_j \leq 0, \quad j = 1, \dots, r. \tag{2.18}$$

It is known that there is an isomorphism P of $\mathbb{C}^n$ onto X such that

$$A = P \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & J_r \end{pmatrix} P^{-1}, \tag{2.19}$$

where J_j is the Jordan block corresponding to λ_j. Therefore

$$T(t) = P \begin{pmatrix} e^{tJ_1} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & e^{tJ_r} \end{pmatrix} P^{-1}. \tag{2.20}$$

Set

$$N_j = J_j - \lambda_j I_{(k_j)}, \quad 1 \leq j \leq r, \tag{2.21}$$

where k_j is the order of J_j. Then N_j is a k_jth nilpotent matrix with $k_j \leq n$ for each $1 \leq j \leq r$. According to (2.20) and (2.18), we have

$$\begin{aligned} \|T(t)\| &\leq \|P\| \cdot \left\|P^{-1}\right\| \max_{1\leq j\leq r} \left\|e^{tJ_j}\right\| \\ &= \|P\| \cdot \left\|P^{-1}\right\| \max_{1\leq j\leq r} \left\|e^{t\lambda_j} e^{tN_j}\right\| \\ &\leq \|P\| \cdot \left\|P^{-1}\right\| \max_{1\leq j\leq r} \left\|e^{tN_j}\right\|. \end{aligned} \tag{2.22}$$

Observing

$$\begin{aligned} \left\|e^{tN_i}\right\| &= \left\| I_{(k_i)} + tN_i + \cdots + \frac{t^{k_i-1}}{(k_i-1)!} N_i^{k_i-1} \right\| \\ &= O\left(t^{k_i-1}\right) \quad (t \longrightarrow +\infty), \end{aligned} \tag{2.23}$$

we see that

$$\left\| e^{tN_i} \right\| = O\left(t^{n-1}\right) \quad (t \longrightarrow +\infty). \tag{2.24}$$

Thus,

$$g(t) = \|T(t)\| = O\left(t^{n-1}\right) \quad (t \longrightarrow +\infty), \tag{2.25}$$

which is a contradiction to (2.16). □

Open Problem 1. Is it possible that there exists an X with $\dim X = \infty$ and a C_0 semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\| = g(t)$ for all $t \geq 0$?

3. The Critical Point of Norm-Continuous (Compact, Differentiable) Semigroups

The following definitions are basic [1–6].

Definition 3.1. A C_0-semigroup $(T(t))_{t\geq 0}$ is called norm-continuous for $t > t_0$ if $t \mapsto T(t)$ is continuous in the uniform operator topology for $t > t_0$.

Definition 3.2. A C_0-semigroup $(T(t))_{t\geq 0}$ is called compact for $t > t_0$ if $(T(t))_{t\geq 0}$ is a compact operator for $t > t_0$.

Definition 3.3. A C_0-semigroup $(T(t))_{t\geq 0}$ is called differentiable for $t > t_0$ if for every $x \in X$, $t \mapsto T(t)x$ is differentiable for $t > t_0$.

It is known that if a C_0-semigroup $(T(t))_{t\geq 0}$ is norm continuous (compact, differentiable) at $t = t_0$, then it remains so for all $t > t_0$. For instance, the following holds.

Proposition 3.4. *If the map $t \rightarrow T(t)x$ is right differentiable at $t = t_0$, then it is also differentiable for $t > t_0$.*

Therefore, if we write

$$\begin{aligned} J_n &:= \{t_0 \geq 0;\ T(t) \text{is norm-continuousfor} \ \ t > t_0\}, \\ J_c &:= \left\{t_0 \geq 0;\ T(t) \text{ is compact for } \ t > t_0\right\}, \\ J_d &:= \{t_0 \geq 0;\ T(t) \text{ is differentiable for } \ t > t_0\}, \end{aligned} \tag{3.1}$$

and suppose $J_n \neq \emptyset$ ($J_c \neq \emptyset$, $J_d \neq \emptyset$), then J_n (J_c, J_d) takes the form of $[\tau_0, \infty)$ for a nonnegative real number τ_0. In other words, if $J_n \neq \emptyset$ ($J_c \neq \emptyset$, $J_d \neq \emptyset$), then $(T(t))_{t\geq 0}$ is norm continuous (compact, differentiable) on the interval (τ_0, ∞) but not at any point in $[0, \tau_0)$. We call τ_0 the *critical point* of the norm continuity (compactness, differentiability) of operator semigroup $(T(t))_{t\geq 0}$.

A natural question is the following

Suppose that τ_0 is the critical point of the norm continuity (compactness, differentiability) of the operator semigroup $(T(t))_{t\geq 0}$. Is $(T(t))_{t\geq 0}$ also norm continuous (compact, differentiable) at τ_0? Of course, concerning norm continuity or differentiability at τ_0 we only mean right continuity or right differentiability.

We show that the answer is "yes" in some cases and "no" for other cases.

Example 3.5. Let $X = L^2[0,1]$ *and*

$$T(t)f(s) = \begin{cases} f(s+t), & s+t \leq 1, \\ 0, & s+t > 1, \end{cases} \quad t \geq 0. \tag{3.2}$$

Then clearly $T(t) = 0$ for $t > 1$. Moreover, $(T(t))_{t\geq 0}$ is not norm continuous (not compact, not differentiable) for any $0 \leq t < 1$ since

$$\|T(t+h) - T(t)\| \geq \|(T(t+h) - T(t))f_h\| = 2 \tag{3.3}$$

for sufficiently small $h > 0$, where

$$f_h(s) = \begin{cases} h^{-1/2}, & t+h \leq s \leq t+2h, \\ 0, & \text{otherwise}. \end{cases} \tag{3.4}$$

Therefore, in this case we have $\tau_0 = 1$. Since $T(1) = 0$, we see that $T(1)$ is compact and $(T(t))_{t\geq 0}$ is differentiable at $t = 1$ from the right.

Example 3.6. Let

$$X = c_0 = \left\{ (x_1, \ldots, x_n, \ldots);\ x_n \in \mathbb{C},\ \lim_{n\to\infty} x_n \text{ exists} \right\} \tag{3.5}$$

with supremum norm. For any $x = (x_1, \ldots, x_n, \ldots) \in X$ set

$$T(t)x = \left(e^{-t}(\cos(te) + \mathrm{i}\ \sin(te))x_1, \ldots, e^{-nt}(\cos(te^n) + i\ \sin(te^n))x_n, \ldots\right), \quad t \geq 0. \tag{3.6}$$

Then, $(T(t))_{t\geq 0}$ is compact (hence norm-continuous) for $t > 0$ since $T(t)$ is the operator-norm limit of a sequence $(T_k(t))_{k\in\mathbb{N}}$ of finite-rank operators:

$$T_k(t)x = \left(y_1(t), \ldots, y_n(t), \ldots\right), \quad \text{for each } x = (x_1, \ldots, x_n, \ldots) \in X,\ k \in \mathbb{N}, \tag{3.7}$$

where

$$y_n(t) = \begin{cases} e^{-nt}(\cos(te^n) + i\ \sin(te^n))x_n, & n \leq k, \\ 0, & n > k. \end{cases} \tag{3.8}$$

So the critical point for compactness and norm continuity is $\tau_0 = 0$. However, the infinitesimal generator of $(T(t))_{t\geq 0}$ is given by

$$Ax = ((-1 + \mathrm{i}e)x_1, \ldots, (-n + \mathrm{i}e^n)x_n, \ldots) \quad \text{for } x = (x_1, \ldots, x_n, \ldots) \in D(A) \tag{3.9}$$

with

$$D(A) = \{x \in X;\ Ax \in X\}. \tag{3.10}$$

In view of that A is unbounded, we know that $(T(t))_{t\geq 0}$ is not norm continuous at $t = 0$.

For differentiability, we note that $T(t)x$ is differentiable at $t = t_0$ if and only if $T(t_0)x \in D(A)$ for each $x \in X$. From

$$\begin{aligned} AT(t)x = \Big(&\Big(e^{(1-t)}(-\sin(te) + i\cos(te)) - e^{-t}(\cos(te) + i\sin(te))\Big)x_1, \ldots, \\ &\Big(e^{n(1-t)}(-\sin(te^n) + i\cos(te^n)) - ne^{-nt}(\cos(te^n) + i\sin(te^n))\Big)x_n, \ldots\Big), \\ &\qquad \text{for each } x = (x_1, \ldots, x_n, \ldots) \in X,\ t \geq 0, \end{aligned} \tag{3.11}$$

it follows that when $t > 1$, $T(t)x \in D(A)$ for every $x \in X$. On the other hand, when $0 \leq t \leq 1$ and x is any nonzero constant sequence, $T(t)x \notin D(A)$. Therefore the critical point for differentiability is $\tau_0 = 1$. But $(T(t))_{t\geq 0}$ is not differentiable at $t = 1$.

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