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Research Article

Employing of Some Basic Theory for Class of Fractional Differential Equations

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Basic theory on a class of initial value problem of some fractional differential equation involving Riemann-Liouville differential operators is discussed by employing the classical approach from the work of Lakshmikantham and A. S. Vatsala (2008). The theory of inequalities, local existence, extremal solutions, comparison result and global existence of solutions are considered. Our work employed recent literature from the work of (Lakshmikantham and A. S. Vatsala, (2008)).

1. Introduction

Differential equations of fractional order have recently proven to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so forth [1–5]. There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monographs of Kilbas et al. [6], Lakshmikantham et al. [7], Podlubny [4], and the survey by Agarwal et al. [8]. For some recent contributions on fractional differential equations, see [9–20] and the references therein. Very recently in [10, 11, 21, 22], the author and other researchers studied the existence and uniqueness of solutions of some classes of fractional differential equations with delay. For more details on the geometric and physical interpretation for fractional derivatives of both the Caputo types see [5, 23]. Băleanu and Mustafa [16] allowed for immediate applications of a general comparison-type result from the recent literature [24].

Lakshmikantham and Vatsal have discussed basic theory of fractional differential equations for initial value problem fractional differential equations type [24]

$$D^{\alpha}x(t) = f(t, x(t)), \quad x(0) = x_0, \ t \in [0, T]. \tag{1.1}$$

This paper deals with the basic theory of initial value problem (IVP) for a generalized class of fractional order differential equation of the form

$$\mathcal{L}(\mathfrak{D})[x(t) - x_0] = f(t, x(t)), \quad x(0) = x_0, \ t \in [0, T], \tag{1.2}$$

where $\mathcal{L}(\mathfrak{D}) = D_0^{\alpha} - t^n D_0^{\beta}, \ f \in C([0, T], \mathbb{R})$ and $0 < \beta \le \alpha < 1$. Recall that $C([0, T], \mathbb{R})$ is the Banach space of continuous functions from the interval $[0, \ T]$ into $\mathbb{R}$ endowed with uniform norm.

We begin in this section with the recall of some definitions and results for fractional calculus which are used throughout this paper [4, 19, 20].

The left-sided Riemann-Liouville fractional integral of a function x of order $\alpha > 0$ is defined as

$$I_0^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) ds, \quad t > 0, \tag{1.3}$$

and the left sided Riemann-Liouville fractional derivative operator of order $0 < \alpha < 1$ is defined by

$$D_0^{\alpha}x(t) = \frac{d}{dt}\left\{I_0^{1-\alpha}x(t)\right\}, \tag{1.4}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt$. We denote $D_{0^+}^{\alpha}x(t)$ by $D_0^{\alpha}x(t)$ and $I_{0^+}^{\alpha}x(t)$ by $I_0^{\alpha}x(t)$. Also $D^{\alpha}x(t)$ and $I^{\alpha}x(t)$ refer to $D_{0^+}^{\alpha}x(t)$ and $I_{0^+}^{\alpha}x(t)$, respectively.

Assume that $0 < \beta < 1, \ \beta \le \alpha$, if the fractional derivative $D_0^{\beta}x(t)$ is integrable, then [4, page 72]

$$I_0^{\alpha}\left(D_0^{\beta}x(t)\right) = I_0^{\alpha-\beta}x(t) - \left[I_0^{1-\beta}x(t)\right]_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{1.5}$$

If $x(t)$ is continuous on $[0, T]$ then $D_0^{\beta}x(t)$ is integrable, $I^{1-\beta}x(t)|_{t=0} = 0$ [25], and (1.5) reduces to

$$I_0^{\alpha}\left(D_0^{\beta}x(t)\right) = I_0^{\alpha-\beta}x(t), \quad 0 < \beta < 1, \ \beta \le \alpha. \tag{1.6}$$

Proposition 1.1. *Let* $x \in C([0, T], \mathbb{R})$ *and* $0 < \beta < 1, \ \beta \le \alpha$. *Then* $\sum_{k=0}^{n}$

(i) $I^{\alpha}(t^n x(t)) = \sum_{k=0}^{n}\binom{-\alpha}{k}[D^k t^n][I^{\alpha+k}x(t)] = \sum_{k=0}^{n}\binom{-\alpha}{k}(n!t^{n-k}/(n-k)!)I^{\alpha+k}x(t)$,

(ii) $I^{\alpha}(t^n D^{\beta}x(t)) = \sum_{k=0}^{n}\binom{-\alpha}{k}(n!t^{n-k}/(n-k)!)I^{\alpha-\beta+k}x(t)$, *where n is a nonnegative integer, and*

$$\binom{-\alpha}{k} = (-1)^k \frac{\Gamma(\alpha + k)}{k!\,\Gamma(\alpha)}. \tag{1.7}$$

Proof. (i) can be found in [19, page 53] and (ii) is an immediate consequence of (1.6) and (i). □

2. Strict and Nonstrict Inequalities

Since f is assumed to be continuous, the initial value problem (1.2) is equivalent to the following Volterra fractional integral [10, 25]:

$$\begin{aligned} x(t) &= x(0) + \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k}\{x(t) - x(0)\} + I^{\alpha} f(t, x(t)) \\ &= x(0) + \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!\,\Gamma(\alpha-\beta+k)} \int_0^t (t-s)^{\alpha-\beta+k-1} x(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds, \quad 0 \le t \le T. \end{aligned} \tag{2.1}$$

Note. Let us consider the notation of $(\mathcal{J}x)(t)$ for second term in (2.1) which is used in throughout of text and so that

$$\begin{aligned} (\mathcal{J}x)(t) &= \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!} I^{\alpha-\beta+k}\{x(t) - x(0)\} \\ &= \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n!t^{n-k}}{(n-k)!\,\Gamma(\alpha-\beta+k)} \int_0^t (t-s)^{\alpha-\beta+k-1} x(s)ds. \end{aligned} \tag{2.2}$$

We now first discuss a fundamental result relative to the fractional inequalities.

Theorem 2.1. *Let $v, w \in C([0,T], \mathbb{R})$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and*

(i) $v(t) \le v(0) + (\mathcal{J}v)(t) + 1/\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} f(s, v(s))ds$,

(ii) $w(t) \ge w(0) + (\mathcal{J}w)(t) + 1/\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} f(s, w(s))ds$,

one of the foregoing inequalities being strict. Moreover, if $f(t,x)$ is nondecreasing in x for each t and $v(0) < w(0)$, then one has $v(t) < w(t)$, $t \in [0,T]$.

Proof. Suppose that for each $t \in [0,T]$, the conclusion $v(t) < w(t)$ is not true. Then, because of the continuity of the functions involved and $v(0) < w(0)$ it follows that there exists a t_1 such that $0 < t_1 \le T$ and

$$v(t_1) = w(t_1), \qquad v(t) < w(t), \qquad 0 < t \le t_1. \tag{2.3}$$

Let us suppose that the inequality (ii) is strict. Then using the nondecreasing nature of f and (2.3) we get

$$\begin{aligned} w(t_1) &> w(0) + (\mathcal{I}w)(t_1) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} f(s, w(s)) ds \\ &\geq v(0) + (\mathcal{I}v)(t_1) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} f(s, v(s)) ds \geq v(t_1), \end{aligned} \tag{2.4}$$

which is a contradiction in view of (2.3). Hence the conclusion of this theorem holds and the proof is complete. □

The next result is for nonstrict inequalities, which require a one-sided Lipschitz type condition.

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold with nonstrict inequalities (i) and (ii). Suppose further that*

$$f(t,x) - f(t,y) \leq \frac{L}{1+t^2}(x-y), \tag{2.5}$$

whenever $x \geq y$ *and* $L > 0$. *Then,* $v(0) \leq w(0)$, *and* $L < \Gamma(\alpha+1)$ *implies*

$$v(t) \leq w(t), \quad 0 \leq t \leq T. \tag{2.6}$$

Proof. Set $w_\epsilon(t) = w(t) + \epsilon(1+t^\alpha)$, for small $\epsilon > 0$ so that we have,

$$w_\epsilon(0) = w(0) + \epsilon > w(0), \quad w_\epsilon(t) > w(t), 0 \leq t \leq T. \tag{2.7}$$

Now,

$$\begin{aligned} w_\epsilon(t) \geq w(0) &+ \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n! t^{n-k}}{(n-k)!} I^{\alpha-\beta+k} \{w_\epsilon(t) - \omega(0)\} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, w(s)) ds + \epsilon(1+t^\alpha). \end{aligned} \tag{2.8}$$

In view of (2.7), using one-sided Lipschitz condition (2.5) we see that

$$\begin{aligned} w_\epsilon(t) &\geq w_\epsilon(0) + (\mathcal{I}w_\epsilon)(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s, w_\epsilon(s)) - \epsilon \frac{L(1+s^\alpha)}{(1+s^\alpha)} \right) ds + \epsilon t^\alpha \\ &= w_\epsilon(0) + (\mathcal{I}w_\epsilon)(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, w_\epsilon(s)) ds - \frac{\epsilon L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \epsilon t^\alpha. \end{aligned} \tag{2.9}$$

Now, since $\int_0^t (t-s)^{\alpha-1}ds = t^\alpha \int_0^1 (1-\tau)^{\alpha-1}d\tau = (\Gamma(\alpha)/\Gamma(\alpha+1))t^\alpha$, we arrive at

$$w_\epsilon(t) > w_\epsilon(0) + (\mathcal{I}w_\epsilon)(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, w_\epsilon(s))ds, \tag{2.10}$$

in view of the condition $L < \Gamma(\alpha+1)$. We now apply Theorem 2.1 to the inequalities (i), (2.9), and (2.10) to get $v(t) < w_\epsilon(t)$, $t \in [0,T]$. Since $\epsilon > 0$ is arbitrary, we conclude that (2.6) is true, and we are done. □

3. Local Existence and Extremal Conditions

In this section, we will consider the local existence and the existence of extremal solutions for the IVP (1.2). We now first discuss Peano's type existence result.

Theorem 3.1. *Assume that $f \in (\Omega_0, \mathbb{R})$, where $\Omega_0 = \{(t,x) : 0 \le t \le r$ and $|x - x_0| \le b\}$, and let $|f(t,x)| \le M$ on Ω. Then the IVP* (1.2) *has at least one solution $x(t)$ on $[0,\lambda]$, where*

$$\lambda = \min\left\{ r, \left[\frac{b\ \Gamma(\alpha+1)}{(n+2)\ M}\right]^{1/\alpha}, \left[\frac{b\ \Gamma(\alpha-\beta+k+1)}{(n+2)\ \rho\ b}\right]^{1/(\alpha-\beta+k)}, k = \overline{0,\ n} \right\}, \tag{3.1}$$

so that ρ and $\|x_0\|$ will be observing in the proof of this theorem.

Proof. Let $x_0(t)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$ such that $x_0(0) = x_0$, $|x_0(t) - x_0| \le b$. For $\epsilon \le \delta$, we define a function $x_\epsilon(t) = x_0(t)$ on $[-\delta, 0]$ and

$$x_\epsilon(t) = x_0 + (\mathcal{I}x_\epsilon)(t-\epsilon) + I^\alpha f(t, x_\epsilon(t-\epsilon)) \tag{3.2}$$

on $[0,\lambda_1]$, where $\lambda_1 = \min\{\lambda, \epsilon\}$. We observe that

$$\begin{aligned} |x_\epsilon(t) - x_0| &\le \sum_{k=0}^{n} \left|\binom{-\alpha}{k}\right| \frac{n! t^{n-k}}{(n-k)!} I^{\alpha-\beta+k} |x_\epsilon(t-\epsilon) - x_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_\epsilon(s-\epsilon))ds, \\ &\le \rho \sum_{k=0}^{n} I^{\alpha-\beta+k} |x_\epsilon(t-\epsilon) - x_0| \ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\ x_\epsilon(s-\epsilon))ds, \end{aligned} \tag{3.3}$$

where

$$\rho = \max\left\{ \left|\binom{-\alpha}{k}\right| \frac{n!\mu_1^{n-k}}{(n-k)!}, \left|\binom{-\alpha}{k}\right| \frac{n!}{(n-k)!} : k = \overline{0,n} \right\}. \tag{3.4}$$

Note that, $0 < s \le t \le \lambda_1 \le \epsilon$, as $\lambda_1 = \min\{\lambda, \epsilon\}$. Then $-\delta \le -\epsilon < s - \epsilon \le 0$, and hence we can consider $x_\epsilon(s-\epsilon)(t) = x_0(s-\epsilon)$ in the last above inequality. Thus $|x_\epsilon(s-\epsilon) - x_0| \le b$.

$$\begin{aligned}
|x_\epsilon(t) - x_0| &\le \sum_{k=0}^{n} \frac{\rho b}{\Gamma(\alpha-\beta+k)} \int_0^t (t-s)^{\alpha-\beta+k-1} + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&= \sum_{k=0}^{n} \frac{\rho b}{\Gamma(\alpha-\beta+k+1)} t^{\alpha-\beta+k} + \frac{M}{\Gamma(\alpha+1)} t^{\alpha} \\
&= \sum_{k=0}^{n} \frac{\rho b}{\Gamma(\alpha-\beta+k+1)} \lambda^{\alpha-\beta+k} + \frac{M}{\Gamma(\alpha+1)} \lambda^{\alpha} \le b,
\end{aligned} \tag{3.5}$$

because of the choice of λ_1. If $\lambda_1 < \lambda$, we can employ (3.2) to extend as a continuous function on $[-\delta, \lambda_2], \lambda_2 = \min\{\lambda, 2\epsilon\}$, such that $|x_\epsilon - x_0| \le b$ holds. Continuing this process, we can define $x_\epsilon(t)$ over $[-\delta, \lambda]$ so that $|x_\epsilon(t) - x_0| \le b$ is satisfied on $[-\delta, \lambda]$. Furthermore, letting $0 \le t_1 \le t_2 \le \lambda$ we see that

$$\begin{aligned}
|x_\epsilon(t_1) - x_\epsilon(t_2)| &= |(\mathcal{J}x_\epsilon)(t_1) - (\mathcal{J}x_\epsilon)(t_2)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \frac{f(s, x_\epsilon(s-\epsilon))}{(t_1-s)^{1-\alpha}} ds - \int_0^{t_2} \frac{f(s, x_\epsilon(s-\epsilon))}{(t_2-s)^{1-\alpha}} ds \right\} \\
&= |(\mathcal{J}x_\epsilon)(t_1) - (\mathcal{J}x_\epsilon)(t_2)| + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha} \right] f(s, x_\epsilon(s-\epsilon)) ds \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{1-\alpha} f(s, x_\epsilon(s-\epsilon)) ds \right| \\
&\le |(\mathcal{J}x_\epsilon)(t_1) - (\mathcal{J}x_\epsilon)(t_2)| \\
&\quad + \left| \frac{M}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha} \right] ds - \frac{M}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{1-\alpha} ds \right|.
\end{aligned} \tag{3.6}$$

Notice that $\binom{-\alpha}{2k} > 0$ and $\binom{-\alpha}{2k+1} < 0$, when $0 < \alpha < 1$. Let,

$$\begin{aligned}
\Lambda_1 &:= \max\left\{ \frac{n! \lambda^{n-2k} \binom{-\alpha}{2k}}{(n-2k)!\, \Gamma(\alpha-\beta+2k)},\ k = 0, 1,\ 2,\ \dots, \left[\frac{n}{2}\right] \right\} > 0 \\
\Lambda_2 &:= \min\left\{ \frac{n! \lambda^{n-2k-1} \binom{-\alpha}{2k+1}}{(n-2k-1)!\, \Gamma(\alpha-\beta+2k+1)},\ k = 0, 1, \dots, 1 + \left[\frac{n}{2}\right] \right\} < 0,
\end{aligned} \tag{3.7}$$

firstly we have

$$
\begin{aligned}
&(\mathcal{J}x_\epsilon)(t_1)-(\mathcal{J}x_\epsilon)(t_2)\\
&\quad=\sum_{k=0}^{n}\binom{-\alpha}{k}\frac{n!t_1^{n-k}}{(n-k)!\,\Gamma(\alpha-\beta+k)}\int_0^{t_1}(t_1-s)^{\alpha-\beta+k-1}\{x_\epsilon(s-\epsilon)-x_0\}ds\\
&\quad\quad-\sum_{k=0}^{n}\binom{-\alpha}{k}\frac{n!{t_2}^{n-k}}{(n-k)!\,\Gamma(\alpha-\beta+k)}\int_0^{t_2}(t_2-s)^{\alpha-\beta+k-1}\{x_\epsilon(s-\epsilon)-x_0\}ds\\
&\quad=\Lambda_1\sum_{k=0}^{[n/2]}\left[\int_0^{t_1}\frac{x_\epsilon(s-\epsilon)-x_0}{(t_1-s)^{\beta-\alpha-2k+1}}ds-\int_0^{t_2}\frac{x_\epsilon(s-\epsilon)-x_0}{(t_2-s)^{\beta-\alpha-2k+1}}ds\right]\\
&\quad\quad+\Lambda_2\sum_{k=0}^{[n/2]+1}\left[\int_0^{t_1}\frac{x_\epsilon(s-\epsilon)-x_0}{(t_1-s)^{\beta-\alpha-2k}}ds-\int_0^{t_2}\frac{x_\epsilon(s-\epsilon)-x_0}{(t_2-s)^{\beta-\alpha-2k}}ds\right]\\
&\quad\le b\Lambda_1\sum_{k=0}^{[n/2]}\left[\int_0^{t_1}\left((t_1-s)^{\alpha-\beta+2k-1}-(t_2-s)^{\alpha-\beta+2k-1}\right)ds\right]\\
&\quad\quad+b\Lambda_1\sum_{k=0}^{[n/2]}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta+2k-1}ds\\
&\quad\quad+b\Lambda_2\sum_{k=0}^{[n/2]+1}\left[\int_0^{t_1}\left((t_1-s)^{\alpha-\beta+2k}-(t_2-s)^{\alpha-\beta+2k}\right)ds\right]\\
&\quad\quad+b\Lambda_2\sum_{k=0}^{[n/2]+1}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta+2k}ds\\
&\quad\le b\Lambda_1\sum_{k=0}^{[n/2]}\frac{(t_2-t_1)^{\alpha-\beta+2k}}{\alpha-\beta+2k}+b|\Lambda_2|\sum_{k=0}^{[n/2]+1}\frac{(t_2-t_1)^{\alpha-\beta+2k+1}}{\alpha-\beta+2k+1}.
\end{aligned}
\tag{3.8}
$$

We, therefore, set $\Lambda=\max\{b|\Lambda_1|,b\ |\Lambda_2|\}$, and get

$$
(\mathcal{J}x_\epsilon)(t_1)-(\mathcal{J}x_\epsilon)(t_2)\le 2\Lambda\left\{\sum_{k=0}^{[n/2]}\frac{(t_2-t_1)^{\alpha-\beta+2k}}{\alpha-\beta+2k}+\sum_{k=0}^{[n/2]+1}\frac{(t_2-t_1)^{\alpha-\beta+2k+1}}{\alpha-\beta+2k+1}\right\}. \tag{3.9}
$$

Finally, inequality (3.6) from inequality (3.9) becomes

$$
\begin{aligned}
|x_\epsilon(t_1) - x_\epsilon(t_2)| &\le \left|\frac{M}{\Gamma(\alpha+1)}\left\{2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha\right\}\right| \\
&\quad + \left|2\Lambda\left\{\sum_{k=0}^{[n/2]}\frac{(t_2-t_1)^{\alpha-\beta+2k}}{\alpha-\beta+2k} + \sum_{k=0}^{[n/2]+1}\frac{(t_2-t_1)^{\alpha-\beta+2k+1}}{\alpha-\beta+2k+1}\right\}\right| \\
&\le 2\Lambda\left\{\sum_{k=0}^{[n/2]}\frac{(t_2-t_1)^{\alpha-\beta+2k}}{\alpha-\beta+2k} + \sum_{k=0}^{[n/2]+1}\frac{(t_2-t_1)^{\alpha-\beta+2k+1}}{\alpha-\beta+2k+1}\right\} \\
&\quad + \frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha < \epsilon,
\end{aligned}
\tag{3.10}
$$

provided that

$$
\begin{aligned}
&|t_2 - t_1| < \delta_0 \\
&\quad = \min\left\{\left(\frac{\epsilon\,\Gamma(\alpha+1)}{2(n+1)M}\right)^{1/\alpha}, \left(\frac{\alpha-\beta+2k}{2(n+1)\Lambda}\right)^{1/(\alpha-\beta+2k)}, \left(\frac{\alpha-\beta+2k+1}{2(n+1)\Lambda}\right)^{1/(\alpha-\beta+2k+1)}\right\}_{k=0}^{n}.
\end{aligned}
\tag{3.11}
$$

It then follows from (3.4) and (3.6) that the family $\{x_\epsilon(t)\}$ forms an equicontinuous and uniformly bounded functions. Ascoli-Arzela theorem implies that the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_m \to 0$ as $m \to \infty$, and $x(t) = \lim_{m\to\infty} x_{\epsilon_m}(t)$ exists uniformly on $[-\delta, \lambda]$. Since f is uniformly continuous, we obtain that $f(t, x_{\epsilon_n}(t-\epsilon_m))$ tends uniformly to $f(t, x(t))$ as $m \to \infty$, and, hence, term by term integration of (3.2) with $\epsilon = \epsilon_m, \lambda_1 = \lambda$ yields

$$
x(t) = x_0 + (\mathcal{J}x)(t) + I^\alpha f(t, x(t)). \tag{3.12}
$$

This proves that $x(t)$ is a solution of the IVP (1.2), and the proof is complete. □

Theorem 3.2. *Under the hypothesis of Theorem 3.1, there exists extremal solution for the IVP* (1.2) *on the interval* $0 \le t \le \lambda_0$, *provided* $f(t,x)$ *is nondecreasing in* x *for each* t, *where* λ_0 *will be observed in the proof of this theorem.*

Proof. We will prove the existence of maximal solution only, since the case of minimal solution is very similar. Let $0 < \epsilon \le b/2$ and consider the fractional differential equation with an initial condition

$$
\mathcal{L}(\mathfrak{D})[x(t) - x_0] = f(t, x(t)) + \epsilon, \qquad x(0) = x_0 + \epsilon. \tag{3.13}
$$

We observe that $f_\epsilon(t,x) = f(t,x) + \epsilon$ is defined and continuous on

$$
\Omega_\epsilon = \left\{(t,x) : 0 \le t \le \lambda, |x - (x_0 + \epsilon)| \le \frac{b}{2}\right\}, \tag{3.14}
$$

$\Omega_\epsilon \subset \Omega_0$ and $|f_\epsilon(t,x) \le M + b/2$ on Ω_ϵ. We then deduce from Theorem 3.1 that IVP (3.6) has a solution $x(t,\epsilon)$ on the interval $o \le t \le \lambda_0$. Now for $0 < \epsilon_2 < \epsilon_1 \le \epsilon$, we have

$$\begin{aligned} x(0,\ \epsilon_2) &= x(0) + \epsilon_2 < x(0) + \epsilon_1 = x(0,\ \epsilon_1), \\ x(t,\epsilon_2) &\le x(0,\epsilon_2) + \mathcal{I}x(t,\epsilon_2) + I^\alpha f_{\epsilon_2}(t,x(t,\epsilon_2)) \\ x(t,\epsilon_1) &\ge x(0,\epsilon_1) + \mathcal{I}x(t,\epsilon_1) + I^\alpha f_{\epsilon_1}(t,x(t,\epsilon_1)), \end{aligned} \tag{3.15}$$

where $\mathcal{I}x$ was introduced by (2.2). In view of Theorem 2.1 to get $x(t,\epsilon_2) < x(t,\epsilon_1),\ t \in (0,\lambda_0)$. Consider the family of functions $\{x(t,\epsilon)\}$ on $0 \le t \le \lambda_0$. We have

$$\begin{aligned} |x(t,\epsilon) - x(0,\epsilon)| &= \left|\mathcal{I}x(t,\epsilon) + I^\alpha f_\epsilon(t,x(t,\epsilon))\right| \\ &= \left|\mathcal{I}x(t,\epsilon_2) + I^\alpha\{f(t,x(t,\epsilon)) + \epsilon\}\right| \\ &\le \sum_{k=0}^{n} \frac{\rho(1+\|x\|)}{\Gamma(\alpha-\beta+k)} \int_0^t (s-t)^{\alpha-\beta+k-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left(\left|f(t,x(t,\epsilon))\right| + \epsilon\right) ds \\ &= \sum_{k=0}^{n} \frac{\rho(1+\|x\|)}{\Gamma(\alpha-\beta+k+1)} t^{\alpha-\beta+k} + \frac{M+\epsilon}{\Gamma(\alpha+1)} t^\alpha \\ &\le \sum_{k=0}^{n} \frac{\rho(1+\|x\|)}{\Gamma(\alpha-\beta+k+1)} {\lambda_0}^{\alpha-\beta+k} + \frac{2M+b}{2\Gamma(\alpha+1)} {\lambda_0}^\alpha \le b, \end{aligned} \tag{3.16}$$

where ρ was denoted by (3.4), $\|x\| = \max\{|x(t,\epsilon)| : t \in [0,\lambda_0],\ \epsilon \in [0,b/2]\}$ and

$$\lambda_0 = \min\left\{ r, \left[\frac{2b\Gamma(\alpha+1)}{(n+2)(2M+b)}\right]^{1/\alpha}, \left[\frac{b\Gamma(\alpha-\beta+k+1)}{\rho\ (\|x\|+1)(n+2)}\right]^{1/(\alpha-\beta+k)}, k = \overline{0,\ n} \right\}. \tag{3.17}$$

Showing that the family $\{x(t,\epsilon)\}$ is uniformly bounded. Also, if $0 \le t_1 \le t_2 \le \lambda_0$ then there exists a constant Λ^* such that

$$|x(t_1,\epsilon) - x(t_2,\epsilon)| \le 2\Lambda^* \left\{ \sum_{k=0}^{[n/2]} \frac{(t_2-t_1)^{\alpha-\beta+2k}}{\alpha-\beta+2k} + \sum_{k=0}^{[n/2]+1} \frac{(t_2-t_1)^{\alpha-\beta+2k+1}}{\alpha-\beta+2k+1} \right\} + \frac{2M+b}{2\Gamma(\alpha+1)}(t_2-t_1)^\alpha. \tag{3.18}$$

Following the computation similar to (3.10) with suitable changes. This proves that the family $\{x(t,\epsilon)\}$ is equicontinuous. Hence there exists a sequence $\{\epsilon_m\}$ with $\epsilon_m \to 0$ as $m \to \infty$ and the uniform limit

$$\mu(t) = \lim_{m\to\infty} x(t,\epsilon_m) \tag{3.19}$$

exists on $[0, \lambda_0]$. Clearly $\mu(0) = x_0$. The uniform continuity of f gives argument as before (as in Theorem 3.1), that $\mu(t)$ is a solution of IVP (1.2).

Next, we show that $\mu(t)$ is required maximal solution of (1.2), on $0 \le t \le \lambda_0$. Let $x(t)$ be a any solution of (1.2) on $0 \le t \le \lambda_0$. Then we have

$$\begin{gathered} x_0 < x_0 + \epsilon = x(0, \epsilon) \\ x(t) < x_0 + \mathcal{I}x(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_\epsilon(s, x(s))ds, \\ x(t, \epsilon) \ge x(0, \epsilon) + \mathcal{I}x(t, \epsilon) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_\epsilon(s, x(s, \epsilon)), \end{gathered} \tag{3.20}$$

where $x(0, \epsilon) = x_0 + \epsilon$, $f_\epsilon(t, x(t)) = f(t, x(t)) + \epsilon$, and $f_\epsilon(t, x(t, \epsilon)) = f(t, x(t, \epsilon)) + \epsilon$.

Using Theorem 2.1, we get $x(t) < x(t, \epsilon)$ on $[0, \lambda_0]$ as $\epsilon \to 0$. Therefore, the proof is complete. □

4. Global Existence

We need the following comparison result before we proceed further.

Theorem 4.1. *Assume that* $h : [0, T] \to \mathcal{R}^+$, $g : [0, T] \times \mathcal{R}^+ \to \mathcal{R}^+$ *are continuous, and* $g(t, u)$ *is nondecreasing with respect to the second argument such that*

$$h(t) \le h(0) + \mathcal{I}h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, h(s))ds, \ t \in [0, T], \tag{4.1}$$

where $\mathcal{I}$ *is defined by* (3.5). *Let* $m(t)$ *be the maximal solution of*

$$\mathcal{L}(\mathfrak{D})[u(t) - u(0)] = g(t, u), \quad u(0) = u_0 \ge 0, \tag{4.2}$$

existing on $[0, T)$ *such that* $h(0) \le u(0)$. *Then we have*

$$h(t) \le m(t), \quad t \in [0, T]. \tag{4.3}$$

Proof. In view of the definition of the maximal solution $m(t)$, it is enough to prove, to conclude (4.3), that

$$h(t) < u(t, \epsilon), \quad t \in [0, T], \tag{4.4}$$

where $u(t, \epsilon)$ is any solution of

$$\mathcal{L}(\mathfrak{D})[u(t) - u(0)] = g(t, u) + \epsilon, \quad u(0) = u_0 + \epsilon, \ \epsilon > 0. \tag{4.5}$$

Now, it follows from (4.4) that

$$u(t,\epsilon) > u(0,\epsilon) + \mathcal{I}u(t,\epsilon) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_\epsilon(s,u(s,\epsilon))ds, \tag{4.6}$$

where $u(0,\epsilon) = u_0 + \epsilon$. Then applying Theorem 2.1, we get immediately (4.1) and since $\lim_{\epsilon\to 0} u(t,\epsilon) = m(t)$ uniformly on each $t \in [0,T_0], T_0 < T$, the proof is complete. □

We are now in position to prove global existence result.

Theorem 4.2. *Assume that* $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *continuous. Let there exists function* $g : [0,\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ *continuous and nondecreasing with respect to the second argument such that*

$$|f(t,x)| \leq g(t,|x|) \quad \forall t \geq 0,\ x \in \mathbb{R}. \tag{4.7}$$

If the maximal solution of the initial value problem

$$\mathcal{L}(\mathfrak{D})[u(t) - u(0)] = g(t,u(t)), \qquad u(0) = u_0,\ t > 0 \tag{4.8}$$

exists in $[0,\infty)$ *then all solutions of the initial value problem*

$$\mathcal{L}(\mathfrak{D})[x(t) - x(0)] = f(t,x(t)), \qquad x(0) = x_0 \tag{4.9}$$

with $|x_0| \leq u_0$ *exist in* $[0,\infty)$.

Proof. Let $x(t,x_0)$ be any solution of IVP (4.8) such that $|x_0| \leq u_0$, which exists on $[0,\gamma)$ for $\gamma \in (0,\infty)$, and the value of γ cannot be increased further. Set $h(t) = |x(t,x_0)|$ for $0 \leq t < \gamma$. Then using the assumption (4.4), we get

$$h(t) \leq |x_0| + \mathcal{I}h(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,h(s))ds. \tag{4.10}$$

Applying the comparison Theorem 4.1, we obtain

$$h(t) = |x(t,x_0)| \leq m(t), \qquad 0 \leq t < \gamma. \tag{4.11}$$

Since $m(t)$ is assumed to exist on $[0,\infty)$, it follows that

$$|g(t,m(t))| \leq M, \qquad 0 \leq t \leq \gamma. \tag{4.12}$$

Now, let $0 \leq t_1 \leq t_2 < \gamma$. Then employing the arguments similar to estimating (3.19) and using (4.7) and the bounded M of g we arrive at

$$|x(t_2, x_0) - x(t_1, x_0)| \leq \frac{2M}{\Gamma(\alpha+1)}(t_2 - t_1)^{\alpha} + 2\Lambda \left\{ \sum_{k=0}^{[n/2]} \frac{(t_2 - t_1)^{\alpha-\beta+2k}}{\alpha - \beta + 2k} + \sum_{k=0}^{[n/2]+1} \frac{(t_2 - t_1)^{\alpha-\beta+2k+1}}{\alpha - \beta + 2k + 1} \right\}. \tag{4.13}$$

Letting $t_1, t_2 \to \gamma^-$ and using Cauchy criterion, it follows that $\lim_{t\to\gamma^-} x(t, x_0)$ exists. We define $x(\gamma, x_0) = \lim_{t\to\gamma^-} x(t, x_0)$ and consider the new IVP

$$\mathcal{L}(\mathfrak{D})[x(t) - x(0)] = f(t, x(t)), \quad x(\gamma) = x(\gamma, x_0). \tag{4.14}$$

By the assumed local existence, we find that $x(t, x_0)$ can be continued beyond γ, contradicting our assumption. Hence, every solution $x(t, x_0)$ of (4.9) exists on $[0, \infty)$, and the proof is complete. □

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