

Research Article

A Criterium for the Strict Positivity of the Density of the Law of a Poisson Process

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We translate in semigroup theory our result (Léandre, 1990) giving a necessary condition so that the law of a Markov process with jumps could have a strictly positive density. This result express, that we have to jump in a finite number of jumps in a “submersive” way from the starting point x to the end point y if the density of the jump process $p_1(x, z)$ is strictly positive in (x, y) . We use the Malliavin Calculus of Bismut type of (Léandre, (2008;2010)) translated in semi-group theory as a tool, and the interpretation in semi-group theory of some classical results of the stochastic analysis for Poisson process as, for instance, the formula giving the law of a compound Poisson process.

1. Introduction

We are interested in this paper in the following problem.

*Problem**. Let X be a random variable given by the solution of a stochastic differential equation, with law $p(dy)$. For what y $p(dy)$ is bounded below by $q(y)dy$, where $q(\cdot)$ is strictly positive continuous near y ?

This problem was solved by using the Malliavin Calculus. See the survey paper of Léandre [1] on that. For various applications of the Malliavin Calculus on heat kernels, we refer to the review of Kusuoka [2], Léandre [3], and Watanabe [4].

Let us explain the state of the art in the case of a diffusion. We consider $m + 1$ smooth vectors fields with bounded derivatives at each order X_i on \mathbb{R}^d and the diffusion generator $L = 1/2 \sum_{i>0} X_i^2 + X_0$. It generates a linear semigroup P_t acting on differentiable bounded functions f on \mathbb{R}^d :

$$\frac{\partial}{\partial t} P_t f(x) = L P_t f(x). \quad (1.1)$$

It is a semigroup in probability measures. It has a probabilistic representation [5]. Let w_t^i be a \mathbb{R}^m valued Brownian motion. Let us use the notation of formal path integrals of physics. The law of the Brownian motion is given formally as the Gaussian measure

$$d\mu = \frac{1}{Z} \exp \left[- \sum_{i>0} \int_0^1 \frac{|d/ds w_s^i|^2}{2ds} \right] dD(w.), \quad (1.2)$$

where $dD(w.)$ is a kind of formal Lebesgue measure. We introduce the stochastic differential equation in the Stratonovitch case issued from x :

$$dx_t(x) = \sum_{i>0} X_i(x_t(x)) dw_t^i + X_0(x_t(x)) dt. \quad (1.3)$$

Then,

$$P_t f(x) = E[f(x_t(x))]. \quad (1.4)$$

If $w. \rightarrow x_1(x)$ is a submersion in some sense in \bar{w} , then we can apply in some sense the implicit function theorem in order to get a lower bound of the law of $x_1(x)$ by a measure having a strictly positive density in the values of $x_1(x)$ in \bar{w} with respect of the Lebesgue measure on \mathbb{R}^d . The problem is that the solution of the stochastic differential equation (1.3) is only almost surely defined. So the use of the implicit theorem leads to some difficulties which were overcome by Bismut in [6]. The use of Bismut's procedure allows to [7] to solve Problem*. See [8] for a translation of the proof of [7] in semigroup theory.

Plenty of the standard tools of stochastic analysis were translated recently by Léandre in semigroup theory. See the review [9, 10] on that. Problem* was solved for a diffusion by using the Malliavin Calculus of Bismut type in semigroup theory in [8].

We are interested in solving Problem* in the case of a jump process. Let us consider a generator of Lévy type. If f is a differentiable function

$$Lf(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x)) \mu(dz), \quad (1.5)$$

it generates a linear semigroup P_t satisfying the parabolic equation

$$\frac{\partial}{\partial t} P_t f(x) = LP_t f(x). \quad (1.6)$$

It is a semigroup in probability measures [11–13]. It is represented by a jump process with independent increments z_t :

$$P_t f(x) = E[f(x + z_t)]. \quad (1.7)$$

Tortrat [14] studied the support S of the law of z_t : if $y_1 \in S$ and $y_2 \in S$, then $y_1 + y_2$ belong to S . If μ has a finite mass λ , then the process z_t has the law of a compound Poisson process: z_t is sum of his jumps. There is only a finite number of jumps. The jumps are all independents with law $\mu(dz)/\lambda$ and the times where the jumps occur follow the law of a standard Poisson process with parameter λ . We will give a proof, uniquely based upon algebraic computations on semi groups, of this fact in the paper.

Problem* was solved in [1] by using the Malliavin Calculus for jump processes (see [15–17] for related works). μ is called the Lévy measure. For that we need some regularity on the Lévy measure $\mu(dz)$. Under regularity assumption on $\mu(dz)$, [1] used another time the implicit function theorem, when we can jumps in a finite number of jumps in a “submersive” way between the starting point and the end point. Recently we have translated in semigroup theory plenty of tools of the stochastic analysis for Poisson processes [18–23]. Our goal is to translate in semigroup theory the result of [1].

For material on stochastic differential equations driven by jump processes, we refer to the books [13, 24, 25]. For the analytic side of the theory of Markov processes with jumps, we refer to the books [11–13].

This paper enters in a general program which would like that stochastic analysis tools become available for partial differential equation different of the parabolic equations whose generators satisfy the maximum principle [26, 27].

2. Statement of the Main Theorems

The goal of this paper is to give the proof of the two next theorems originally proved by [1] by using stochastic analysis and the Malliavin Calculus of Bismut type for jump processes of [28].

Let us consider m functions $g_j(z)$ positive with compact support on \mathbb{R} continuous except in 0 equal to $|z|^{-1-\alpha_j}$ near 0 with $\alpha_j \in]0, 1[$.

Let us introduce m functions $\gamma_j(z)$ with bounded derivatives at each order, equal to 0 in 0 with values in \mathbb{R}^d .

We consider the Markov generator

$$Lf(x) = \sum \int_{\mathbb{R}} (f(x + \gamma_j(z)) - f(x))g_j(z)dz. \quad (2.1)$$

We do the following hypothesis.

Hypothesis 2.1. There exists an r such that the family of vectors $\{\cup_{j,k \leq r} (d^k/dz^k)\gamma_j(0)\}$ generates \mathbb{R}^d .

L generates a convolution linear semigroup P_t in probability measures acting on differentiable bounded functions f . P_t satisfies the parabolic equation

$$\frac{\partial}{\partial t} P_t f(x) = LP_t f(x). \quad (2.2)$$

Under Hypothesis 2.1, [20, 29, 30] proved that P_1 has a smooth heat kernel $p_1(x, y)$:

$$P_1 f(x) = \int_{\mathbb{R}^d} f(y) p_1(x, y) dy. \quad (2.3)$$

We denote

$$F_{k,j}(z) = \sum \gamma_{j_i}(z_i), \quad (2.4)$$

where $j = \{j_1, \dots, j_k\}$.

Theorem 2.2. *If $p_1(x, y) > 0$, then there exists $k, j_i, z_i^0 \neq 0$ such that $g_{j_i}(z_i^0) > 0$ such that,*

- (i) $F_{k,j}(z_i^0) = y - x$,
- (ii) $(z) \rightarrow F_{k,j}(z)$ is a submersion in z_i^0 .

Remark 2.3. Let us explain heuristically the theorem. Let z_t^j be the process with independent increments associated to the generator

$$L_j f(y) = \int_{\mathbb{R}} (f(y+z) - f(y)) g_j(z) dz, \quad (2.5)$$

where $y \in \mathbb{R}$. The processes z_t^j are independents, and the time of their jumps are disjoint. We put

$$x_t(x) = x + \sum_{s \leq t, j} \gamma_j(\Delta z_s^j). \quad (2.6)$$

Then,

$$P_t f(x) = E[f(x_t(x))]. \quad (2.7)$$

The theorem explains that we have to jump in a finite numbers of jumps in a submersive way from x to y if we want $p_1(x, y) > 0$. Let us give some explanations what we mean about this fact, because the jump process has in fact an infinite number of jumps because the Lévy measure is of infinite mass. We take

$$z_t^{j,\epsilon} = \sum_{s \leq t} \mathbb{I}_{[-\epsilon, \epsilon]^c}(\Delta z_s^j) \Delta z_s^j. \quad (2.8)$$

$z_t^{j,\epsilon}$ has generator

$$L_j^\epsilon f(y) = \int_{|z| > \epsilon} (f(y+z) - f(y)) g_j(z) dz. \quad (2.9)$$

The jump process

$$x_t^\epsilon(x) = x + \sum_{s \leq t, j} \gamma_j \left(\Delta z_s^{j, \epsilon} \right) \tag{2.10}$$

has only a finite number of jumps because its Lévy measure is of finite mass and its law gives a good approximation of the law of $x_t(x)$ if ϵ is small enough!

We consider some vectors e_j and a smooth vector fields X_0 with bounded derivatives at each order. We consider the generator

$$Lf(x) = \sum \int_{\mathbb{R}} (f(x + e_j z) - f(x)) g_j(z) dz + \langle df(x), X_0(x) \rangle. \tag{2.11}$$

It generates a Markov semigroup P_t ,

$$\frac{\partial}{\partial t} P_t f(x) = LP_t f(x) \tag{2.12}$$

if is bounded differentiable. If $g_j(z) = |z|^{1-\alpha_j}$, the L is classically related to fractional powers of the Laplacian [31].

We do the following Hypothesis.

Hypothesis 2.4. Consider $\inf_{x \in \mathbb{R}^d, |\xi|=1} \sum_j |\langle \xi, e_j \rangle| + |\langle \xi, (\partial/\partial x) X_0(x) e_j \rangle| > 0$.

In such a case, [19, 29, 30] has proven that there exists a smooth heat kernel $p_1(x, y)$:

$$P_1 f(x) = \int_{\mathbb{R}^d} f(y) p_1(x, y) dy. \tag{2.13}$$

We consider $t_1 < t_2 < \dots < t_k < 1$ and we denote by $t. = \{t_1, \dots, t_k\}$. We introduce the differential impulsive equation starting from x :

$$dx_s(j., t., z.) (x) = X_0(x_s(j., t., z.) (x)) ds, \quad \Delta x_{t_l}(j., t., z.) (x) = e_{j_l} z_l. \tag{2.14}$$

We denote

$$F_{k,j,t.}(z.) = x_1(j., t., z.) (x). \tag{2.15}$$

Theorem 2.5. *The condition $p_1(x, y) > 0$ implies that there exists $j, k, t.$, and $z_l^0 \neq 0, g_{j_l}(z_l^0) > 0$ such that:*

- (i) $F_{k,j,t.}(z^0) = y,$
- (ii) $z \rightarrow F_{k,j,t.}(z.)$ is a submersion in $z_l^0.$

Remark 2.6. Let us explain heuristically this theorem. We consider the processes with independent increments z_t^j . We consider the stochastic differential equation

$$x_t(x) = x + \sum z_t^j e_j + \int_0^t X_0(x_s(x)) ds. \quad (2.16)$$

Then,

$$P_t f(x) = E[f(x_t(x))]. \quad (2.17)$$

It has since $\int_{\mathbb{R}} g_j(z) = \infty$ an infinite number of jumps. We take

$$z_t^{j,\epsilon} = \sum_{s \leq t} \mathbb{I}_{[-\epsilon, \epsilon]^c}(\Delta z_s^j) \Delta z_s^j. \quad (2.18)$$

$z_t^{j,\epsilon}$ has a finite number of jumps and has generator

$$L_j^\epsilon f(y) = \int_{|z| > \epsilon} (f(y+z) - f(y)) g_j(z) dz. \quad (2.19)$$

We consider the stochastic differential equation

$$x_t^\epsilon(x) = x + \sum z_t^{j,\epsilon} e_j + \int_0^t X_0(x_s^\epsilon(x)) ds. \quad (2.20)$$

The law of $x_1^\epsilon(x)$ is a good approximation of the law of $x_1(x)$ if ϵ is small enough. This express the fact that by a finite number of jumps, $x_s^\epsilon(x)$ has to pass from x to y in a submersive way if $p_1(x, y) > 0$.

3. Two Results on Jump Processes Translated in Semigroup Theory

We consider $\mathbb{R}^{\hat{d}}$, $\hat{x} \in \mathbb{R}^{\hat{d}}$, $\hat{z} \in \mathbb{R}^{\hat{d}}$, and $\hat{\mu}$ a positive measure on $\mathbb{R}^{\hat{d}}$ such that $\hat{\lambda} = \int \hat{\mu}(d\hat{z}) < \infty$. We introduce the expression

$$\hat{G}^n \hat{f}(\hat{x}) = \hat{\lambda}^{-n} \int_{(\mathbb{R}^{\hat{d}})^n} \hat{f}\left(\hat{x} + \sum_i \hat{z}_i\right) \prod d\hat{\mu}(\hat{z}_i). \quad (3.1)$$

We consider the generator

$$\hat{L}(\hat{f})(\hat{x}) = \int_{\mathbb{R}^{\text{hstd}}} (\hat{f}(\hat{x} + \hat{z}) - \hat{f}(\hat{x})) d\hat{\mu}(\hat{z}). \quad (3.2)$$

It is a bounded operator on the space of continuous bounded functions endowed with the uniform norm. It generates therefore a semigroup \hat{P}_t .

Theorem 3.1 (compound Poisson process). *We have the formula*

$$\hat{P}_t \hat{f}(\hat{x}) = \exp[-\hat{\lambda}t] \sum_n \frac{(\hat{\lambda}t)^n}{n!} \hat{G}^n \hat{f}(\hat{x}). \quad (3.3)$$

Proof. In order to simplify the exposition, we suppose $\hat{\lambda} = 1$.

We have the recursion formula

$$\hat{G}^n \hat{f}(\hat{x}) - \hat{G}^{n-1} \hat{f}(\hat{x}) = \hat{L} \hat{G}^{n-1} \hat{f}(\hat{x}) \quad (3.4)$$

such that

$$\hat{G}^n = (I + \hat{L})^n \quad (3.5)$$

But \hat{L} is a bounded operator on the set of continuous functions endowed with the uniform norm. Therefore, the semigroup \hat{P}_t satisfies to

$$\hat{P}_t \hat{f}(\hat{x}) = \sum_{n \geq 0} t^n / n! (\hat{L})^n \hat{f}(\hat{x}). \quad (3.6)$$

We write

$$(\hat{L})^n = (\hat{L} + I - I)^n \quad (3.7)$$

such that

$$(\hat{L})^n = \sum_p (-1)^p C_n^p \hat{G}^{n-p}. \quad (3.8)$$

Therefore,

$$\hat{P}_t \hat{f}(\hat{x}) = \sum_{n \geq 0, p \leq n} t^n / n! (-1)^p C_n^p \hat{G}^{n-p} \hat{f}(\hat{x}, V) = \sum t^n / n! \hat{G}^{n,\epsilon} \hat{f}(\hat{x}) \exp[-t]. \quad (3.9)$$

□

Let us consider now a generator

$$\hat{L}(\hat{f})(\hat{x}) = \int_{\mathbb{R}^d} (\hat{f}(\hat{x} + \hat{z}) - \hat{f}(\hat{x})) d\hat{\mu}_{\hat{x}}(\hat{z}) + \langle d\hat{f}(\hat{x}), \hat{X}_0(\hat{x}) \rangle. \quad (3.10)$$

We suppose that the total mass of $\hat{\mu}_{\hat{x}}$ is finite and is equal to the constant quantity $\hat{\lambda}$ and that $\hat{\mu}_{\hat{x}}$ depends continuously of \hat{x} for the strong topology. \hat{L} generates a semigroup on the space of continuous functions endowed with the uniform norm.

Let \widehat{L}^1 be the generator

$$\widehat{L}^1(\widehat{f})(\widehat{x}) = \langle d\widehat{f}(\widehat{x}), \widehat{X}_0(\widehat{x}) \rangle. \quad (3.11)$$

It generates a semigroup \widehat{P}_t^1 . We suppose that $\widehat{P}_t^1(1) = 1$. It is the same to suppose that the solution of the ordinary differential equation

$$d\widehat{x}_s^1 = \widehat{X}_0(\widehat{x}_s^1)ds \quad (3.12)$$

does not blow up

$$\widehat{L}^2(\widehat{f})(\widehat{x}) = \int_{\mathbb{R}^d} (\widehat{f}(\widehat{x} + \widehat{z}) - \widehat{f}(\widehat{x})) d\widehat{\mu}_{\widehat{x}}(\widehat{z}). \quad (3.13)$$

It is a bounded operator on the set of uniformly bounded functions endowed with the uniform topology. Therefore it generates a semigroup on the set of bounded continuous functions. We get the following translation of (2.20) in semigroup theory.

Theorem 3.2. *We have if \widehat{f} is bounded continuous*

$$\widehat{P}_t(\widehat{f})(\widehat{x}) = \widehat{f}(\widehat{x}) + \exp[-\widehat{\lambda}t] \times \int_{0 < s_1 < s_2 < \dots < s_r < t} \widehat{P}_{s_1}^1 \widehat{G}^2 \dots \widehat{G}^2 \widehat{P}_{t-s_r}^1 \widehat{f}(\widehat{x}) ds_1 \dots ds_r. \quad (3.14)$$

Proof. We suppose to simplify that $\widehat{\lambda} = 1$. By the classical Volterra expansion, we get

$$\widehat{P}_t(\widehat{f})(\widehat{x}) = \widehat{f}(\widehat{x}) + \int_{0 < s_1 < s_2 < \dots < s_r < t} \widehat{P}_{s_1}^1 \widehat{L}^2 \dots \widehat{L}^2 \widehat{P}_{t-s_r}^1 \widehat{f}(\widehat{x}) ds_1 \dots ds_r. \quad (3.15)$$

We write

$$\widehat{L}^2 = \widehat{G}^2 - I. \quad (3.16)$$

The previous Volterra expansion can be written as

$$\widehat{P}_t(\widehat{f})(\widehat{x}) = \widehat{f}(\widehat{x}) + \int_{0 < s_1 < s_2 < \dots < s_r < t} \widehat{P}_{s_1}^1 (\widehat{G}^2 - I) \dots (\widehat{G}^2 - I) \widehat{P}_{t-s_r}^1 \widehat{f}(\widehat{x}) ds_1 \dots ds_r. \quad (3.17)$$

We distribute in the last expression, and we use the two formulas

$$\widehat{P}_{s_1}^1 \widehat{P}_{s_2}^1 = \widehat{P}_{s_1+s_2}^1, \quad (3.18)$$

$$\int_{t_1 < s_2 < \dots < s_r < t_2} ds_1 \dots ds_r = \frac{(t_2 - t_1)^r}{r!}. \quad (3.19)$$

We recognize

$$\begin{aligned} \widehat{P}_t &= I + \sum_r \sum_{n_1, \dots, n_r} (-1)^{\sum n_i} \times \int_{0 < s_1 < \dots < s_r < t} \frac{s_1^{n_1}}{n_1!} \widehat{P}_{s_1}^1 \widehat{G}^2 \frac{(s_2 - s_1)^{n_2}}{n_2!} \dots \widehat{G}^2 \widehat{P}_1^{t-s_r} \frac{(t - s_r)^{n_r}}{n_r!} ds_1 \dots ds_r \\ &= I + \sum_r \int_{0 < s_1 < \dots < s_r < t} \times \exp[-s_1] \widehat{G}^2 \widehat{P}_{s_1}^1 \exp[-(s_2 - s_1)] \dots \\ &\quad \times \widehat{G}^2 \widehat{P}^1(t - s_r) \exp[-(t - s_r)] ds_1 \dots ds_r. \end{aligned} \tag{3.20}$$

The result follows from the fact that

$$\exp[-t] = \exp[-s_1] \exp[-(s_2 - s_1)] \dots \exp[-(t - s_r)]. \tag{3.21}$$

□

4. Proof of Theorem 2.2

Let \widehat{L} the Malliavin generator acting on smooth function on $\mathbb{R}^d \times \mathbb{M}_d$, where \mathbb{M}_d is the space of symmetric matrices on \mathbb{R}^d :

$$\widehat{L}\widehat{f}(x, V) = \sum \int_{\mathbb{R}} \left(\widehat{f}\left(x + \gamma_j(z), V + \nu(z) \langle \cdot, \gamma'_j(z) \rangle^2\right) - \widehat{f}(x, V) \right) g_j(z) dz. \tag{4.1}$$

$\nu(z)$ is a smooth positive function with compact support equal to z^4 on a neighborhood of 0. V is called the Malliavin matrix. \widehat{L} generates a semigroup \widehat{P}_t called the Malliavin semigroup. Under Hypothesis 2.1, we have for all p [20]

$$\widehat{P}_1[V^{-p}](x, 0) < \infty. \tag{4.2}$$

Let g be a smooth positive function equals to 1 if $|V| < 1$ and equal to 0 if $|V| > 2$. We consider the measure μ_K

$$f \longrightarrow \widehat{P}_1 \left[g \left(\frac{V^{-1}}{K} \right) f \right] (x, 0). \tag{4.3}$$

Proposition 4.1. *The measure μ_K has a smooth density $p^K(x, y)$, and, when $K \rightarrow \infty$, $p^K(x, y) \rightarrow p_1(x, y)$ uniformly.*

Proof. We follow the argument of [20]. We put $\mathbb{E}^l = \mathbb{R}^d \times \mathbb{M}_d \times \mathbb{R}^{d_2} \times \dots \times \mathbb{R}^{d_r}$, $x^l = (x_1, v, x_2, \dots, x_r)$. We consider a bounded map g , $g(0) = 0$, from \mathbb{R}^m into \mathbb{E}^l . Its values in \mathbb{R}^d is $\gamma(z) = \sum \gamma_j(z_j)$ and its values in \mathbb{M}_d is $\sum v(z_j) \langle \cdot, \gamma'_j(z_j) \rangle^2$. $d\mu(z) = \sum g_j(z_j) dz_j$. We consider the generator

$$L^l f(x^l) = \int_{\mathbb{R}^m} (f^l(x^l + g(z)) - f^l(x^l)) d\mu(z). \quad (4.4)$$

This generates a semigroup P_t^l . We put $g_K(V) = g(V/K)$. By using the integration by parts formulas of [20],

$$\widehat{P}_1 \left[(1 - g_K(V^{-1})) D^\alpha f \right] (x, 0) = P_1^l [f\theta] (x, 0, \dots), \quad (4.5)$$

where P_t^l is a semi group of the previous type, θ a polynomial in the components, V^{-1} , and of valuation 1 in $(1 - g_K)(V^{-1})$ and the derivatives of $g_K(V^{-1})$. α is a multi-index. By Theorem 3 of [20], we deduce that

$$\left| \widehat{P}_1 \left[(1 - g_K(V^{-1})) D^\alpha f \right] (x, 0) \right| \leq C(K) \|f\|_\infty \quad (4.6)$$

when $C(K) \rightarrow 0$ when $K \rightarrow \infty$. Therefore the result is obtained. \square

Let $\epsilon > 0$. Let

$$\widehat{L}_\epsilon \widehat{f}(x, V) = \sum \int_{|z| > \epsilon} \left(\widehat{f} \left(x + \gamma_j(z), V + v(z) \langle \cdot, \gamma'_j(z) \rangle^2 \right) - \widehat{f}(x, V) \right) g_j(z) dz. \quad (4.7)$$

By the same procedure, we define analog generators L_ϵ^l . We deduce several semigroups \widehat{P}_t^ϵ and $P_t^{\epsilon, l}$. We consider the measure μ_K^ϵ

$$f \longrightarrow \widehat{P}_1^\epsilon \left[g_K(V^{-1}) f \right] (x, 0). \quad (4.8)$$

Proposition 4.2. μ_K^ϵ has a density $p_1^{\epsilon, K}(x, y)$, and, when $\epsilon \rightarrow 0$, $p_1^{\epsilon, K}(x, y)$ tends uniformly to $p_1^K(x, y)$.

Proof. Let (α) be a multi-index. We have

$$\mu_K [D^\alpha f] - \mu_K^\epsilon [D^\alpha f] = P_1^l [f\theta] (x, 0, \dots) - P_1^{\epsilon, l} [f\theta] (x, 0, \dots), \quad (4.9)$$

where θ is a polynomial in u^l , V^{-1} and of valuation 1 in $g_K(V^{-1})$ and its derivatives. The result will come from the next lemma. \square

Lemma 4.3. Let θ be a polynomial in u^l , V^{-1} and of valuation 1 in $g_K(V^{-1})$ and its derivatives. Then when $\epsilon \rightarrow 0$

$$P_1^{\epsilon, l} [f\theta] (x, 0, \dots) \longrightarrow P_1^l [f\theta] (x, 0, \dots). \quad (4.10)$$

Proof. If θ is smooth bounded, we have by Duhamel formula

$$P_1^l[f\theta](x, 0) = P_1^{l,\epsilon}[f\theta](x, 0, \dots) + \int_0^1 P_s^{l,\epsilon}(L^l - L_\epsilon^l)P_{1-s}[f\theta](x, 0, \dots), \quad (4.11)$$

and the result goes. It remains to remark that under the previous condition $(L^l - L_\epsilon^l)P_{1-s}[f\theta](x, 0, \dots)$ has a polynomial behaviour whose component tends to zero and to apply Theorem 3 of [20]. This comes from the fact that $P_{1-s}[f\theta]$ is a polynomial in x_1, \dots, x_d and is differentiable bounded in v because we keep only bounded values of V^{-1} due to the apparition of $g(V^{-1}/K)$. \square

Proof of Theorem 2.2. If $p_1(x, y) > 0$, there exists a K, ϵ such that $p^{\epsilon,K}(x, y) > 0$.

Let us introduce $\epsilon > 0$. We put

$$\widehat{L}_\epsilon f(x, V) = \int_{|z|>\epsilon} (f((x, V) + \widehat{\gamma}(z)) - f(x, V))d\mu(z). \quad (4.12)$$

To simplify the exposition, we suppose that $\int_{|z|>\epsilon} d\mu(z) = 1$.

We put

$$\widehat{G}^{n,\epsilon} \widehat{f}(x, V) = \int_{|z^i|>\epsilon} f(\widehat{F}_n(z^1, \dots, z^n; x, V)) \prod d\mu(z^i), \quad (4.13)$$

where \widehat{F} is defined as in (2.4) but with $\widehat{\gamma}$. By Theorem 3.1,

$$\widehat{P}_t^\epsilon f(x, V) = \sum t^n/n! \widehat{G}^{n,\epsilon} \widehat{f}(x, V) \exp[-t]. \quad (4.14)$$

Since $p^{\epsilon,K}(x, y) > 0$, the measure $f \rightarrow \widehat{G}^{n,\epsilon}(f(\cdot)g(V^{-1}/K))$ has a strictly positive density in y for some n . This measure is equal to the measure

$$f \rightarrow \sum_{j_1, \dots, j_n} \int \dots \int_{|z_l|>0} f(F_{n,j}) g\left(\frac{V^{-1}}{K}\right) \prod g_{j_l}(z_l) dz_l. \quad (4.15)$$

One of the measure in the above sum has a stricly positive density in y , and, therefore, nearby y . So there exists for y' close from $yn, j_l, |z_l| > \epsilon, g_{j_l}(z_l) > 0$ such that

(i) $F_{n,j}(z) = y' - x,$

(ii) The matrix $V_{n,j}(z) = \sum v(z_l) \langle \cdot, \gamma'_{j_l}(z_l) \rangle^2$ has an inverse bounded by K .

It remains to remark that the Gram matrix associated to $F_{n,j}(z)$ is equal to $\sum \langle \cdot, \gamma'_{j_l}(z_l) \rangle^2$ is larger to $CV_{n,\epsilon}(z)$ and to apply the implicit function theorem. \square

5. Proof of Theorem 2.5

Let us consider the Malliavin generator

$$\begin{aligned} \widehat{L}\widehat{f}(x, U, V) &= \sum \int_{\mathbb{R}} \left(\widehat{f}\left(x + ze_j, U, V + \nu(z)\langle \cdot, U^{-1}e_j \rangle^2\right) - \widehat{f}(x, U, V) \right) g_j(z) dz \\ &+ \left\langle d_x \widehat{f}(x, U, V), X_0(x) \right\rangle + \left\langle d_U \widehat{f}(x, U, V), \frac{\partial}{\partial x} X_0(x) U \right\rangle. \end{aligned} \quad (5.1)$$

U belong to \mathbb{G}_d , the space of invertible matrices, and V belong to \mathbb{M}_d . V is called the Malliavin matrix.

As in the previous part, we approximate \widehat{L} by a generator whose Lévy measure is of finite mass. We get for $\epsilon > 0$,

$$\begin{aligned} \widehat{L}_\epsilon \widehat{f}(x, U, V) &= \sum \int_{|z|>\epsilon} \left(\widehat{f}\left(x + ze_j, U, V + \nu(z)\langle \cdot, U^{-1}e_j \rangle^2\right) - \widehat{f}(x, U, V) \right) g_j(z) dz \\ &+ \left\langle d_x \widehat{f}(x, U, V), X_0(x) \right\rangle + \left\langle d_U \widehat{f}(x, U, V), \frac{\partial}{\partial x} X_0(x) U \right\rangle. \end{aligned} \quad (5.2)$$

\widehat{L} and \widehat{L}_ϵ generate Markov semigroup \widehat{P}_t and \widehat{P}_t^ϵ .

We repeat with some algebraic modifications due to [19] the considerations of the previous part. Let $K > 0$. We consider the measure μ_K^ϵ

$$f \longrightarrow \widehat{P}_1^\epsilon \left[g \left(\frac{V^{-1}}{K} \right) f \right] (x, I, 0). \quad (5.3)$$

It has a density $p^{\epsilon, K}(x, y)$. When $K \rightarrow \infty$ the density $p^{0, K}(x, z)$ of μ_K^0 tends uniformly to $p_1(x, z)$ in z . When $\epsilon \rightarrow \infty$, the density $p^{\epsilon, K}(x, z)$ tends uniformly in z to $p^{0, K}(x, z)$. Therefore, if $p_1(x, y) > 0$, we can find ϵ and K such that $p^{\epsilon, K}(x, y) > 0$.

Let \overline{P}_s be the semi group generated by \overline{L} :

$$\overline{L}\widehat{f}(x, U, V) = \left\langle d_x \widehat{f}(x, U, V), X_0(x) \right\rangle + \left\langle d_U \widehat{f}(x, U, V), \frac{\partial}{\partial x} X_0(x) U \right\rangle. \quad (5.4)$$

Let \widehat{L}_ϵ defined by:

$$\widehat{L}_\epsilon^1 \widehat{f}(x, U, V) = \sum \int_{|z|>\epsilon} \left(\widehat{f}\left(x + ze_j, U, V + \nu(z)\langle \cdot, U^{-1}e_j \rangle^2\right) - \widehat{f}(x, U, V) \right) g_j(z) dz. \quad (5.5)$$

We suppose to simplify the exposition that $\sum \int_{|z|>\epsilon} g_j(z) dz = 1$.

We put

$$\widehat{G}_\epsilon \widehat{f}(x, U, V) = \sum \int_{|z|>\epsilon} \left(\widehat{f}\left(x + ze_j, U, V + \nu(z)\langle \cdot, U^{-1}e_j \rangle^2\right) \right) g_j(z) dz. \quad (5.6)$$

Let us use Theorem 3.2. If $p^{\epsilon, K}(x, y) > 0$, then there exists a k such that the measure

$$f \longrightarrow \int_{0 < s_1 < \dots < s_k < 1} \bar{P}_{s_1} \hat{G}_\epsilon \dots \hat{G}_\epsilon \bar{P}_{1-s_k} \left[g \left(\frac{V^{-1}}{K} \right) f \right] (x, I, 0) ds_1 \dots ds_k \quad (5.7)$$

has a density $p_k^{\epsilon, K}(x, y) > 0$. Therefore, there exist $0 < t_1 < \dots < t_k < 1$ such that the measure

$$f \longrightarrow \bar{P}_{t_1} \hat{G}_\epsilon \dots \hat{G}_\epsilon \bar{P}_{t_k} \left[g \left(\frac{V^{-1}}{K} \right) f \right] (x, I, 0) \quad (5.8)$$

has a strictly positive density near y . We consider the system of impulsive equation issued from $(x, I, 0)$:

$$\begin{aligned} dx_s(j, t, z)(x) &= X_0(x_s(j, t, z)(x)) ds; & \Delta x_{t_l}(j, t, z)(x) &= e_{j_l} z_l, \\ dU_s(j, t, z) &= \frac{\partial}{\partial x} X_0(x_s(j, t, z)(x)) U_s(j, t, z) ds, & & \\ \Delta V_{t_l}(j, t, z) &= v(z_l) \langle \cdot, U_{t_l}^{-1} e_{j_l} \rangle^2. & & \end{aligned} \quad (5.9)$$

We remark that

$$\begin{aligned} f &\longrightarrow \bar{P}_{t_1} \hat{G}_\epsilon \dots \hat{G}_\epsilon \bar{P}_{t_k} \left[g \left(\frac{V^{-1}}{K} \right) f \right] (x, I, 0) \\ &= \sum_{j_1, \dots, j_k} \int \dots \int_{|z_{j_l}| > \epsilon} f \left(x_1(j, t, z) g \left(\frac{V_1(j, t, z)}{K} \right) \right) \prod g_{j_l}(z_l) dz_l. \end{aligned} \quad (5.10)$$

Therefore, the density of one of the measure

$$f \longrightarrow \int \dots \int_{|z_{j_l}| > \epsilon} f \left(x_1(j, t, z) g \left(\frac{V_1(j, t, z)}{K} \right) \right) \prod g_{j_l}(z_l) dz_l \quad (5.11)$$

is strictly positive in y !

From (5.11), we see that there exists j, t , such that, for some $|z_j| > \epsilon$, $g_{j_l}(z_k) > 0$ we have for y' close from y

- (i) $x_1(j, t, z)(x) = y'$,
- (ii) $V_1(j, t, z)^{-1}$ is bounded by K .

But the Gram matrix associated to $x_1(j, t, z)(x)$ is equal to $\sum \langle \cdot, U_1 U_{t_l}^{-1} e_{j_l} \rangle^2$. It has therefore an inverse bounded by CK . The result arises by the implicit function theorem.

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